Introduction to Graphs

2 is a real number, and 3 is a real number. We can take those two numbers and write them as a pair of real numbers: $(2, 3)$. When we write a pair of real numbers, the order is important. That is to say that $(2, 3)$ is not the same pair as $(3, 2)$.

Unfortunately, $(2, 3)$ is also the way we write the interval of real numbers between 2 and 3. We have to try hard to never confuse a pair of numbers for an interval, but it's usually clear from the context of a problem whether $(2, 3)$ refers to a pair of numbers or to an interval.

$\mathbb{R}^2$ is the set of all pairs of real numbers. So $(2, 3) \in \mathbb{R}^2$, and $(3, 2) \in \mathbb{R}^2$, and $(\sqrt{2}, -7) \in \mathbb{R}^2$, etc.. Any pair of real numbers is called a *point* in $\mathbb{R}^2$.

Suppose $f : A \to B$ is a function with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. The *graph* of f is the subset of $\mathbb{R}^2$ consisting of all points of the form $(a, f(a))$.

Examples.

• If $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = 5x$, then $f(3) = 5(3) = 15$. We put 3 in, and got 15 out. That means the point $(3, 15)$ is in the graph of f.

Also, $f(1) = 5$, so $(1, 5)$ is a point in the graph of f, and $(2, f(2)) = (2, 10)$ is a point in the graph of f as well.

• Suppose $g : \mathbb{R} \to \mathbb{R}$ is the function where $g(x) = x - 2$. If you put 2 in to g, then 0 comes out. That means the point $(2, 0)$ is in the graph of g.

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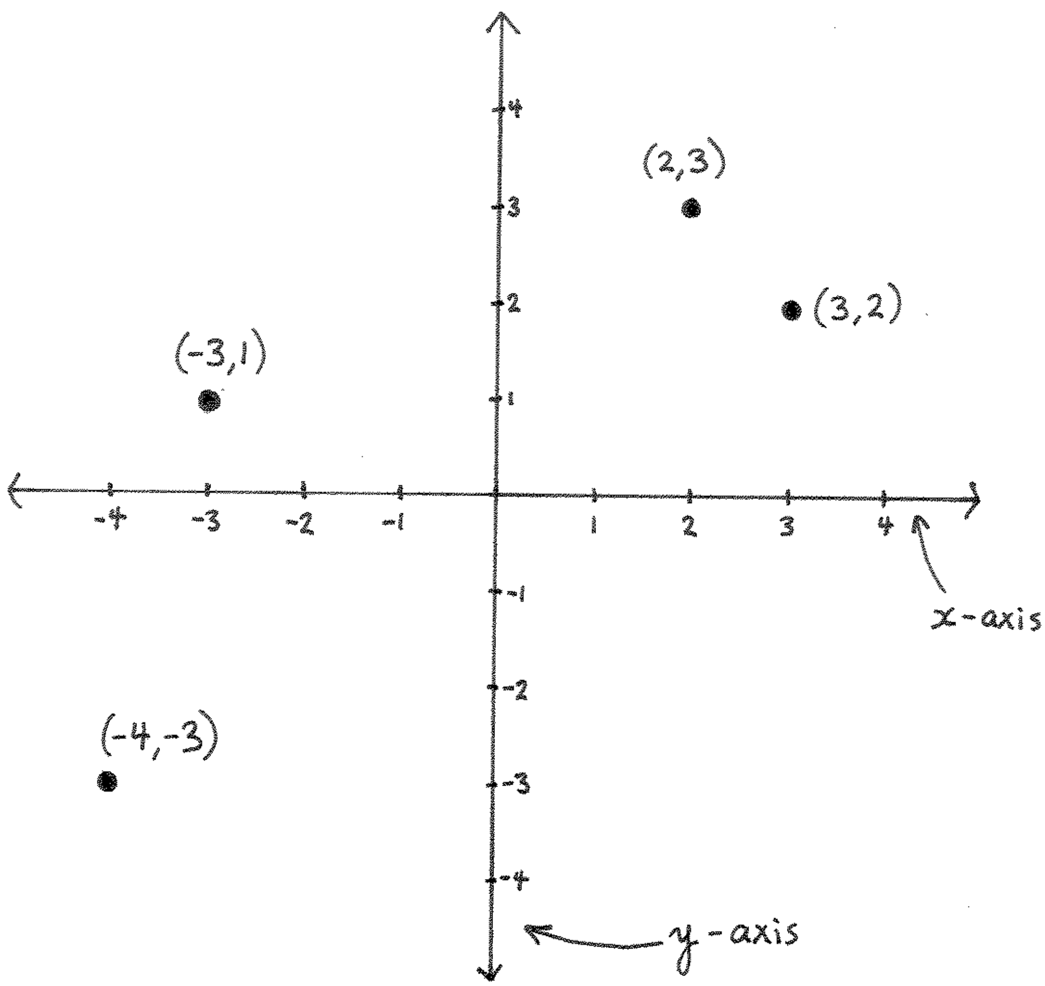
Drawing $\mathbb{R}^2$

The set $\mathbb{R}^2$ is a plane. The first coordinate of a point in $\mathbb{R}^2$ measures the horizontal. The second number measures the vertical.

The set of points in $\mathbb{R}^2$ of the form $(x, 0)$ creates a horizontal line called the *x-axis.*

The set of points in $\mathbb{R}^2$ of the form $(0, y)$ creates a vertical line called the *y-axis.*

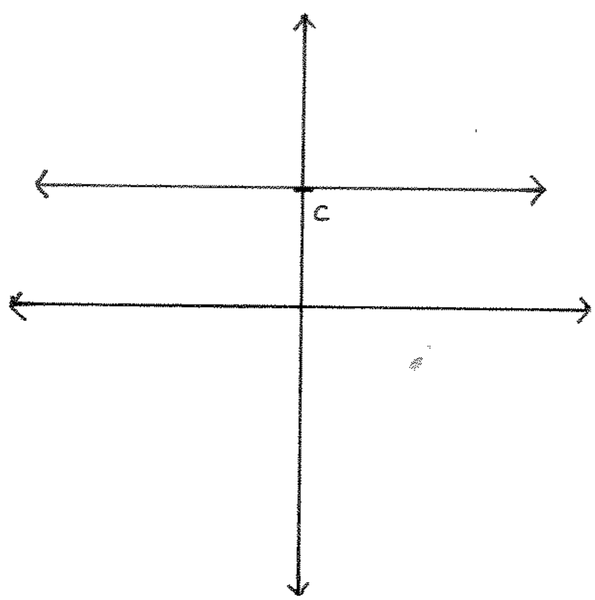


Drawing graphs

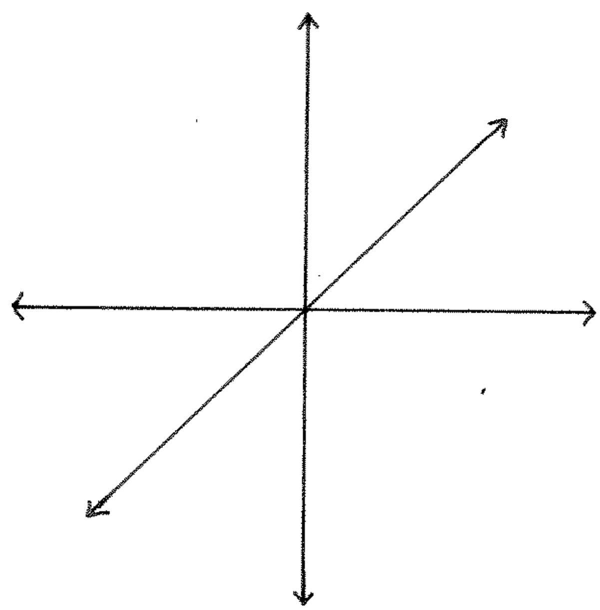
The graph of a function is a subset of $\mathbb{R}^2$. You draw it by marking all of the points in the graph.

Graphs of important functions

Some functions are important enough in mathematics that you should be able to draw their graphs quickly (and you will be required to do so on exams). A list of these important functions includes constant functions; the identity function id; $f(x) = x^n$ for an even $n \in \mathbb{N}$; $f(x) = x^n$ for $n \in \mathbb{N}$ odd and $n \geq 3$; $f(x) = \frac{1}{x^n}$ for odd $n \in \mathbb{N}$; and $f(x) = \frac{1}{x^n}$ for even $n \in \mathbb{N}$.

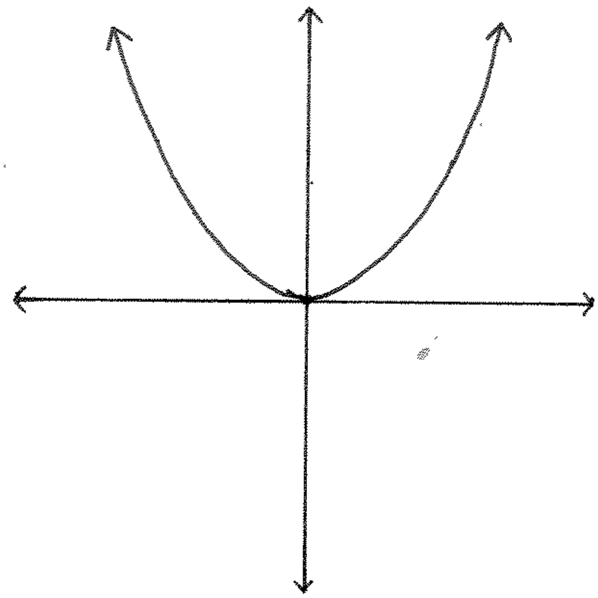


constant function
$f(x)=c$ for $c \in \mathbb{R}$

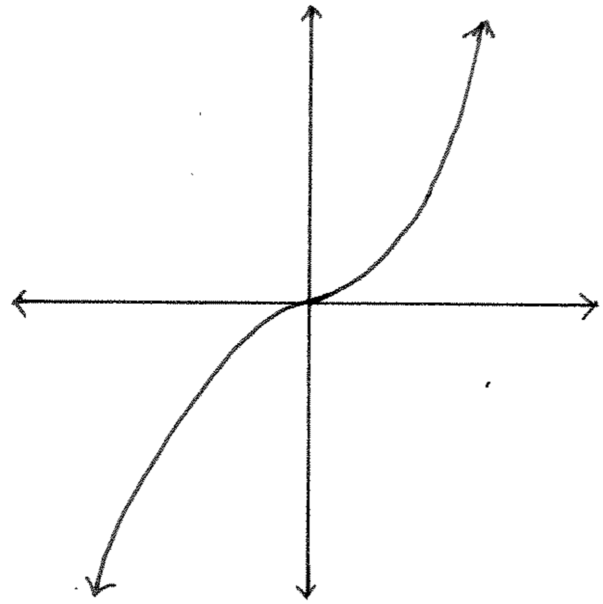


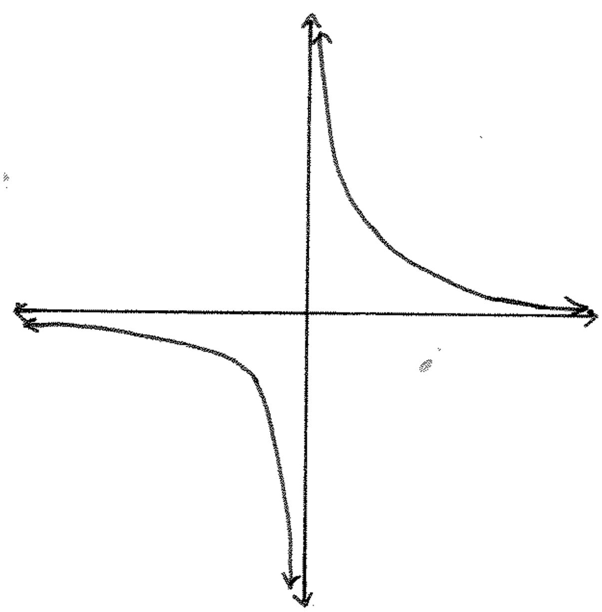
identity function

x^n for $n \in \mathbb{N}$ even

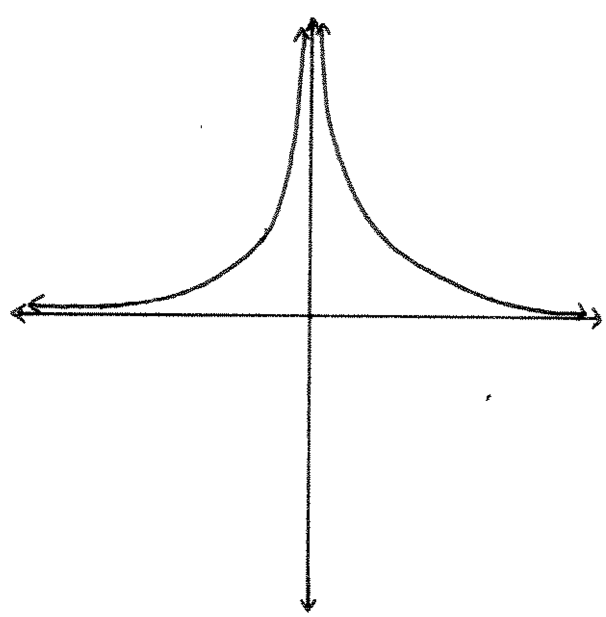


x^n for $n \in \mathbb{N}$ odd and $n \geq 3$





$\frac{1}{x^n}$ for $n \in \mathbb{N}$ odd



$\frac{1}{x^n}$ for $n \in \mathbb{N}$ even

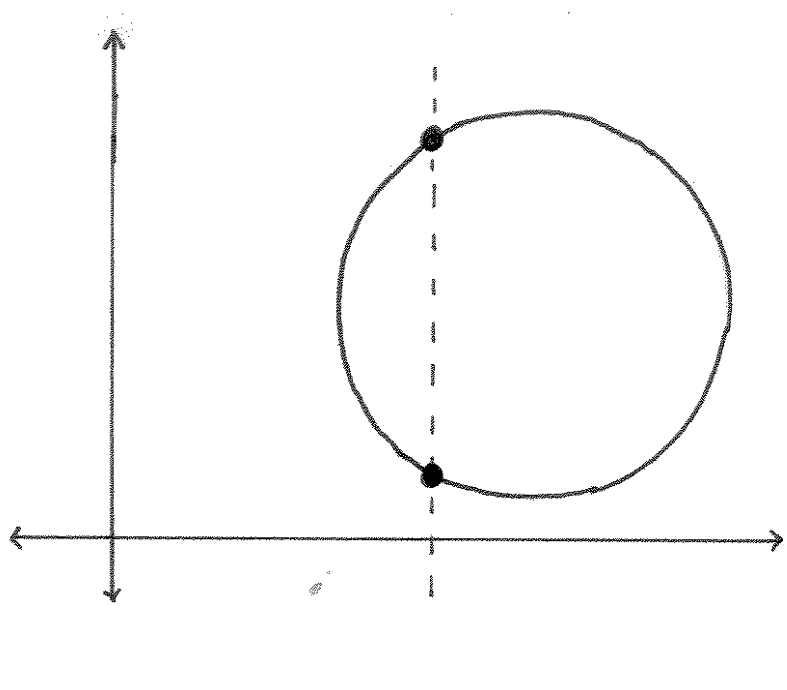
Vertical line test

Sometimes you'll see something drawn in $\mathbb{R}^2$ that looks like it might be the graph of a function. To know for sure if it is, use the *vertical line test*:

> If a vertical line intersects a thing in more than one point, then that thing is not a graph of a function.

The reason such a thing is not a graph of a function, is because if a vertical line intersects it in two different points, then the thing would include two different points with the same first coordinate – for example, $(1, 4)$ and $(1, 9)$. This could not be the graph of a function, because if it were, then the function would assign two different numbers — 4 and 9 — to the same object of the domain — 1. Functions can't do that.

Example. A circle is *not* the graph of a function. It fails the vertical line test.

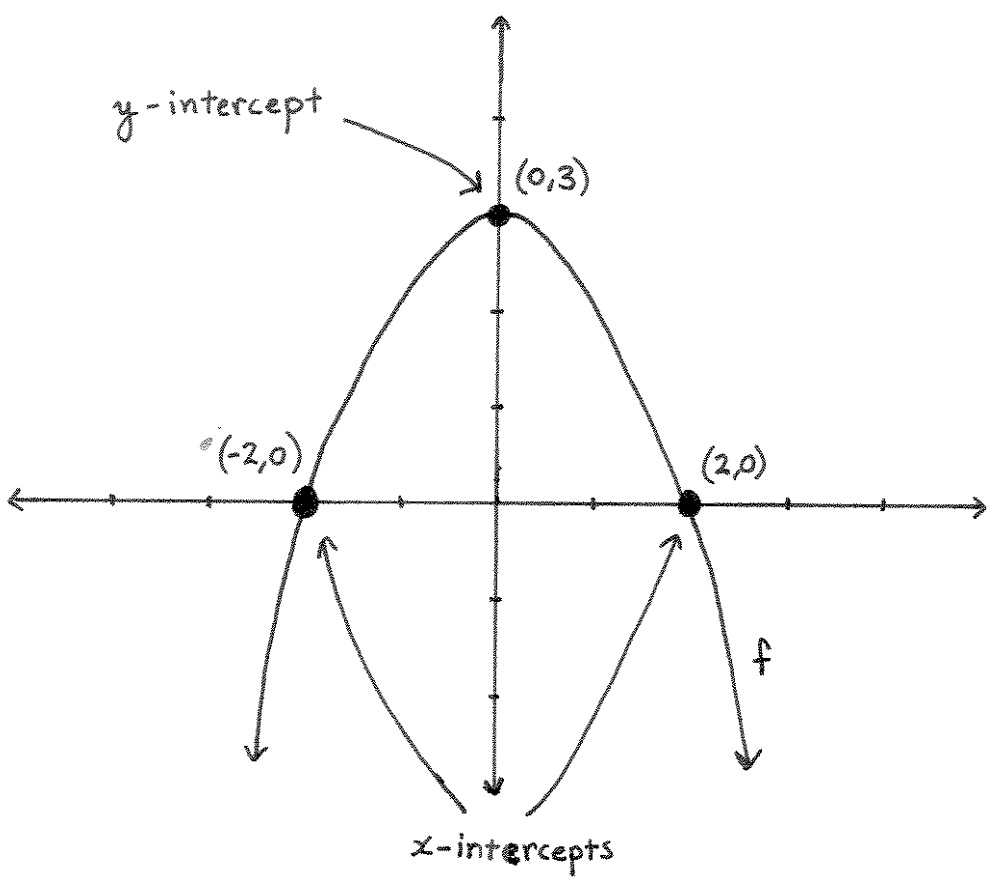


Intercepts

If the graph of a function contains a point of the form $(a, 0)$ for some $a \in \mathbb{R}$, then a is called an x-intercept of the graph.

If the graph of a function contains a point of the form $(0, b)$ for some $b \in \mathbb{R}$, then b is called the y-intercept of the graph.

Example. Below is the graph of a function f. The x-intercepts of the graph are 2 and -2. The y-intercept of the graph is 3.



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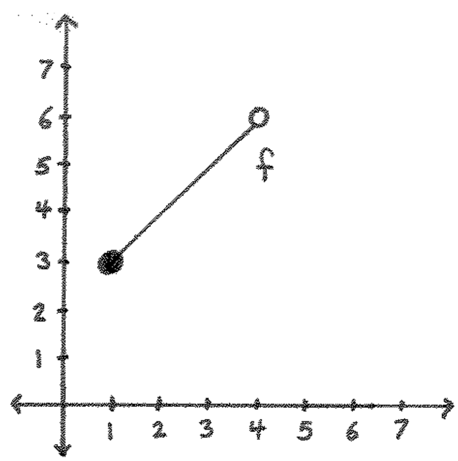
Little circles vs. giant dots

Drawing a giant dot in a graph means that point is in the graph of the function.

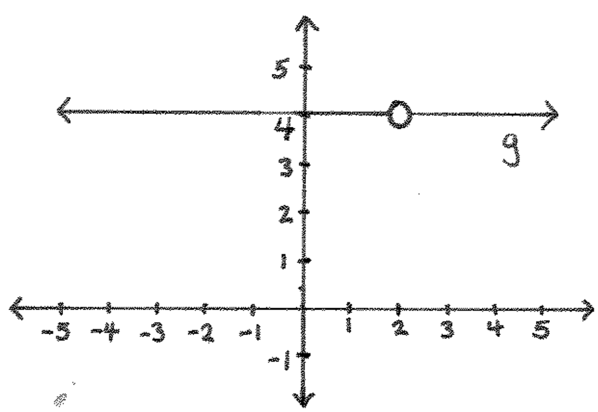
Drawing a little circle in a graph means that point is not in the graph of the function, but some nearby points are.

Example. Below is the graph of the function $f : [1, 4) \to \mathbb{R}$ where $f(x) = x+2$. The number 1 is in the domain of f, and $f(1) = 3$, so the point $(1, 3)$ is in the graph of f. We can label it with a giant dot.

The number 4 is not in the domain of f, but some numbers really close to 4 are. If 4 was in the domain, then $f(4) = 6$, and $(4, 6)$ would be a point in the graph of f. But 4 isn't in the domain, so $(4, 6)$ isn't a point in the graph of f. The graph does go all the way up to the point $(4, 6)$, but it doesn't include the point $(4, 6)$. So we label the point $(4, 6)$ with a little circle to remind us that it's not actually in the graph.



Example. Below is the graph of the function $g : \mathbb{R} - \{2\} \to \mathbb{R}$ where $g(x) = 4$. Since 2 is not in the domain of g, the point $(2, 4)$ is not in the graph of g, so we label it with a little circle to remind us that it's not in the graph.



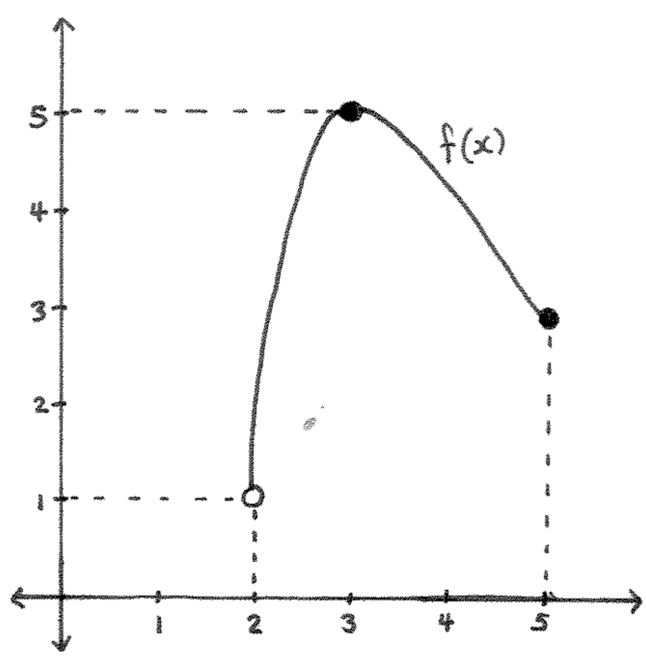
Domains and Ranges for graphs

Suppose you are given a graph, and you're told that it is the graph of a function. To find the domain of the function, draw its "shadow" on the x-axis.

To find the range of the function, draw its "shadow" on the y-axis.

Example. Drawn below is the graph of the function f. The domain of f is the set of real numbers in the x-axis that lie directly below the graph. Those are all of the numbers between 2 and 5. Because there is a giant dot on the point $(5,3) \in \mathbb{R}^2$, we know that 5 is in the domain. But since there is a little circle on the point $(2,1) \in \mathbb{R}^2$, we know that 2 is not in the domain. That is, the domain of f is the interval $(2,5]$.

The range of f is the set of real numbers on the y-axis that lie directly to the left of the graph of f. The range of f is the interval $(1,5]$.



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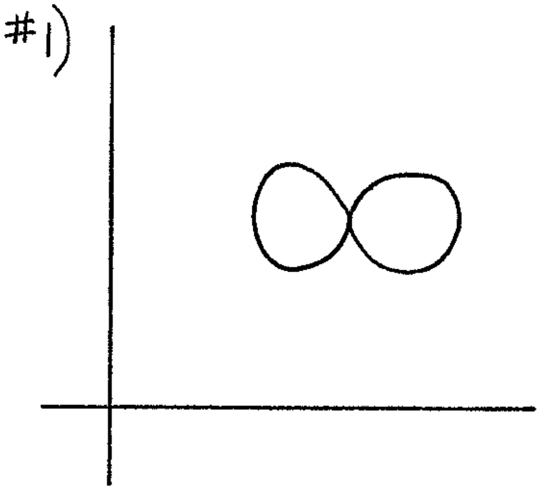
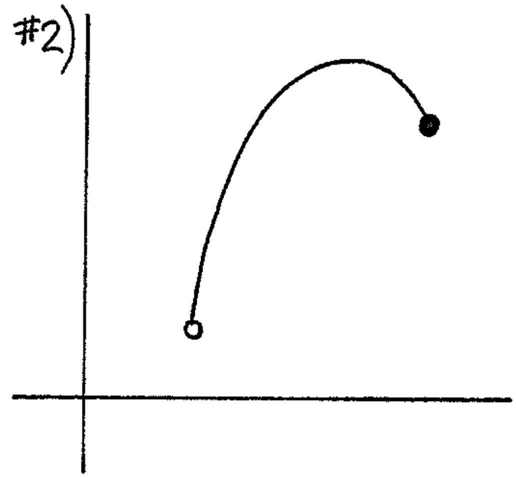
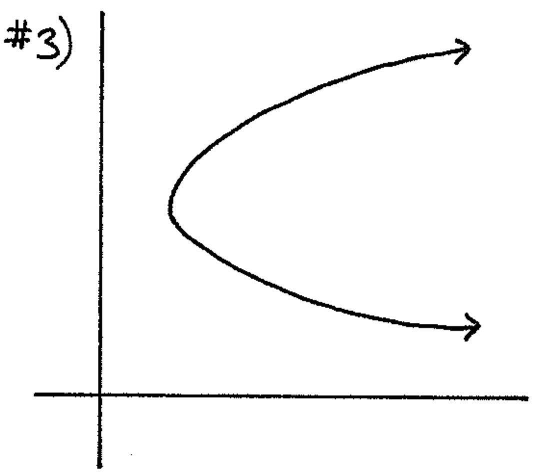
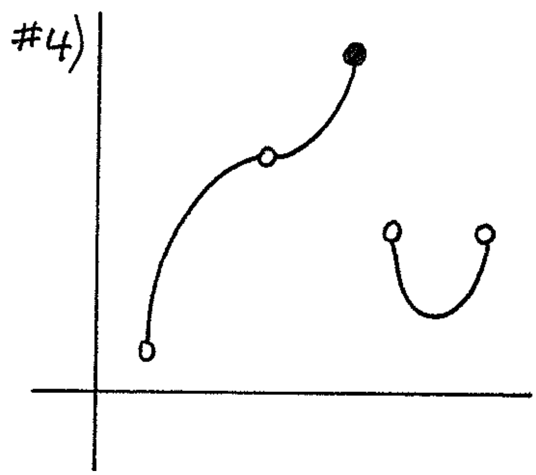
Exercises

For #1-4, decide whether or not each of the drawings in $\mathbb{R}^2$ is the graph of a function.

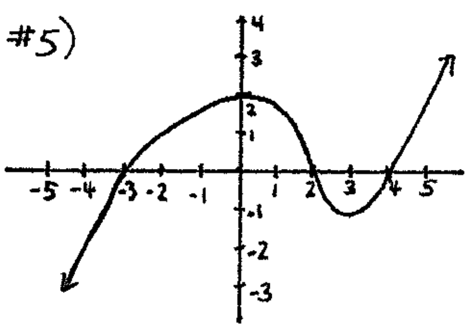


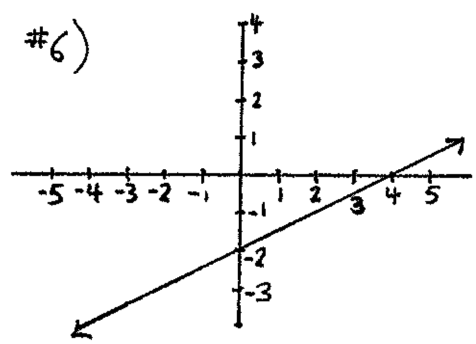






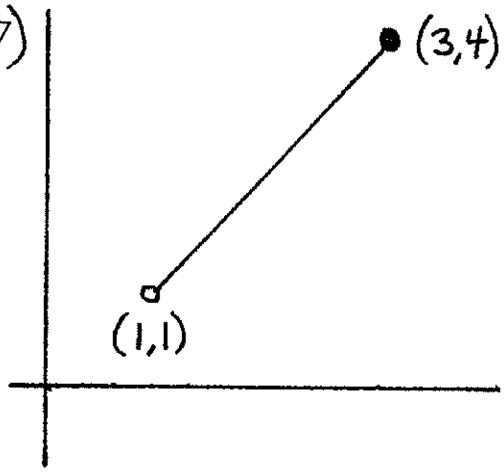
For #5-6, list the x- and y-intercepts of the graphs below.





For #7-12, determine the domains and ranges for the functions that are drawn.

#7)



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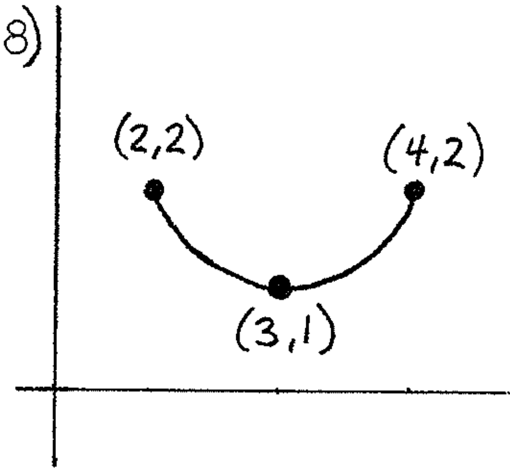

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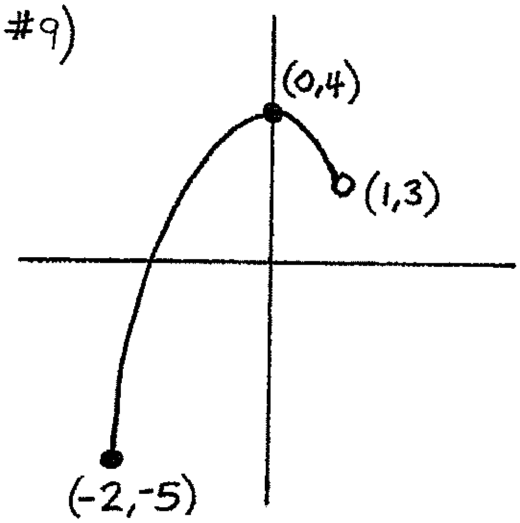

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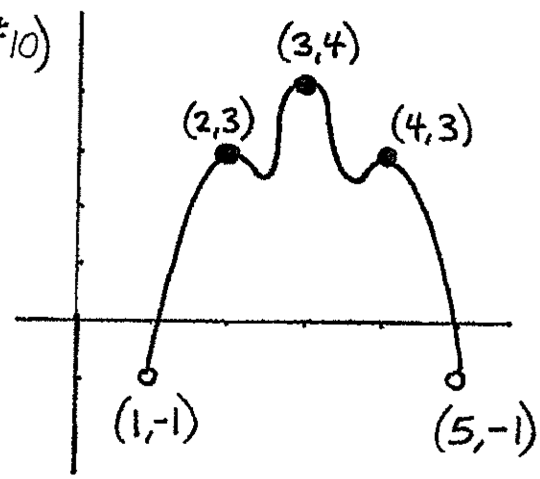

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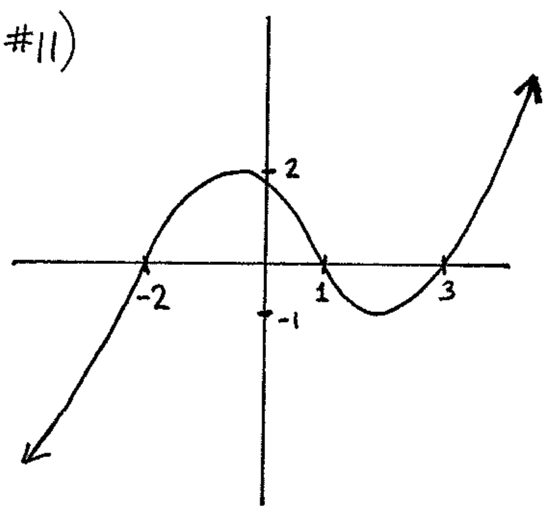
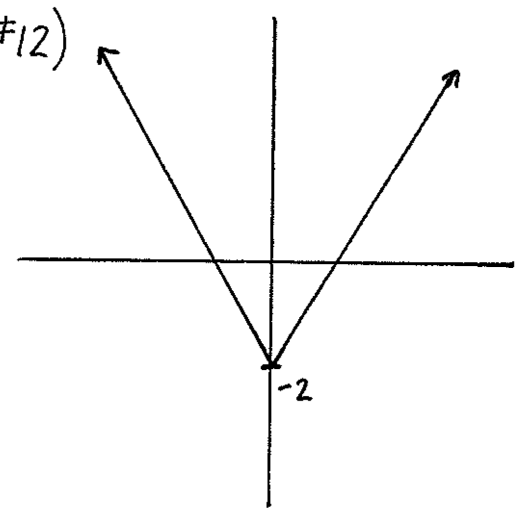


#12)



13.) Write out the product $(x+y)^4$.

For #14-19, assume that $f(x)=x+1$, $g(x)=4$, and that $h(x)=x^2-2$. Match each of the numbered functions with one of the lettered formulas.

14.) $f \circ g(x)$ 15.) $g \circ f(x)$ 16.) $g \circ h(x)$

17.) $h \circ g(x)$ 18.) $f \circ h(x)$ 19.) $h \circ f(x)$

A.) x^2-1 B.) x^2+2x-1 C.) 4

D.) 5 E.) 14

20.) What is the implied domain of $f(x)=2x^2-3x+7$?

21.) What is the implied domain of $g(x)=\frac{2x}{x-8}$?