A STREAMLINED COURSE ON THE FUNDAMENTALS OF PRECALCULUS

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Beginnings

Sets & Numbers

Sets

A *set* is a collection of objects. For example, the set of days of the week is a set that contains 7 objects: Mon., Tue., Wed., Thur., Fri., Sat., and Sun..

Set notation. Writing $\{2,3,5\}$ is a shorthand for the set that contains the numbers 2, 3, and 5, and no objects other than 2, 3, and 5.

The order in which the objects of a set are written doesn't matter. For example, $\{5,2,3\}$ and $\{2,3,5\}$ are the same set. Alternatively, the previous sentence could be written as "For example, $\{5,2,3\} = \{2,3,5\}$."

If B is a set, and x is an object contained in B, we write $x \in B$. If x is not contained in B then we write $x \notin B$.

Examples.

- $5 \in \{2,3,5\}$
- $1 \notin \{2,3,5\}$

Subsets. One set is a *subset* of another set if every object in the first set is an object of the second set as well. The set of weekdays is a subset of the set of days of the week, since every weekday is a day of the week.

A more succinct way to express the concept of a subset is as follows:

The set B is a *subset* of the set C if every $b \in B$ is also contained in C.

Writing $B \subseteq C$ is a shorthand for writing "B is a subset of C". Writing $B \nsubseteq C$ is a shorthand for writing "B is *not* a subset of C".

Examples.

- $\{2,3\} \subseteq \{2,3,5\}$
- $\{2,3,5\} \nsubseteq \{3,5,7\}$

Set minus. If A and B are sets, we can create a new set named $A - B$ (spoken as "A minus B") by starting with the set A and removing all of the objects from A that are also contained in the set B.

Examples.

- $\{1,7,8\} - \{7\} = \{1,8\}$
- $\{1,2,3,4,5,6,7,8,9,10\} - \{2,4,6,8,10\} = \{1,3,5,7,9\}$

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Numbers

Among the most common sets appearing in math are sets of numbers. There are many different kinds of numbers. Below is a list of those that are most important for this course.

Natural numbers. $\mathbb{N} = \{1, 2, 3, 4, ...\}$

Integers. $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, 3, ...\}$

Rational numbers. $\mathbb{Q}$ is the set of fractions of integers. That is, the numbers contained in $\mathbb{Q}$ are exactly those of the form $\frac{n}{m}$ where n and m are integers and $m \neq 0$.

For example, $\frac{1}{3} \in \mathbb{Q}$ and $\frac{-7}{12} \in \mathbb{Q}$.

Real numbers. $\mathbb{R}$ is the set of numbers that can be used to measure a distance, or the negative of a number used to measure a distance. The set of real numbers can be drawn as a line called "the number line".

$\sqrt{2}$ and π are two of very many real numbers that are not rational numbers.

(Aside: the definition of $\mathbb{R}$ above isn't very precise, and thus isn't a very good definition. The set of real numbers has a better definition, but it's outside the scope of this course. For this semester we'll make due with this intuitive notion of what a real number is.)

Numbers as subsets. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

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Exercises

Decide whether the following statements are true or false.

1) $3 \in \{7, 4, -10, 17, 3, 9, 67\}$

2) $4 \in \{14, 44, 43, 24\}$

3) $\frac{1}{3} \in \mathbb{Z}$

4) $-5 \in \mathbb{N}$

5) $\frac{-271}{113} \in \mathbb{Q}$

6) $-37 \in \mathbb{Z}$

7) $5 \in \mathbb{R} - \{4, 6\}$

8) $\{2, 4, 7\} \subseteq \{-3, 2, 5, 4, 7\}$

9) $\{2, 3, 5\} \subseteq \{2, 5\}$

10) $\{2, 5, 9\} \subseteq \{2, 4, 9\}$

11) $\{-15, \frac{3}{4}, \pi\} \subseteq \mathbb{R}$

12) $\{-15, \frac{3}{4}, \pi\} \subseteq \mathbb{Q}$

13) $\{-2, 3, 0\} \subseteq \mathbb{N}$

14) $\{-2, 3, 0\} \subseteq \mathbb{Z}$

15) $\{\sqrt{2}, 271\} \subseteq \mathbb{R}$

16) $\{\sqrt{2}, 271\} \subseteq \mathbb{Q}$

Rules for Numbers

The real numbers are governed by a collection of rules that have to do with addition, multiplication, and inequalities. In the rules below, $x, y, z \in \mathbb{R}$. (In other words, x, y, and z are real numbers.)

Rules of addition.

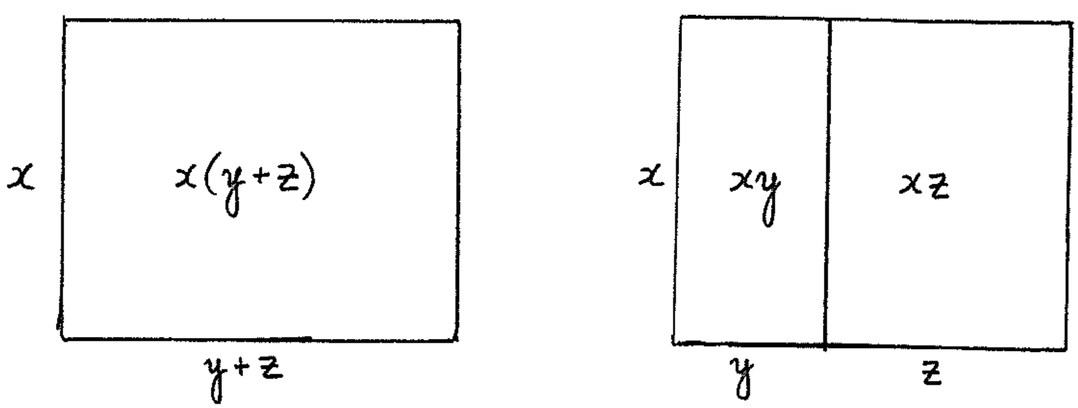
- $(x + y) + z = x + (y + z)$ (Law of associativity)
- $x + y = y + x$ (Law of commutativity)
- $x + 0 = x$ (Law of identity)
- $-x + x = 0$ (Law of inverses)

Rules of multiplication.

- $(xy)z = x(yz)$ (Law of associativity)
- $xy = yx$ (Law of commutativity)
- $x1 = x$ (Law of identity)
- If $x \neq 0$ then $\frac{1}{x}x = 1$ (Law of inverses)

Distributive Law. There is a rule that combines addition and multiplication: the distributive law. Of all the rules listed so far, it's arguably the most important.

- $x(y + z) = xy + xz$ (Distributive Law)



Here are some other forms of the distributive law that you will have to be comfortable with:

- $(y+z)x = yx + zx$
- $x(y-z) = xy - xz$
- $(x+y)(z+w) = xz + xw + yz + yw$
- $x(y+z+w) = xy + xz + xw$
- $x(y_1 + y_2 + y_3 + \cdots + y_n) = xy_1 + xy_2 + xy_3 + \cdots + xy_n$

Examples. Sometimes you'll have to use the distributive law in the "forwards" direction, as in the following three examples:

- $3(y+z) = 3y + 3z$
- $(-2)(4y-5z) = (-2)4y - (-2)5z = -8y + 10z$
- $2(3x - 2y + 4z) = 6x - 4y + 8z$

Sometimes you'll have to use the distributive law in "reverse". This process is sometimes called *factoring out a term.* The three equations below are examples of factoring out a -4, factoring out a 3, and factoring out a 2.

- $-4y - 4z = -4(y+z)$
- $3x + 6y = 3(x + 2y)$
- $10x - 8y + 4z = 2(5x - 4y + 2z)$

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In addition to the algebra rules above, the real numbers are governed by laws of inequalities.

Rules of inequalities.

- If $x > 0$ and $y > 0$ then $x + y > 0$
- If $x > 0$ and $y > 0$ then $xy > 0$
- If $x \in \mathbb{R}$, then either $x > 0$, or $x < 0$, or $x = 0$

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Intervals

The following chart lists 8 important types of subsets of $\mathbb{R}$. Any set of one of these types is called an *interval*.

Name of set	Those $x \in \mathbb{R}$ contained in the set
$[a, b]$	$a \leq x \leq b$
(a, b)	$a < x < b$
$[a, b)$	$a \leq x < b$
$(a, b]$	$a < x \leq b$
$[a, \infty)$	$a \leq x$
$(-\infty, b]$	$x \leq b$
(a, ∞)	$a < x$
$(-\infty, b)$	$x < b$

Notice that for every interval listed in the chart above, the least of the two numbers written in the interval is always written on the left, just as they appear in the real number line. For example, $3 < 7$, so 3 is drawn on the left of 7 in the real number line, and the following intervals are legitimate intervals to write: $[3, 7]$, $(3, 7)$, $[3, 7)$, and $(3, 7]$. You must *not* write an interval such as $(7, 3)$, because the least of the two numbers that define an interval has to be written on the left.

Similarly, $(\infty, 2)$ or $[5, -\infty)$ are *not* proper ways of writing intervals.

Exercises

Decide whether the following statements are true or false.

1) $5x + 5y = 5(x + y)$

2) $3x + y = 3(x + y)$

3) $4x - 6y = 2(2x - 3y)$

4) $x + 7y = 7(x + y)$

5) $36x - 9y + 81z = 3(12x - 3y + 9z)$

6) $2 \in (2, 5]$

7) $0 \in (-4, 0]$

8) $-3 \in [-3, 1)$

9) $156, 345, 678 \in (-1, \infty)$

10) $2 \in (-\infty, -3]$

11) $[7, 10) \subseteq [7, 10]$

12) $[-17, \infty) \subseteq (-17, \infty)$

13) $(-4, 0] \subseteq [-4, 0)$

14) $(-\infty, 20] \subseteq (-\infty, -7]$

15) $[0, \infty) \subseteq \mathbb{R} - \{\pi\}$

16) $\{ 3, 10, 7 \} \subseteq (2, 8)$

17) $\{ 0, 2, \frac{4}{5}, \sqrt{2} \} \subseteq [0, \infty)$

Solving Some Simple Equations

You probably already know how to solve the equations that we'll see in this section. The point of this section isn't to learn how to solve these equations, but rather to focus on a method for solving these equations that will be helpful for us later in the semester.

First example. Let's begin with the equation $x + 2 = 3$. To solve this equation, we want to isolate the variable x on one side of the equal sign, and to have a known number on the other side of the equal sign (a known number like 4, $\frac{7}{2}$, or π, but nothing with an x or a y etc.).

To isolate the x, we'll want to "erase" the $+2$. You can erase a $+2$ by writing its "opposite" on the other side of the equation. The opposite of adding 2 is subtracting 2. That means

$$x = 3 - 2$$

In other words,

$$x = 1$$

Second example. Let's solve $3x = 6$. To isolate the variable x, we can erase multiplication by 3 if we write its opposite—division by 3—on the other side of the equation. That is,

$$x = \frac{6}{3}$$

More simply,

$$x = 2$$

Third example. To isolate x in the equation $4x + 1 = 8$, we'll have to erase multiplication by 4 and adding 1, and the order in which we perform those two tasks matters. Always start with what happens to x LAST.

Let me explain. If you are given a number to use for x, say if $x = 3$, what would you do to find $4x + 1$? First you would multiply x by 4, then you would add 1. Multiply by 4 happens first. What happened last was adding 1. We always start by erasing what happened last.

So, take $4x + 1 = 8$, and erase $+1$ by writing -1 on the other side of the equation:

$$4x = 8 - 1$$

which is

$$4x = 7$$

Now erase the multiplication by 4 by writing division by 4 on the other side of the equation:

$$x = \frac{7}{4}$$

Fourth example. In the equation $2x - 3 = 5$, the first thing that happens to x is that it is multiplied by 2. The second, and last, thing that happens is subtraction by 3.

Start by erasing what happened last:

$$2x = 5 + 3 = 8$$

Then erase the multiplication by 2:

$$x = \frac{8}{2} = 4$$

Functions

A *function* is a way of describing a relationship between two sets.

To have a function we first need two sets, so lets suppose that D and T are sets. Then a *function* is something that assigns every $x \in D$ to a single object in T.

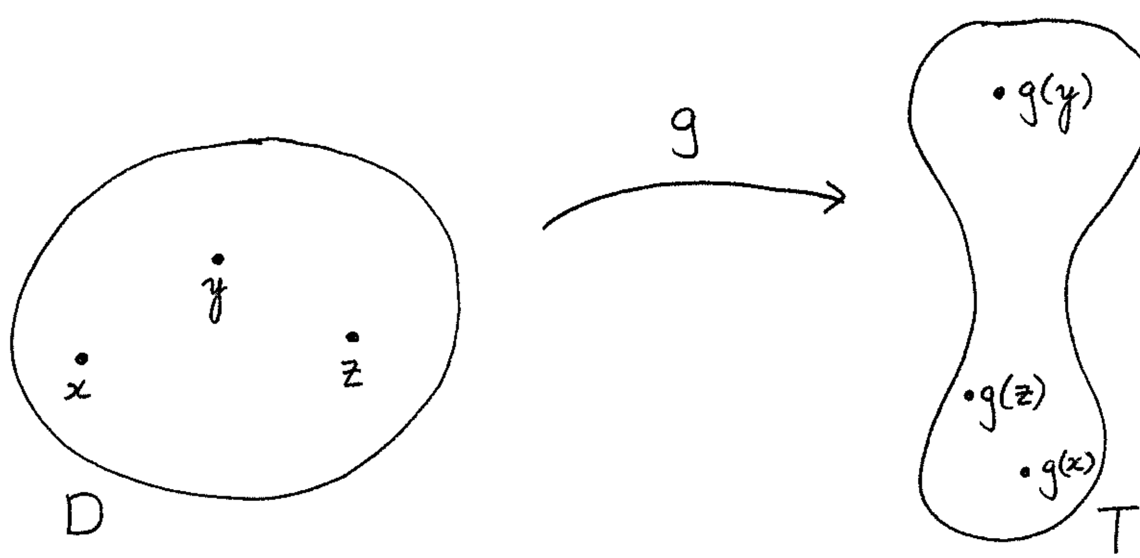
D is called the *domain* of the function, and T is called the *target* of the function. We usually assign names to our functions — though usually simple and generic names — like g, for example. Naming the function lets us give a specific name to the object in the target that the function assigns to an object in the domain as follows:

If $x \in D$, then $g(x) \in T$ is the object that g assigns to x.

Writing the symbols

$$g : D \to T$$

is a shorthand for writing that g is a function that assigns every $x \in D$ to a single object in T.



Example 1. $f : \mathbb{N} \to \mathbb{R}$ where $f(n) = 2^n$

- $f(3) = 2^3 = 2 \cdot 2 \cdot 2 = 8$
- $f(4) = 2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$
- $f(1) = 2^1 = 2$

Example 2. $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = 3x - 4$

- $g(2) = 3 \cdot 2 - 4 = 2$
- $g(-1) = 3 \cdot (-1) - 4 = -3 - 4 = -7$

Example 3. *Constant functions* are functions that assign every object in the domain to the same object in the target. For example, $h : \mathbb{R} \to \mathbb{R}$ where $h(x) = \frac{-5}{3}$.

Example 4. The *identity function* is the function that assigns every object in the domain to itself. (To have an identity function, the domain and target have to be the same set.) Identity functions are important enough that they get to have a name that is reserved only for identity functions: *id*. In other words, the identity function is described by

$$id : \mathbb{R} \to \mathbb{R} \quad \text{where} \quad id(x) = x.$$

(Sometimes it will make sense for us to use a different domain for the function *id*, but that's mostly a cosmetic change; it doesn't affect the way the function acts.)

Not a function I.
Assign to every $n \in \mathbb{N}$ the number $x \in \mathbb{R}$ such that $x^2 = n$.

Not a function II.
Assign to every $x \in \mathbb{R}$ the real number $\frac{3}{x-2}$.

Question: Why aren't the two examples above functions?

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Exercises

Suppose $f : \mathbb{N} \to \mathbb{R}$ is defined by $f(n) = \frac{1}{n^2}$.

1) What is $f(5)$?

2) What is $f(10)$?

Suppose $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = \frac{2x^2 - x + 1}{x^2 + 1}$.

3) What is $g(1)$?

4) What is $g(-3)$?

Suppose $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(x) = 14$.

5) What is $h(0)$?

6) What is $h(\frac{\pi^2}{6})$?

7) What is $id(15)$?

8) What is $id(-4)$?

You have a stubborn puppy who won't come when she's called unless you give her three puppy treats.

9) How many puppy treats do you need to bring with you on a walk if you expect to call the puppy seven times?

10) How many puppy treats do you need to bring with you on a walk if you expect to call the puppy ten times?

What's the domain and target of this "puppy function"?

Discrete Mathematics

Sequences

A *sequence* is an infinite list of numbers.

Sequences are written in the form

$$a_1, a_2, a_3, a_4, ...$$

where $a_1 \in \mathbb{R}$, and $a_2 \in \mathbb{R}$, and $a_3 \in \mathbb{R}$, and $a_4 \in \mathbb{R}$, and so on.

A shorter way to write what's above is to say that a sequence is an infinite list

$$a_1, a_2, a_3, a_4, ... \text{ with } a_n \in \mathbb{R} \text{ for every } n \in \mathbb{N}.$$

A sequence is different from a set in that the order the numbers are written is important in a sequence. For example, $2, 3, 4, 4, 4, 4, 4, 4, ...$ is the sequence where $a_1 = 2$, $a_2 = 3$, and $a_n = 4$ if $n \geq 3$. This is a different sequence than $3, 2, 4, 4, 4, 4, 4, 4, ...$.

$1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, ...$ and $2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...$ are also different sequences. The former is the sequence with $a_n = 1$ if n is odd and $a_n = 2$ if n is even. The latter is the sequence with $a_n = 2$ if n is odd and $a_n = 1$ if n is even.

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Arithmetic sequences

An *arithmetic sequence* is a sequence $a_1, a_2, a_3, ...$ where there is some number $d \in \mathbb{R}$ such that

$$a_{n+1} = a_n + d$$

for every $n \in \mathbb{N}$.

Examples.

- If $a_1 = -10$ and $a_{n+1} = a_n + 5$, then the sequence $a_1, a_2, a_3, ...$ is

$$-10, -5, 0, 5, 10, 15, 20, 25, ...$$

- Each number in the sequence $3, 13, 23, 33, 43, 53$ is 10 more than the term directly preceding it. Therefore, it is an arithmetic sequence with

$d = 10$. If you're already convinced of this, that's all you need to do. You don't have to check it any further.

If you're not convinced of that, let's check that it's true: The sequence starts with 3, so $a_1 = 3$. The second number in the sequence is 13, so $a_2 = 13$.

The definition of an arithmetic sequence states that $a_2 = a_1 + d$, and we're checking that $d = 10$ works. That means that we have to check that

$$13 = 3 + 10$$

and of course it does, so our formula that $a_{n+1} = a_n + 10$ works when $n = 1$.

Also, $a_3 = 23$ and $a_2 + 10 = 13 + 10 = 23$, so $a_{n+1} = a_n + 10$ when $n = 2$.

Notice too that $a_4 = 33$ and $a_3 + 10 = 23 + 10 = 33$, so $a_{n+1} = a_n + 10$ when $n = 3$.

Similarly, $a_5 = 43$ and $a_4 + 10 = 33 + 10 = 43$, so $a_{n+1} = a_n + 10$ when $n = 4$.

Finally, $a_6 = 53$ and $a_5 + 10 = 43 + 10 = 53$, so $a_{n+1} = a_n + 10$ when $n = 5$.

We have now competed our check that $3, 13, 23, 33, 43, 53$ is an arithmetic sequence, since $a_{n+1} = a_n + 10$.

(We really haven't checked that $a_{n+1} = a_n + 10$ for every $n \in \mathbb{N}$, but we've checked it for every n that we possibly could have checked it for, given that we are only told the first six numbers of the sequence. If ever someone writes the first few terms of a sequence, and those first few terms follow an arithmetic pattern, they mean to imply that that pattern will continue forever. Thus, you only ever have to check the numbers of the sequence that were given explicitly to determine if the entire sequence is arithmetic or not.)

- $6, 3, 0, -3, -6, -9, -12, ...$ is an arithmetic sequence because each number in the sequence is 3 less than the number directly preceding it. That is, $a_{n+1} = a_n - 3$, which is the same as $a_{n+1} = a_n + (-3)$. So in this example, $d = -3$.

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Geometric sequences

A *geometric sequence* is a sequence $a_1, a_2, a_3, ...$ where there is some number $r \in \mathbb{R}$ such that

$$a_{n+1} = ra_n$$

for every $n \in \mathbb{N}$.

Examples.

• If $a_1 = 4$ and $a_{n+1} = 3a_n$, then the sequence $a_1, a_2, a_3, ...$ is

$$4, 12, 36, 108, ...$$

• Each number in the sequence $8, 32, 128, 512, ...$ is exactly 4 times the number directly preceding it. To check that, notice that

$$32 = 4(8),$$
$$128 = 4(32), \text{ and}$$
$$512 = 4(128).$$

Since each number is 4 times the number before it, $8, 32, 128, 512, ...$ is a geometric series with $r = 4$. That is, $a_{n+1} = 4a_n$.

• An important kind of geometric sequence is one where $a_{n+1} = ra_n$ for a number r with $0 < r < 1$. Then each number in the sequnce will be smaller than the ones that came before it. For instance, $a_1 = 8$ and $r = \frac{1}{2}$ describes the geometric sequence that starts with 8, and such that every number is half of the number that came before it: $8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} ...$

• The geometric sequence with $a_1 = 5$ and $r = -2$ is $5, -10, 20, -40, 80, ...$

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Predicting numbers in an arithmetic sequence

Let's suppose that $a_1, a_2, a_3...$ is an arithmetic sequence. Then for some $d \in \mathbb{R}$, the equation $a_{n+1} = a_n + d$ holds for every $n \in \mathbb{N}$. If we let $n = 1$ then

$$a_2 = a_1 + d$$

If $n = 2$ then

$$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$$

If $n = 4$ then

$$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$$

If $n = 4$ then

$$a_5 = a_4 + d = (a_1 + 3d) + d = a_1 + 4d$$

We could continue this forever, but there's no need to because there is a pattern emerging. That pattern is

$$\boxed{a_n = a_1 + (n-1)d}$$

Examples.

- What is the 201^{st} number in the sequence $-10, -5, 0, 5, 10, 15, 20, 25, ...$?

We saw before that $a_1 = -10$ and $a_{n+1} = a_n + 5$. Hence,

$$a_{201} = a_1 + (201-1)5 = a_1 + (200)5 = -10 + 1000 = 990$$

- What is the 27^{th} number in the sequence $3, 13, 23, 33, 43, 53, ...$?

Because $a_1 = 3$ and $a_{n+1} = a_n + 10$,

$$a_{27} = a_1 + (27-1)10 = 3 + (26)10 = 263$$

- The 14^{th} number in the sequence $6, 3, 0, -3, -6, -9, -12, ...$ is

$$6 + 13(-3) = -33$$

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Predicting numbers in a geometric sequence

If $a_1, a_2, a_3 ...$ is a geometric sequence then there is an $r \in \mathbb{R}$ such that $a_{n+1} = ra_n$ for all $n \in \mathbb{N}$. Using this formula we see that

$$\begin{aligned} a_2 &= ra_1 \\ a_3 &= ra_2 = r(ra_1) = r^2a_1 \\ a_4 &= ra_3 = r(r^2a_1) = r^3a_1 \\ a_5 &= ra_4 = r(r^3a_1) = r^4a_1 \\ &\vdots \\ a_n &= r^{n-1}a_1 \end{aligned}$$

Examples.

• What is the 7^{th} number in the sequence $4, 12, 36, 108, ...$?

$a_1 = 4$ and $a_{n+1} = 3a_n$, so

$$a_7 = 3^{(7-1)}\, a_1 = 3^6\, 4 = (729)4 = 2916$$

• What is the 14^{th} number in the sequence $8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}...$?

$$a_{14} = \left(\frac{1}{2}\right)^{13} 8 = \frac{1}{1024}$$

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Sequences as functions

Let's go back to the definition of a sequence. A fancier way to define a sequence is to say that it is a function whose domain is $\mathbb{N}$ and whose target is $\mathbb{R}$, because a sequence assigns to every natural number n a single real number, namely a_n.

Like many functions, sequences are sometimes described using explicit formulas.

Examples.

• Suppose that $a_1, a_2, a_3, ...$ is a sequence that is defined by the formula $a_n = 25n^2$. Then

$$a_{10} = (25)10^2 = (25)100 = 2500$$

• If a sequence is defined by the formula $b_n = \frac{n+13}{n+1}$ then

$$b_7 = \frac{7+13}{7+1} = \frac{20}{8} = \frac{5}{2}$$

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Exercises

Decide whether the following six sequences are either arithmetic, geometric, or neither.

1.) $2, 7, 14, 28, ...$

2.) $-11, -7, -3, 1, 5, ...$

3.) $3, -3, 3, -3, 3, ...$

4.) $2, 3, 4, 5, 6, ...$

5.) $1, 6, 12, 24, ...$

6.) $1000, 100, 10, 1, \frac{1}{10}, ...$

In the next four problems you are given an arithmetic sequence. For each one, what is a_1, and what is the number d such that $a_{n+1} = a_n + d$?

7.) $-1, 4, 9, 14, ...$

8.) $2, -10, -22, -34, ...$

9.) $17, 15, 13, 11, ...$

10.) $3, 7, 11, 15, ...$

In the next four problems you are given a geometric sequence. For each one, what is a_1, and what is the number r such that $a_{n+1} = ra_n$?

11.) $15, 5, \frac{5}{3}, \frac{5}{9}, ...$

12.) $2, 6, 18, 54, ...$

13.) $-5, -25, -125, -625, ...$

14.) $4, -8, 16, -32, 64, ...$

The next three problems involve arithmetic sequences.

15.) What is the 301^{st} number in the sequence $10, 16, 22, 28, ...$?

16.) What is the 4223^{rd} number in the sequence $5, 7, 9, 11, ...$?

17.) What is the 5224^{th} number in the sequence $4, 1, -2, -5, ...$?

Arithmetic sequences don't grow very fast, so it's a doable problem in most cases to find future numbers in a sequence. For example, if you are asked to find a_{327854} in an arithmetic sequence where $a_{n+1} = a_n + d$, the main computational step is to find $(327853)d$. If you know what d is, you could probably do this by hand. If not, a calculator could certainly do this.

In contrast, it is difficult to find large powers of a number. Powers of numbers greater than 1 tend to be so large that calculators can't display the number on their screen, even if they could calculate them. For example, if you are dealing with a sequence where $a_{n+1} = ra_n$, then to find a_{327854} involves finding r^{327853}. That's a really, really big number, even if r isn't so big. For example if $r = 3$, then $r^{327853} = 3^{327853}$, and hand-held calculators won't be able to help you with that.

So for the next three problems, you are asked to find numbers in the sequence, but because they are geometric sequences, you are only asked to find some of the first few numbers in the sequences.

18.) What is the 7^{th} number in the sequence $54, 18, 6, 2, ...$?

19.) What is the 6^{th} number in the sequence $-11, 22, -44, 88, ...$?

20.) What is the 8^{th} number in the sequence $2000, 200, 20, 2, ...$?

The last three problems deal with sequences that are defined using explicit equations.

21.) Suppose the sequence $a_1, a_2, a_3, ...$ is defined by $a_n = 3n - 4$. Find a_{200}.

22.) Let $b_1, b_2, b_3, ...$ be the sequence where $b_n = \frac{n^2+1}{n}$. Find b_{20}.

23.) If $c_1, c_2, c_3, ...$ is the sequence $c_n = (3 - n)(n + 2)$, what is c_8?

Sums & Series

Suppose $a_1, a_2, \ldots$ is a sequence.

Sometimes we'll want to sum the first k numbers (also known as *terms*) that appear in a sequence. A shorter way to write $a_1 + a_2 + a_3 + \cdots + a_k$ is as

$$\sum_{i=1}^{k} a_i$$

There are four rules that are important to know when using $\sum$. They are listed below. In all of the rules, $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are sequences and $c \in \mathbb{R}$.

Rule 1. $c\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} ca_i$

Rule #1 is the distributive law. It's another way of writing the equation

$$c(a_1 + a_2 + \cdots + a_k) = ca_1 + ca_2 + \cdots + ca_k$$

Rule 2. $\sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i = \sum_{i=1}^{k}(a_i + b_i)$

This rule is essentially another form of the commutative law for addition. It's another way of writing that

$$a_1 + a_2 + \cdots + a_k + b_1 + b_2 + \cdots + b_k = a_1 + b_1 + a_2 + b_2 + \cdots + a_k + b_k$$

Rule 3. $\sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i = \sum_{i=1}^{k}(a_i - b_i)$

Rule #3 is a combination of the first two rules. To see that, remember that $-b_i = (-1)b_i$, so we can use Rule #1 (with $c = -1$) followed by Rule #2 to derive Rule #3, as is shown below:

$$\begin{aligned}\sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i &= \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} -b_i \\ &= \sum_{i=1}^{k} (a_i + (-b_i)) \\ &= \sum_{i=1}^{k} (a_i - b_i)\end{aligned}$$

Rule 4. $\displaystyle\sum_{i=1}^{k} c = kc$

The fourth rule can be a little tricky. The number c does not depend on i — it's a constant — so $\sum_{i=1}^{k} c$ is taken to mean that you should add the first k terms in the sequence $c, c, c, c, ...$ That is to say that

$$\sum_{i=1}^{k} c = c + c + \cdots + c = kc$$

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Sum of first k terms in an arithmetic sequence

If $a_1, a_2, a_3, ...$ is an arithmetic sequence, then $a_{n+1} = a_n + d$ for some $d \in \mathbb{R}$. We want to show that

$$\sum_{i=1}^{k} a_i = \frac{k}{2}(a_1 + a_n)$$

To show this, let's write the sum in question in two different ways: front-to-back, and back-to-front. That is,

$$\sum_{i=1}^{k} a_i = a_1 \qquad\qquad + (a_1 + d) + (a_1 + 2d) + \cdots + (a_k - 2d) + (a_k - d) + a_k$$

and

$$\sum_{i=1}^{k} a_i = a_k \qquad\qquad + (a_k - d) + (a_k - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1$$

Add the two equations above "top-to-bottom" to get

$$2\sum_{i=1}^{k} a_i = [a_1 + a_k] + [a_1 + a_k] + [a_1 + a_k] + \cdots + [a_1 + a_k] + [a_1 + a_k] + [a_1 + a_k]$$

Count and check that there are exactly k of the $[a_1 + a_k]$ terms in the line above being added. Thus,

$$2\sum_{i=1}^{k} a_i = k[a_1 + a_k]$$

which is equivalent to what we were trying to show:

$$\boxed{\sum_{i=1}^{k} a_i = \frac{k}{2}(a_1 + a_k)}$$

Example. What is the sum of the first 63 terms of the sequence $-1, 2, 5, 8, ...$?

The sequence above is arithmetic, because each term in the sequence is 3 plus the term before it, so $d = 3$. The first term of the sequence is -1, so $a_1 = -1$. Our formula $a_n = a_1 + (n-1)d$ tells us that $a_{63} = -1 + (62)3 = 185$. Therefore,

$$\sum_{i=1}^{63} a_i = \frac{63}{2}(-1 + 185) = \frac{63}{2}(184) = 5,796$$

Example. The sum of the first 201 terms of the sequence $10, 17, 24, 31, ...$ equals $\frac{201}{2}(10 + 1410) = \frac{201}{2}(1420) = 142,710$.

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Geometric series

It usually doesn't make any sense at all to talk about adding infinitely many numbers. But if $a_1, a_2, a_3, ...$ is a geometric sequence where $a_{n+1} = ra_n$ and $-1 < r < 1$, then we can make sense of adding all of the terms of the sequence together. (We'll see why later in the semester.)

We will use the symbols

$$\sum_{i=1}^{\infty} a_i$$

to represent adding all of the numbers in the sequence $a_1, a_2, a_3, ...$, and we call this infinite "sum" a *series*.

For the moment, let $S = a_1 + a_2 + a_3 + a_4 + \cdots$. Then

$$S = a_1 + ra_1 + r^2a_1 + r^3a_1 + \cdots$$

and using the distributive law we have

$$rS = \qquad ra_1 + r^2a_1 + r^3a_1 + \cdots$$

Thus, $S - rS = a_1$. Since the distributive law tells us that $S - rS = S(1-r)$, we have $S(1-r) = a_1$, or in other words, $S = \frac{a_1}{1-r}$. We have shown that

$$\sum_{i=1}^{\infty} a_i = \frac{a_1}{1-r}$$

Examples.

- The sum of the terms in the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$ equals 2. We know the sequence is geometric, follows the rule $a_{n+1} = \frac{1}{2}a_n$, and that the first term in the sequence equals 1. Thus

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

- The sum of the terms in the sequence $5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27}, ...$ equals

$$\frac{5}{1-\frac{1}{3}} = \frac{5}{\frac{2}{3}} = \frac{15}{2}$$

Caution. If $a_1, a_2, a_3, ...$ isn't geometric, or if it is but either $r \geq 1$ or $r \leq -1$, then

$$\sum_{i=1}^{\infty} a_i$$

probably doesn't make sense.

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Exercises

1.) If the sum of the first 3976 terms of the sequence $a_1, a_2, a_3, ...$ equals 114, then what is the sum of the first 3976 terms of the sequence $\frac{3}{2}a_1, \frac{3}{2}a_2, \frac{3}{2}a_3, ...$?

Determine what each of the following eight series equals.

2.) $\sum_{i=1}^{50} 3$

3.) $\sum_{i=1}^{100} 49$

4.) $\sum_{i=1}^{78} (-2)$

5.) $\sum_{i=1}^{40} i$

6.) $\sum_{i=1}^{100} i$

7.) $\sum_{i=1}^{900} i$

8.) $\sum_{i=1}^{5} (2i - 1)$

9.) $\sum_{i=1}^{4}(i^2 - 2)$

10.) What is the sum of the first 701 terms of the sequence $-5, -1, 3, 7, ...$?

11.) What is the sum of the first 53 terms of the sequence $140, 137, 134, 131, ...$?

12.) What is the sum of the first 100 terms of the sequence $4, 9, 14, 19, ...$?

13.) What is the sum of the first 80 terms of the sequence $53, 54, 55, 56, ...$?

14.) Sum all of the terms of the geometric sequence $20, 5, \frac{5}{4}, \frac{5}{16}, ...$

15.) Sum all of the terms of the geometric sequence $120, 90, \frac{135}{2}, \frac{405}{8}, ...$

16.) Sum all of the terms of the geometric sequence $7, \frac{14}{3}, \frac{28}{9}, \frac{56}{27}, ...$

17.) Sum all of the terms of the geometric sequence $25, 15, 9, \frac{27}{5}, ...$

18.) Sum all of the terms of the geometric sequence $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, ...$

19.) Suppose that you expect to pay \$400 for gas for your car next year, and that each year after that you plan your yearly gas expenditures will increase by \$20. How much will you spend on gas in the next 8 years?

20.) Suppose you are entertaining two different job offers. Job A has a starting salary of \$20,000 and assures you of a raise of \$1,000 per year. Job B offers you a starting salary of \$23,000, with a yearly raise of \$725. Which job will pay you more over the first ten years? How much more?

21.) An oil well currently produces 5 million gallons of oil per year, but the well is drying up, and each year it will produce 60% of what it did the year before. How much oil can be produced from the well before it is completely dry?

Counting I

For this section you'll need to know what *factorials* are.

If $n \in \mathbb{N}$, then n-factorial, which is written as $n!$, is the product of numbers

$$n(n-1)(n-2)(n-3)\cdots(4)(3)(2)(1)$$

Examples. $3! = (3)(2)(1) = 6$, and $5! = (5)(4)(3)(2)(1) = 120$.

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Options multiply

When you have to make one choice, and then another choice, the total number of choices multiply.

Suppose you have to choose a sandwich with one of four types of meat – ham, turkey, pastrami, or roast beef – and one of three kinds of cheese – swiss, cheddar, or gouda.

There are 4 different ways to choose a meat. Once you've made that choice, there are 3 different ways to choose a cheese. The number of choices for meat and cheese can be displayed in a 4×3 rectangle, where it's easy to see that the total number of choices for sandwiches is $4(3) = 12$. (12 is the area of a 4×3 rectangle.)

	Swiss	Cheddar	Gouda
Ham	H/S	H/C	H/G
Turkey	T/S	T/C	T/G
Pastrami	P/S	P/C	P/G
Roast Beef	R/S	R/C	R/G

If, in addition to a meat and cheese option, you are given a bread option of either wheat or white, then that's a third choice to make. The third choice has 2 options, and the number of options multiply, so there would be $4(3)(2) = 24$ total number of sandwiches to choose from. (You could build a rectangular solid of dimensions $4 \times 3 \times 2$, to list out the total number of options, just as we made a rectangle above to list the number of options after making two different choices. The area of a $4 \times 3 \times 2$ rectangular solid is 24.)

Examples.

• You have to wear a tie and jacket for a fancy dinner. You own 3 jackets and 7 ties. There are $3(7) = 21$ possible jacket and tie combinations to choose from.

• You are buying an airplane ticket for a flight with a meal service. When you buy your ticket, you can choose to sit in an aisle seat, or a window seat. You can choose the vegetarian meal, or the chicken. You can sit in any row of the plane you like: 1-32. How many different tickets can you buy?

There are 2 options for type of seat, 2 options for the meal, and 32 options for the row. Thus, there are $2(2)(32) = 128$ total number of different tickets you can buy.

• Suppose you are designing a house for yourself to live in. You can choose the house to be made out of wood, brick, or metal. The roof can be wood shingles, asphalt shingles, or tin. You can paint the house brown, red, yellow, or green. You can choose to have two, three, four, or five bedrooms. How many total number of possibilities are there for the design of your house?

There are 3 building material options, 3 roof options, 4 color options, and 4 bedroom options. Altogether, there are $3(3)(4)(4) = 144$ total options for the design of your house.

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Ordering sets

Remember that the order in which we list the contents of a set doesn't change what the set is. For example, $\{5, 2, 3\} = \{2, 3, 5\}$.

But sometimes it can be useful to *order* the contents of a set: that is, to designate an object of the set as being "first", and another object as being "second", etc.

Examples.

• There are two different ways to order the objects of the set $\{\pi, b\}$. We could either choose to make π the first object, and b the second, or we could take b to be first and π second.

• Here's a list of all the ways to order the set $\{\sqrt{2}, -\frac{2}{5}, f, e\}$. If you count the items on the list, you'll see that there are 24 different ways to order the set $\{\sqrt{2}, -\frac{2}{5}, f, e\}$.

$\sqrt{2},\ -\frac{2}{5},\ f,\ e$	$\sqrt{2},\ f,\ -\frac{2}{5},\ e$	$\sqrt{2},\ e,\ f,\ -\frac{2}{5}$
$\sqrt{2},\ -\frac{2}{5},\ e,\ f$	$\sqrt{2},\ f,\ e,\ -\frac{2}{5}$	$\sqrt{2},\ e,\ -\frac{2}{5},\ f$
$-\frac{2}{5},\ \sqrt{2},\ f,\ e$	$-\frac{2}{5},\ f,\ \sqrt{2},\ e$	$-\frac{2}{5},\ e,\ f,\ \sqrt{2}$
$-\frac{2}{5},\ \sqrt{2},\ e,\ f$	$-\frac{2}{5},\ f,\ e,\ \sqrt{2}$	$-\frac{2}{5},\ e,\ \sqrt{2},\ f$
$e,\ \sqrt{2},\ f,\ -\frac{2}{5}$	$e,\ f,\ \sqrt{2},\ -\frac{2}{5}$	$e,\ -\frac{2}{5},\ f,\ \sqrt{2}$
$e,\ \sqrt{2},\ -\frac{2}{5},\ f$	$e,\ f,\ -\frac{2}{5},\ \sqrt{2}$	$e,\ -\frac{2}{5},\ \sqrt{2},\ f$
$f,\ e,\ \sqrt{2},\ -\frac{2}{5}$	$f,\ \sqrt{2},\ e,\ -\frac{2}{5}$	$f,\ -\frac{2}{5},\ \sqrt{2},\ e$
$f,\ e,\ -\frac{2}{5},\ \sqrt{2}$	$f,\ \sqrt{2},\ -\frac{2}{5},\ e$	$f,\ -\frac{2}{5},\ e,\ \sqrt{2}$

Spelling. Arranging the order of a set of letters is a good example of when order is important. The set $\{\mathbf{e}, \mathbf{t}, \mathbf{a}\}$ can be ordered in six different ways: **eta**, **eat**, **tea**, **tae**, **ate**, and **aet**. Some of the six arrangements are not words. Some of them are words, and the words that do appear have different meanings. So if you have a set of letters, the order in which you write them is very important.

Another look at spelling. Let's find a better way to count the number of ways we can order the objects of the set $\{\mathbf{e}, \mathbf{t}, \mathbf{a}\}$ without having to write out a list of them, and then counting the list.

To arrange the three letters **e**, **t**, and **a** into some order, we need to choose a letter to be first. There are 3 letters, and thus 3 options for which letter can be first.

Once we've decided which letter is first, there are two letters remaining. We choose one of the two to be second, so there are 2 options for which number is second.

After we've chosen a first letter and a second letter, only one letter remains. It must be third, because there are no other letters to choose from. So there is only 1 option for which letter can be third at this point.

Options multiply. There were 3 options for the first letter, followed by 2 options for the second letter, and 1 option for the third letter. So the total number of ways the letters **e**, **t**, and **a** can be arranged is $3! = (3)(2)(1) = 6$.

General Problem. Suppose you have a set that contains exactly n objects. How many different ways are there to order the objects in the set?

General Solution. We have to choose a first object. There are n total objects in the set, so there are n different options for what that first object could be.

Once we've chosen a first object, we remove it from the set, leaving $(n-1)$ options for what the second object could be.

Once we've chosen and removed the first and second objects from the set, there are $(n-2)$ objects from which we could choose a third, so there are $(n-2)$ options for what object we can make third.

After we've selected and removed the first three objects, there are $(n-3)$ options left for what could be fourth.

This pattern continues. Eventually there will be two objects left for us to choose from in deciding which object will be next-to-last. That means we have 2 options at this point for what the next-to-last object will be.

After having chosen and selected what the first $(n-1)$ objects are, there is only one object from the set remaining. That means there is only 1 option for what we can take last.

We just made n different choices: a choice for first, second, third, fourth..., next-to-last, and last. Options multiply, so the total number of ways we can order a set of n objects is

$$n(n-1)(n-2)(n-3)(n-4)\cdots(2)(1) = n!$$

Examples.

• There are exactly 4 objects in the set $\{\sqrt{2}, -\frac{2}{5}, f, e\}$. Therefore, there are $4! = 24$ ways to order the objects in the set $\{\sqrt{2}, -\frac{2}{5}, f, e\}$. (Wasn't that much easier than making a list of all the options, and then counting the items on the list?)

• You have 8 rooms in your house, and 8 different lamps. You want to put a single lamp in each of the rooms, but you're not sure which one to put in which room. So you decide to try out all possible arrangement of lamps in rooms to see which arrangement you like the best.

If it takes you two minutes every time you try out a new arrangement of lamps in rooms, and if you never take a break for sleeping, eating, using the restroom, etc., then you will have finished experimenting with all of the possible arrangements of lamps after 56 days.

There are $8! = 40320$ arrangements, and each arrangement cost you 2 minutes. So the task will require $40320(2) = 80640$ minutes. There are 1440 minutes in a day, so the task will take $\frac{80640}{1440} = 56$ days.

Of course, the amount of time spent arranging lamps increases as the number of possible arrangements increase. If you were unfortunate enough to live in a 20 room mansion, and you wanted to experiment by placing one of 20 different lamps into each room in every possible arrangement, and if it took you two minutes each time you rearranged the lamps, then trying out every possible arrangement would take more than 9 trillion years – if you never stopped for a break of any kind.

• There are 26 letters in the English alphabet. We have assigned those letters an order: A,B,C,D,...,X,Y,Z, which is called the alphabetical order. This is only one of the choices for ordering the alphabet that we could have made as a society. We could have chosen any one of the possible 26! different ways to order the alphabet. Note that

$$26! = 403,291,461,126,605,635,584,000,000$$

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Choosing and ordering some of the objects in a set

In the Olympics you might have 120 people competing in the same event. The goal of the competition is to determine a gold, silver, and bronze athlete, and that's it. The Olympics will choose and order 3 athletes out of the 120.

We want a general formula that will allow us to count the number of different ways that we can choose and order k objects out of a set of n objects. (Of course, for this process to make sense we need to have that $k \leq n$.)

We could choose any of the n objects to be first in our order, leaving us with $(n-1)$ options for a second, then $(n-2)$ options for the third, and so on, until we have chosen the first $k-1$ objects. The last step would be to choose a k^{th} object from the remaining $(n-(k-1)) = (n-k+1)$ objects. Then we multiply the number of options we had for each choice to find that the number of different ways that we can choose and order k objects out of a set of n objects is

$$n(n-1)(n-2)\cdots(n-k+2)(n-k+1)$$

That number is the same as the fraction

$$\frac{n(n-1)(n-2)\cdots(n-k+2)(n-k+1)(n-k)(n-k-1)\cdots(2)(1)}{(n-k)(n-k-1)\cdots(2)(1)}$$

And the above fraction can be written more simply as

$$\frac{n!}{(n-k)!}$$

Examples.

- If 120 athletes are competing for a gold, silver, and bronze medal (and no athlete can win two medals), then there are

$$\frac{120!}{(120-3)!} = \frac{120!}{117!} = \frac{120(119)(118)(117!)}{117!} = 120(119)(118) = 1,685,040$$

different ways that the athletes could be standing on the winner's podium by the end of the competition.

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Exercises

1.) How many ways are there to choose a five-digit PIN code using the numbers 0-9 if no two consecutive numbers in the code are allowed to be the same? (For example, 34556 is not allowed, but 54568 is allowed.)

2.) You have three pairs of shoes, four pairs of pants, six shirts, two jackets, and two hats to choose from. How many different outfits can you put together that use one pair of shoes, one pair of pants, one shirt, one jacket, and one hat?

3.) How many ways can the letters **a**, **f**, **t**, **e**, **r** be arranged?

4.) A couple plans to have five children. They have decided the names of their children in advance: Sam, Sue, Terry, Robin, and Tonie. All they have left is to decide which of their children will receive which name. How many different options are there for which child is given which name?

5.) A basketball team has 12 players. There are five different positions on a basketball team. A starting lineup consists of five players, each assigned to one of the five positions. How many different ways can a coach select a starting lineup?

6.) There are 51 contestants for a beauty pageant, one for every state and the District of Columbia. The judges need to select one contestant as the winner, one as the runner up, and one as the winner for congeniality. How many different ways can the judges distribute the awards?

7.) There are 10 people on a boat. One person needs to be the captain, one needs to be the first mate, and you need a person to swab the decks (no person can do more than one job). How many different ways can those three jobs be filled by the 10 people on board.

8.) A national magazine wants to rank the three most desirable states to live in – first, second, and third – and the three most undesirable states to live in – 50th, 49th, and 48th. How many rankings are possible?

Counting II

Sometimes we will want to choose k objects from a set of n objects, and we won't be interested in ordering them. For example, if you are leaving for vacation and you want to pack your suitcase with three of the seven pairs of shorts that you own, then it doesn't matter in which order you pack the shorts. All that matters is which three pairs you pack.

n choose k

The number of different ways that k objects can be chosen from a set of n objects (when order doesn't matter) is called n *choose* k. It is written in symbol form as $\binom{n}{k}$.

Examples.

• There are four different ways that one letter can be chosen from the set of four letters $\{\mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{a}\}$. One way is to choose the letter **e**. Alternatively, you could also choose the letter **f**, or the letter **g**, or the letter **a**.

Since there are 4 options for choosing one object from a set of 4 objects, we have $\binom{4}{1} = 4$.

• Below is a list of all the possible ways that 2 numbers can be chosen from the set of four numbers $\{3, 7, 2, 9\}$. There are six different ways. Thus, $\binom{4}{2} = 6$.

$3, 7$	$3, 2$	$3, 9$
$7, 2$	$7, 9$	$2, 9$

General formula

To say that we are choosing and ordering k objects from a set of n objects is to say that we are performing 2 separate tasks. First is the task of choosing k objects from the set of n objects, and the number of ways to perform that task is $\binom{n}{k}$. Second is the task of ordering the k objects after we've chosen them. There are $k!$ ways to order k objects.

Let's repeat that. To choose and order k objects: First, choose the k objects, then order the k objects you chose. Options multiply, so the total number of ways that we can choose and order k objects from a set of n objects is $\binom{n}{k}k!$.

We saw in the previous chapter that there are exactly $\frac{n!}{(n-k)!}$ ways to choose and order k objects from a set of n objects. Therefore,

$$\binom{n}{k}k! = \frac{n!}{(n-k)!}$$

Dividing the previous equation by $k!$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Examples.

- There are $\binom{7}{3}$ different ways to choose which 3 of the 7 pairs of shorts that you will take on your vacation.

$$\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7(6)(5)(4!)}{3!4!} = \frac{7(6)(5)}{6} = 7(5) = 35$$

- How many 5 card poker hands are there if you play with a standard deck of 52 cards?

You're counting the number of different collections of 5 cards that can be taken from a set of 52 cards. This number is

$$\binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52(51)(50)(49)(48)(47!)}{5!\,47!} = \frac{52(51)(50)(49)(48)}{5!}$$
$$= \frac{52(51)(50)(49)(48)}{120} = 2,598,960$$

- You are at a DVD rental store. You want to rent 3 DVDs. The store has 3287 different DVDs to choose from. There are $\binom{3287}{3}$ different collections of three movies that you could rent.

$$\binom{3287}{3} = \frac{3287!}{3!(3287-3)!} = \frac{3287(3286)(3285)(3284!)}{3!\,3284!} = \frac{3287(3286)(3285)}{3!}$$
$$= \frac{35,481,554,370}{6} = 5,913,592,395$$

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Pascal's Triangle

We can arrange the numbers $\binom{n}{k}$ into a triangle.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & \binom{0}{0} & & & & & & & \\
 & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & & \\
 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & & \\
 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & \\
 & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & & \\
 & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & & \\
 & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6} & \\
\binom{7}{0} & & \binom{7}{1} & & \binom{7}{2} & & \binom{7}{3} & & \binom{7}{4} & & \binom{7}{5} & & \binom{7}{6} & & \binom{7}{7}
\end{array}
$$

In each row, the "top" number of $\binom{n}{k}$ is the same. The "bottom" number of $\binom{n}{k}$ is the same in each upward slanting diagonal. The triangle continues on forever. The first 8 rows are shown above.

This is called Pascal's triangle. It is named after a French mathematician who discovered it. It had been discovered outside of Europe centuries earlier by Chinese mathematicians. Modern mathematics began in Europe, so it's traditions and stories tend to promote the exploits of Europeans over others.

Some values of $\binom{n}{k}$ to start with

$\binom{n}{n}$ is the number of different ways you can select n objects from a set of n objects. There is only one way to take everything – you just take everything – so $\binom{n}{n} = 1$.

Similarly, there is only one way to take nothing from a set – just take nothing, that's your only option. The number of ways you can select nothing, a.k.a. 0 objects, from a set is $\binom{n}{0}$. That means $\binom{n}{0} = 1$.

Now we can fill in the values for $\binom{n}{n}$ and $\binom{n}{0}$ into Pascal's triangle.

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad \binom{2}{1} \quad 1 \\
1 \quad \binom{3}{1} \quad \binom{3}{2} \quad 1 \\
1 \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad 1 \\
1 \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad 1 \\
1 \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad 1 \\
1 \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad 1
\end{array}
$$

There are n different ways to choose 1 object from a set that has n objects. Thus, $\binom{n}{1} = n$.

Similarly, there are exactly n different options for choosing $n-1$ objects from a set of n objects. To see this, notice that deciding which of the $n-1$ objects that you will take from a set of n objects is the same as deciding which 1 object you will leave behind. So the number of ways you can take $n-1$ objects is the same as the number of ways you can leave 1 object. That is to say, $\binom{n}{n-1} = \binom{n}{1}$. Hence, $\binom{n}{n-1} = n$

Once we fill in this new information on Pascal's triangle it looks like

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad \binom{4}{2} \quad 4 \quad 1 \\
1 \quad 5 \quad \binom{5}{2} \quad \binom{5}{3} \quad 5 \quad 1 \\
1 \quad 6 \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad 6 \quad 1 \\
1 \quad 7 \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad 7 \quad 1
\end{array}
$$

Before moving on, let's look back at the last rule that we found: $\binom{n}{n-1} = \binom{n}{1}$. The argument we gave there generalizes, in that taking k objects from a set of n objects is the same as leaving $n-k$. Therefore,

$$\binom{n}{k} = \binom{n}{n-k}$$

What we saw earlier in the form $\binom{n}{n-1} = \binom{n}{1}$ was the special case of the formula $\binom{n}{k} = \binom{n}{n-k}$ when $k = n-1$.

Add the two numbers above to get the number below

Suppose that you have a set of n different rocks: 1 big red brick, and $n-1$ different little blue marbles. How many different ways are there to choose $k+1$ rocks from the set of n rocks?

Any collection of $k+1$ rocks either includes the big red brick, or it doesn't.

Let's first look at those collections of $k+1$ objects that *do* contain a big red brick. One of the $k+1$ objects we will choose is a big red brick. That's a given. That means that all we have to do is decide which k of the little blue marbles we want to choose along with the big red brick to make up our collection of $k+1$ objects. There are $\binom{n-1}{k}$ different ways we could choose k marbles from the total number of $n-1$ little blue marbles. Thus, there are $\binom{n-1}{k}$ different ways we could choose a set of $k+1$ objects from our set of n rocks if we know that one of the objects we will choose is a big red brick.

Now let's look at those collections of $k+1$ objects that *don't* contain the big red brick. Then all $k+1$ objects that we will choose are little blue marbles. There are $n-1$ little blue marbles, and the number of different ways we could choose $k+1$ of the $n-1$ little blue marbles is $\binom{n-1}{k+1}$.

Any collection of $k+1$ rocks either includes the big red brick, or it doesn't. So to find the number of ways that we could choose $k+1$ objects, we just have to add the number of possibilities that contain a big red brick, to the number of possibilities that don't contain a big red brick. That formula is

$$\binom{n}{k+1} = \binom{n-1}{k} + \binom{n-1}{k+1}$$

If $n = 6$ and $k = 2$, then the above formula says that $\binom{6}{3} = \binom{5}{2} + \binom{5}{3}$. Looking at Pascal's triangle, you'll see that $\binom{5}{2}$ and $\binom{5}{3}$ are the two numbers that are just above the number $\binom{6}{3}$.

Change the values of n and k and check that the above formula always indicates that to find a number in Pascal's triangle, just sum the two numbers that are directly above it.

For example,

$$\binom{2}{1} = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2$$

and

$$\binom{5}{4} = \binom{4}{3} + \binom{4}{4} = 4 + 1 = 5$$

We know the natural numbers that are at the very tip of Pascal's triangle. To find the rest of the numbers in Pascal's triangle, we can let that knowledge trickle down the triangle using this latest formula that any number in the triangle is the sum of the two numbers above it.

$$\begin{array}{ccccccccccccccc} & & & & & & & 1 & & & & & & & \\ & & & & & & 1 & & 1 & & & & & & \\ & & & & & 1 & & 2 & & 1 & & & & & \\ & & & & 1 & & 3 & & 3 & & 1 & & & & \\ & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\ & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\ & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\ 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \end{array}$$

Notice that Pascal's triangle is the same if you read it left-to-right, or right-to-left. Convince yourself that this is a consequence of the formula $\binom{n}{k} = \binom{n}{n-k}$.

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Binomial Theorem

For $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

Example. We can use the binomial theorem and Pascal's triangle to write out the product $(x+y)^3$. The binomial theorem states that

$$\begin{aligned}(x+y)^3 &= \sum_{i=0}^{3} \binom{3}{i} x^{3-i} y^i \\ &= \binom{3}{0} x^{3-0} y^0 + \binom{3}{1} x^{3-1} y^1 + \binom{3}{2} x^{3-2} y^2 + \binom{3}{3} x^{3-3} y^3 \\ &= \binom{3}{0} x^3 + \binom{3}{1} x^2 y + \binom{3}{2} x y^2 + \binom{3}{3} y^3\end{aligned}$$

The numbers $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$, and $\binom{3}{3}$ make up the fourth row of Pascal's triangle, and we can see from the triangle that they equal 1, 3, 3, and 1 respectively. Therefore,

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Binomial coefficients. Because of the binomial theorem, numbers of the form $\binom{n}{k}$ are called *binomial coefficients.*

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Exercises

1.) A small country has just undergone a revolution and they commission you to design a new flag for them. The want exactly 3 colors used in their flag, and they have given you 7 colors of cloth that you are allowed to use. How many different color combinations do you have to decide between?

2.) A sports team is selling season ticket plans. They have 15 home games in a season, and they allow people to purchase tickets for any combination of 7 home games. How many ways are there to choose a collection of 7 home games?

3.) A bagel shop asks customers to create a "Baker's Dozen Variety Pack" by choosing 13 different types of bagels. If the shop has 20 different kinds of bagels to choose between, then how many different variety packs does a customer have to choose from?

4.) To play the lottery you have to select 6 out of 59 numbers. How many different kinds of lottery tickets can you purchase?

5.) Use the Binomial Theorem and Pascal's triangle to write out the product $(x+y)^6$. (Your answer should have a similar form to the example that was done before the exercises.)

6.) Use the Binomial Theorem and Pascal's triangle to write out the product $(x+y)^7$.

7.) Find $(x+2)^3$ using the Binomial Theorem.

8.) Find $(2x-1)^4$ using the Binomial Theorem.

Back to Functions

More on functions

Suppose $f : \mathbb{R} \to \mathbb{R}$ is the function defined by $f(x) = x^5$. The letter x in the previous equation is just a placeholder. You are allowed to replace the x with any number, symbol, or combination of symbols that you like.

$$f(4) = 4^5 \qquad f(-1) = (-1)^5$$

$$f(\pi) = \pi^5 \qquad f\big(-\tfrac{11}{13}\big) = \big(-\tfrac{11}{13}\big)^5$$

$$f(\blacksquare) = \blacksquare^5 \qquad f(\clubsuit) = \clubsuit^5$$

$$f(y) = y^5 \qquad f(x-3) = (x-3)^5$$

$$f(g(x)) = (g(x))^5 \qquad f\left(\tfrac{1-x}{x^2+4}\right) = \left(\tfrac{1-x}{x^2+4}\right)^5$$

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Composition

Suppose that $f : A \to B$ and $g : B \to C$ are functions.

If $a \in A$, then a is in the domain of f and $f(a) \in B$. Since $f(a) \in B$, we have that $f(a)$ is in the domain of g, so $g(f(a))$ is an object in C.

This process defines a third function, named $g \circ f : A \to C$ that is defined by

$$g \circ f(a) = g(f(a))$$

The function $g \circ f$ is pronounced "g composed with f".

Examples.

• Suppose $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are functions and that $g(1) = 3$ and $h(3) = 7$. Then

$$h \circ g(1) = h(g(1)) = h(3) = 7$$

• Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions that are defined by $f(x) = x^2$ and $g(x) = x - 1$.

Then $g \circ f : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$g \circ f(x) = g(f(x)) = g(x^2) = x^2 - 1$$

And $f \circ g : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$f \circ g(x) = f(g(x)) = f(x - 1) = (x - 1)^2$$

Important: Notice in the previous example that $g \circ f(2) = 3$ and $f \circ g(2) = 1$. That means that $g \circ f$ is not the same function as $f \circ g$. In other words, $g \circ f \neq f \circ g$.

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Range

Recall that the target of a function $f : A \to B$ is the set B. That means that for any $a \in A$, we have that $f(a) \in B$.

But it might not be that every object in B has an object from A assigned to it by the function f. For example, you might recall that if you square a real number, the result is never a negative number ($2^2 = 4$, $(-3)^2 = 9$, etc.). Therefore, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ has $\mathbb{R}$ as its target, although none of the negative numbers in the target have a number from the domain assigned to it.

The *range* of a function f is the set of numbers that "come out of" f. For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x - 2$, then $f(3) = 3 - 2 = 1$. We put 3 in to f, and got 1 out, so 1 is an object in the range.

Another way to say what the range is, is to say that it is the smallest set that can serve as the target of the function.

Examples.

- Let $h : \{1, 2, 3, 4\} \to \{3, 4, 7, 8, 9\}$ be the function given by

$$h(1) = 9 \qquad h(2) = 4$$

$$h(3) = 4 \qquad h(4) = 8$$

If we put the numbers from the domain "in to" h, the only numbers that "come out" are 9, 4, and 8. That means that the range of h is $\{9, 4, 8\}$.

• If $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = \frac{x-1}{x^2+1}$, then

$$g(1) = \frac{1-1}{1^2+1} = \frac{0}{2} = 0$$

and

$$g(2) = \frac{2-1}{2^2+1} = \frac{1}{5}$$

Therefore, 0 and $\frac{1}{5}$ are both objects in the range of g.

There are also other numbers in the range of g. For example, $g(4)$, $g(-1)$, $g(\sqrt{2})$, etc.

• The range of $f : \mathbb{N} \to \mathbb{R}$ where $f(n) = (-1)^n$ is the set $\{-1, 1\}$.

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Implied domains

Sometimes we won't go through the trouble of writing the entire name of a function as "$f : D \to T$ where $f(x) = x^5$". This is similar to how we usually call people by their first names, omitting their middle and last names, just because it's easier.

If we are introduced to a function that is given by an equation, and its domain is not specified, we will assume that the domain for that function is the largest subset of the real numbers possible. This set will be called the *implied domain* of the function.

Examples.

• Let $h(x) = 4x - 1$. Then for any real number $r \in \mathbb{R}$, $h(r) = 4r - 1$ makes sense, because it is a real number itself. Therefore, it is safe to put any real number into h. Its implied domain is $\mathbb{R}$.

• Let $f(x) = \frac{5}{x-1}$. If r is a real number, then $r - 1$ is a real number. As long as $r - 1 \neq 0$, $\frac{5}{r-1}$ is also a real number. However, if $r - 1 = 0$, then $\frac{5}{r-1} = \frac{5}{0}$ does not make sense – it is not a real number.

To recap, $f(r)$ is a real number except when $r - 1 = 0$, or equivalently, except when $r = 1$. Therefore, the numbers that it makes sense to "put in to" f are all of the real numbers except for 1. Another way of saying the previous sentence, is that the implied domain of f is the set $\mathbb{R} - \{1\}$.

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Exercises

For #1-6, assume that $f(x) = x^2$, $g(x) = 2x - 1$, and that $h(x) = x - 5$, and match the given function with one of the following:

A. $2x - 11$ B. $x^2 - 10x + 25$ C. $2x - 6$

D. $x^2 - 5$ E. $2x^2 - 1$ F. $4x^2 - 4x + 1$

1) $f \circ g(x)$

2) $g \circ f(x)$

3) $g \circ h(x)$

4) $h \circ g(x)$

5) $f \circ h(x)$

6) $h \circ f(x)$

For #7-12, assume that $f(x) = x+3$, $g(x) = 3x-4$, and that $h(x) = x^2+1$, and match the given function with one of the following:

A. $3x + 5$ B. $9x^2 - 24x + 17$ C. $3x^2 - 1$

D. $x^2 + 6x + 10$ E. $x^2 + 4$ F. $3x - 1$

7) $f \circ g(x)$

8) $g \circ f(x)$

9) $g \circ h(x)$

10) $h \circ g(x)$

11) $f \circ h(x)$

12) $h \circ f(x)$

13) What is the implied domain of $f(x) = 3x - 4$?

14) What is the implied domain of $g(x) = x^3 - 4x^2 - 2x + 1$?

15) What is the implied domain of $h(x) = 15x - 3$?

16) What is the implied domain of

$$f(x) = \frac{1}{x-3} \text{ ?}$$

17) What is the implied domain of

$$g(x) = \frac{2x-4}{x+3} \text{ ?}$$

18) What is the implied domain of

$$h(x) = \frac{3x^2 - 4x + 5}{-3x+4} \text{ ?}$$

Intro to Graphs

2 is a real number, and 3 is a real number. We can take those two numbers and write them as a pair of real numbers: $(2, 3)$. When we write a pair of real numbers, the order is important. That is to say that $(2, 3)$ is not the same pair as $(3, 2)$.

Unfortunately, $(2, 3)$ is also the way we write the interval of real numbers between 2 and 3. We have to try hard to never confuse a pair of numbers for an interval, but it's usually clear from the context of a problem whether $(2, 3)$ refers to a pair of numbers or to an interval.

$\mathbb{R}^2$ is the set of all pairs of real numbers. So $(2, 3) \in \mathbb{R}^2$, and $(3, 2) \in \mathbb{R}^2$, and $(\sqrt{2}, -7) \in \mathbb{R}^2$, etc.. Any pair of real numbers is called a *point* in $\mathbb{R}^2$.

Suppose $f : A \to B$ is a function with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. The *graph* of f is the subset of $\mathbb{R}^2$ consisting of all points of the form $(a, f(a))$.

Examples.

- If $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = 5x$, then $f(3) = 5(3) = 15$. We put 3 in, and got 15 out. That means the point $(3, 15)$ is in the graph of f.

Also, $f(1) = 5$, so $(1, 5)$ is a point in the graph of f, and $(2, f(2)) = (2, 10)$ is a point in the graph of f as well.

- Suppose $g : \mathbb{R} \to \mathbb{R}$ is the function where $g(x) = x - 2$. If you put 2 in to g, then 0 comes out. That means the point $(2, 0)$ is in the graph of g.

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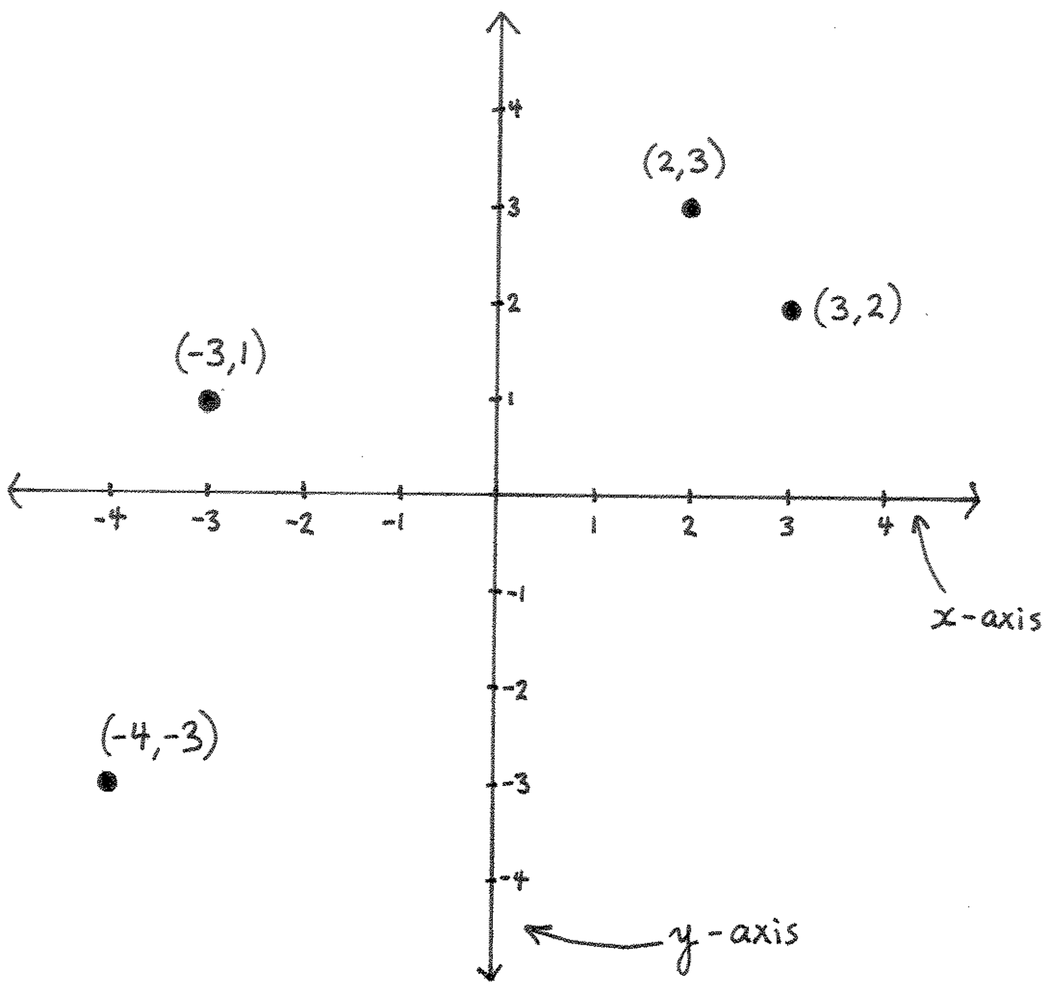
Drawing $\mathbb{R}^2$

The set $\mathbb{R}^2$ is a plane. The first coordinate of a point in $\mathbb{R}^2$ measures the horizontal. The second number measures the vertical.

The set of points in $\mathbb{R}^2$ of the form $(x, 0)$ creates a horizontal line called the *x-axis.*

The set of points in $\mathbb{R}^2$ of the form $(0, y)$ creates a vertical line called the *y-axis.*

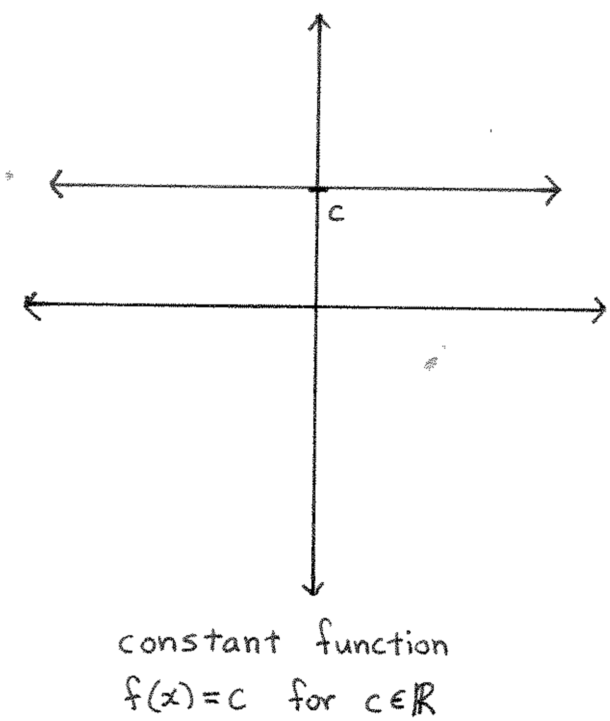


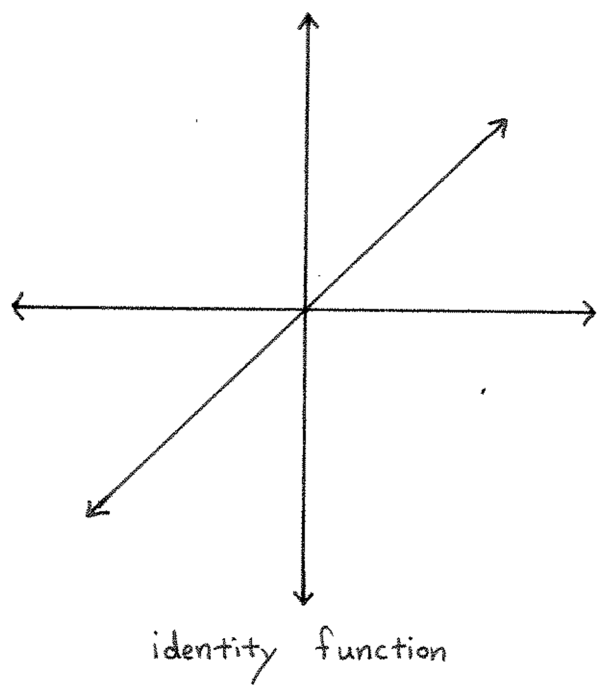
Drawing graphs

The graph of a function is a subset of $\mathbb{R}^2$. You draw it by marking all of the points in the graph.

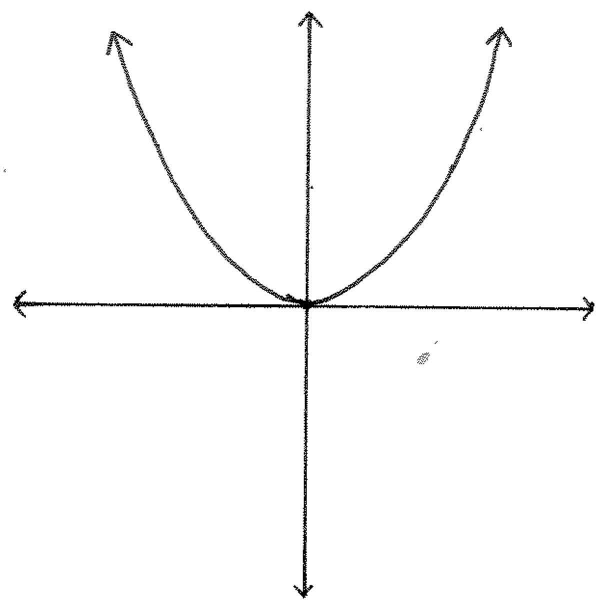
Graphs of important functions

Some functions are important enough in mathematics that you should be able to draw their graphs quickly (and you will be required to do so on exams). A list of these important functions includes constant functions; the identity function id; $f(x) = x^n$ for an even $n \in \mathbb{N}$; $f(x) = x^n$ for $n \in \mathbb{N}$ odd and $n \geq 3$; $f(x) = \frac{1}{x^n}$ for odd $n \in \mathbb{N}$; and $f(x) = \frac{1}{x^n}$ for even $n \in \mathbb{N}$.

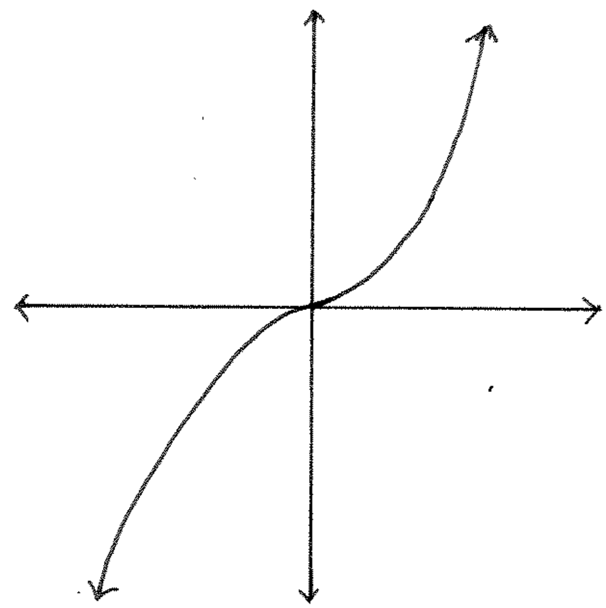


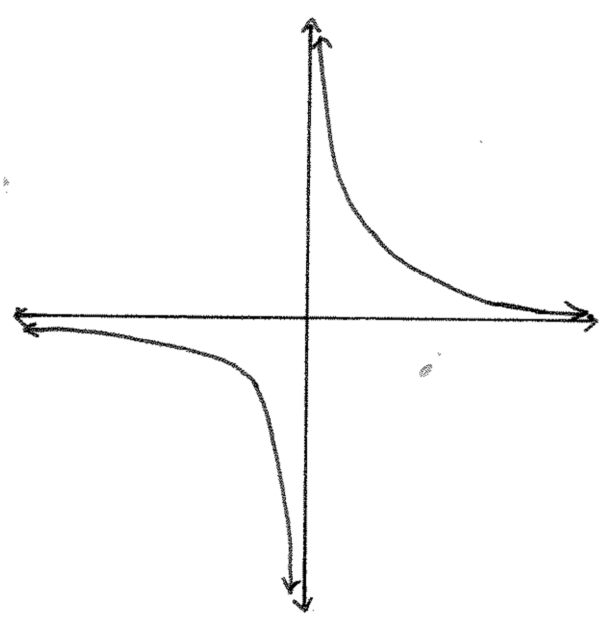


x^n for $n \in \mathbb{N}$ even

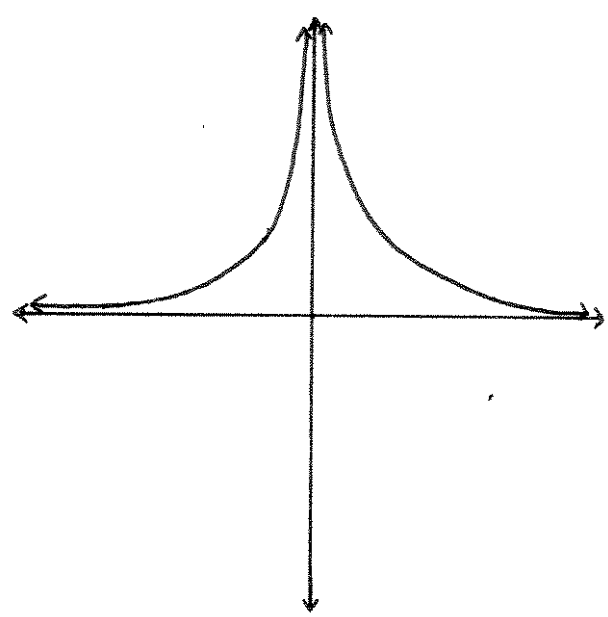


x^n for $n \in \mathbb{N}$ odd and $n \geq 3$





$\frac{1}{x^n}$ for $n \in \mathbb{N}$ odd



$\frac{1}{x^n}$ for $n \in \mathbb{N}$ even

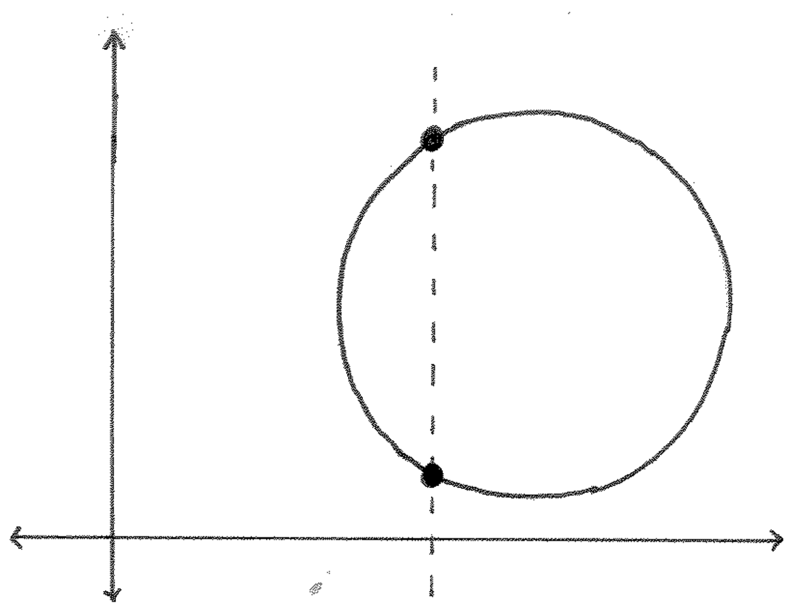
Vertical line test

Sometimes you'll see something drawn in $\mathbb{R}^2$ that looks like it might be the graph of a function. To know for sure if it is, use the *vertical line test*:

> If a vertical line intersects a thing in more than one point, then the thing is not a graph.

The reason such a thing is not a graph, is because if a vertical line intersects it in two different points, then the thing would include two different points with the same first coordinate – for example, $(1, 4)$ and $(1, 9)$. This could not be the graph of a function, because if it were, then the function would assign two different numbers – 4 and 9 – to the same object of the domain – 1. Functions can't do that.

Example. A circle is *not* the graph of a function. It fails the vertical line test.

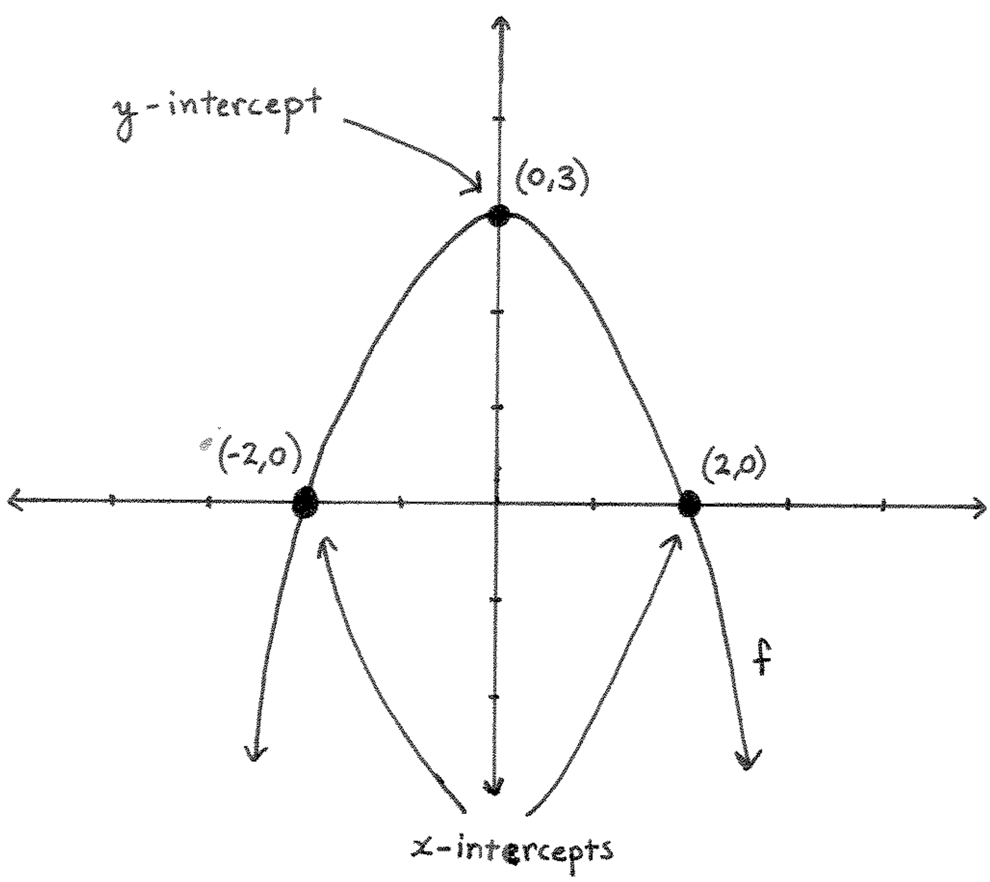


Intercepts

If the graph of a function contains a point of the form $(a, 0)$ for some $a \in \mathbb{R}$, then a is called an x-intercept of the graph.

If the graph of a function contains a point of the form $(0, b)$ for some $b \in \mathbb{R}$, then b is called the y-intercept of the graph.

Example. Below is the graph of a function f. The x-intercepts of the graph are 2 and -2. The y-intercept of the graph is 3.



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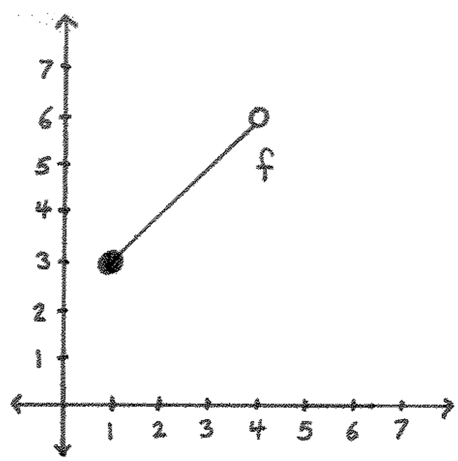
Little circles vs. giant dots

Drawing a giant dot in a graph means that point is in the graph of the function.

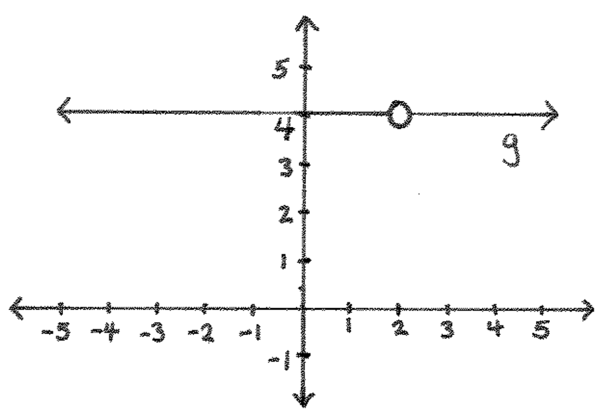
Drawing a little circle in a graph means that point is not in the graph of the function, but some nearby points are.

Example. Below is the graph of the function $f : [1, 4) \to \mathbb{R}$ where $f(x) = x+2$. The number 1 is in the domain of f, and $f(1) = 3$, so the point $(1, 3)$ is in the graph of f. We can label it with a giant dot.

The number 4 is not in the domain of f, but some numbers really close to 4 are. If 4 was in the domain, then $f(4) = 6$, and $(4, 6)$ would be a point in the graph of f. But 4 isn't in the domain, so $(4, 6)$ isn't a point in the graph of f. The graph does go all the way up to the point $(4, 6)$, but it doesn't include the point $(4, 6)$. So we label the point $(4, 6)$ with a little circle to remind us that it's not actually in the graph.



Example. Below is the graph of the function $g : \mathbb{R} - \{2\} \to \mathbb{R}$ where $g(x) = 4$. Since 2 is not in the domain of g, the point $(2, 4)$ is not in the graph of g, so we label it with a little circle to remind us that it's not in the graph.



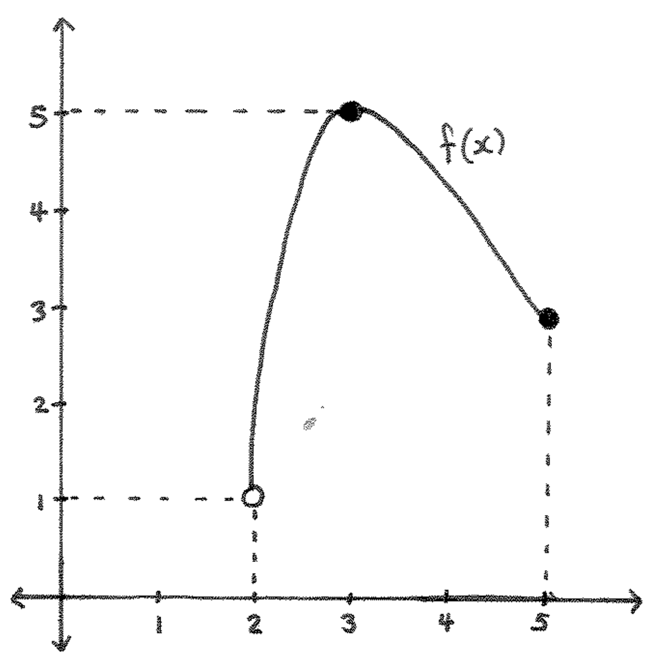
Domains and Ranges for graphs

Suppose you are given a graph, and you're told that it is the graph of a function. To find the domain of the function, draw its "shadow" on the x-axis.

To find the range of the function, draw its "shadow" on the y-axis.

Example. Drawn below is the graph of the function f. The domain of f is the set of real numbers in the x-axis that lie directly below the graph. Those are all of the numbers between 2 and 5. Because there is a giant dot on the point $(5, 3) \in \mathbb{R}^2$, we know that 5 is in the domain. But since there is a little circle on the point $(2, 1) \in \mathbb{R}^2$, we know that 2 is not in the domain. That is, the domain of f is the interval $(2, 5]$.

The range of f is the set of real numbers on the y-axis that lie directly to the left of the graph of f. The range of f is the interval $(1, 5]$.



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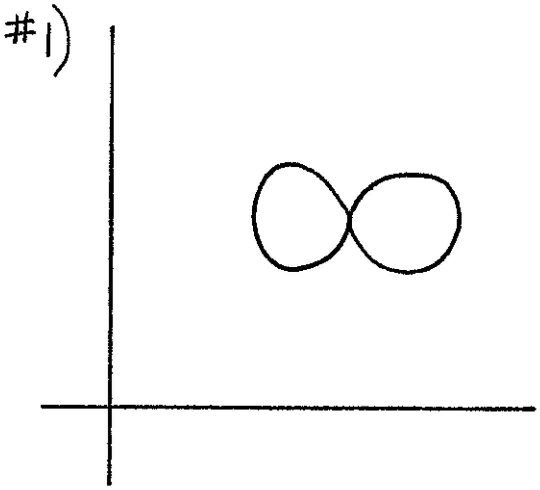
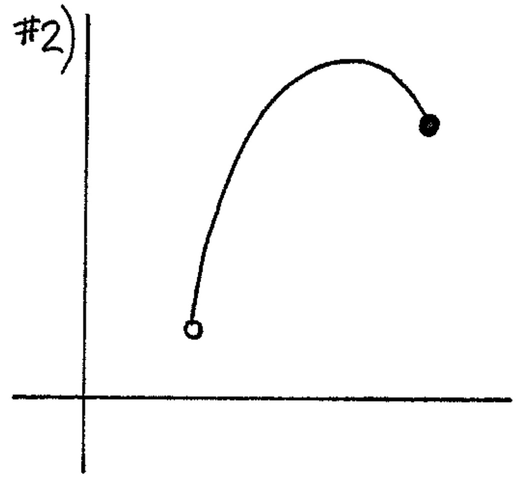
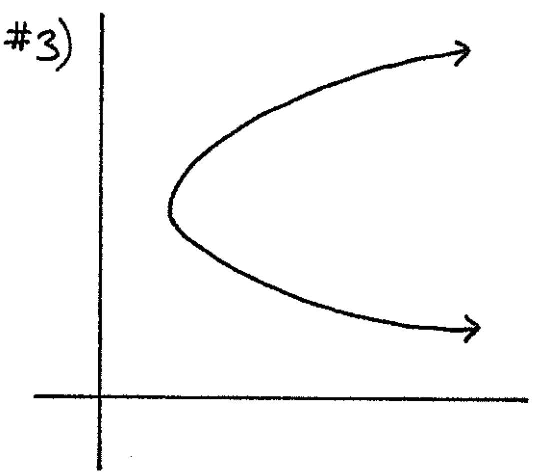
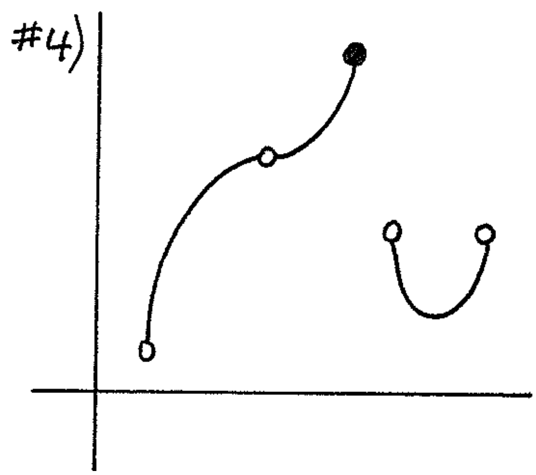
Exercises

For #1-4, decide whether or not each of the drawings in $\mathbb{R}^2$ is the graph of a function.

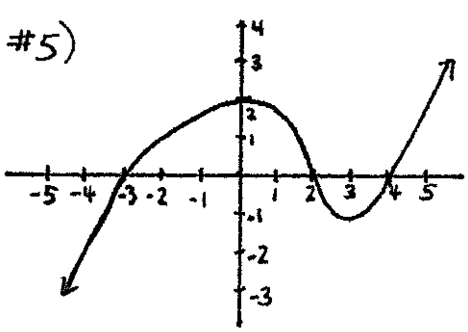


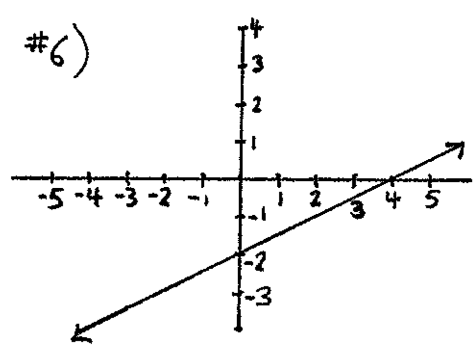




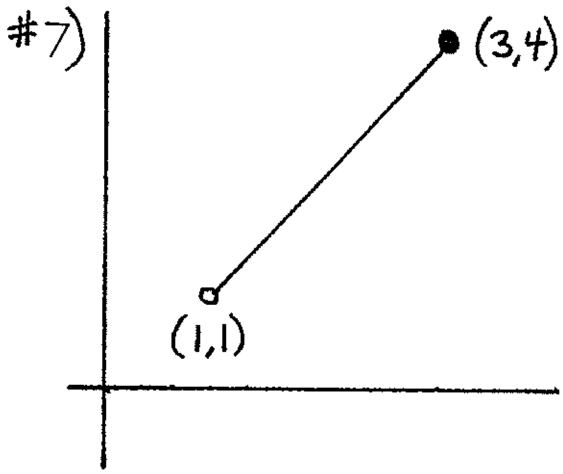


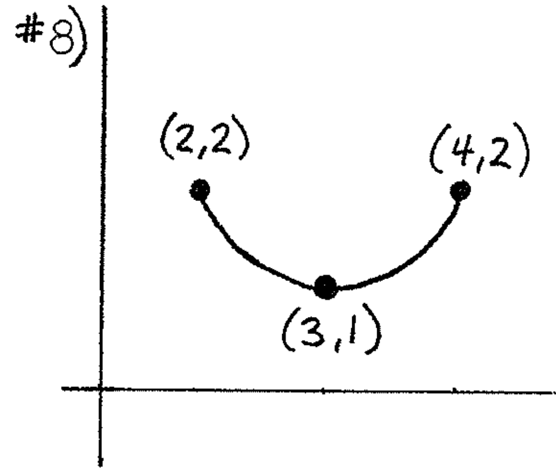
For #5-6, list the x and y-intercepts of the graphs below.

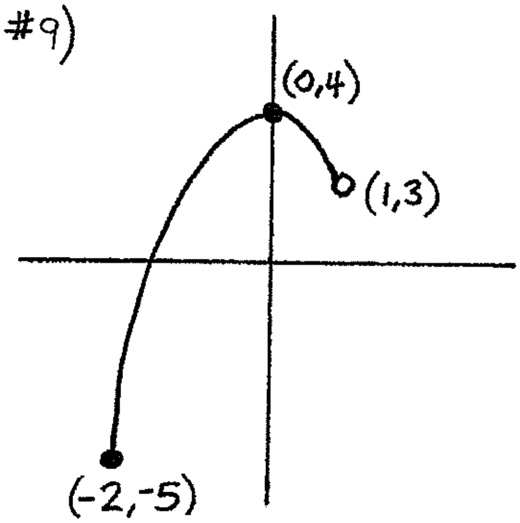


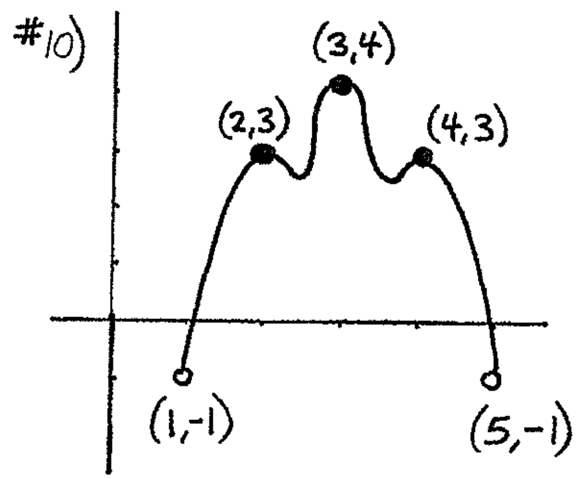


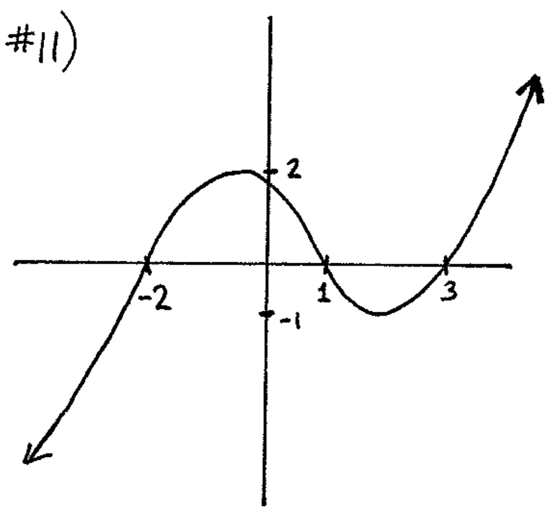
For #7-12, determine the domains and ranges for the functions that are drawn.

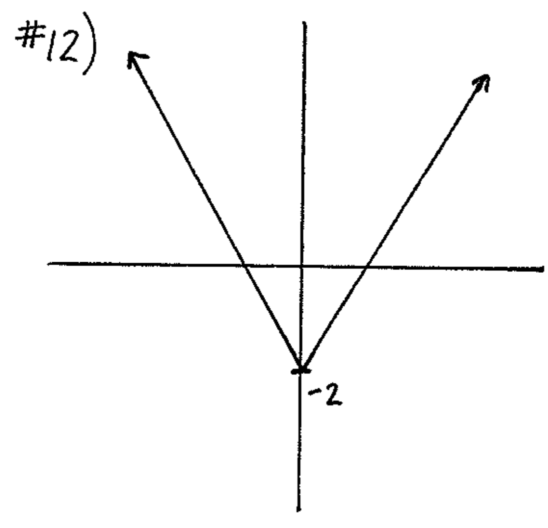












Graph Transformations

There are many times when you'll know very well what the graph of a particular function looks like, and you'll want to know what the graph of a very similar function looks like. In this chapter, we'll discuss some ways to draw graphs in these circumstances.

Transformations "after" the original function

Suppose you know what the graph of a function $f(x)$ looks like. Suppose $d \in \mathbb{R}$ is some number that is greater than 0, and you are asked to graph the function $f(x) + d$. The graph of the new function is easy to describe: just take every point in the graph of $f(x)$, and move it up a distance of d. That is, if (a, b) is a point in the graph of $f(x)$, then $(a, b + d)$ is a point in the graph of $f(x) + d$. Let's see why:

If (a, b) is a point in the graph of $f(x)$, then that means $f(a) = b$. Hence, $f(a) + d = b + d$, which is to say that $(a, b + d)$ is a point in the graph of $f(x) + d$.

The chart on the next page describes how to use the graph of $f(x)$ to create the graph of some similar functions. Throughout the chart, $d > 0$, $c > 1$, and (a, b) is a point in the graph of $f(x)$.

Notice that all of the "new functions" in the chart differ from $f(x)$ by some algebraic manipulation that happens after f plays its part as a function. For example, first you put x into the function, then $f(x)$ is what comes out. The function has done its job. Only after f has done its job do you add d to get the new function $f(x) + d$.

Because all of the algebraic transformations occur after the function does its job, all of the changes to points in the second column of the chart occur in the second coordinate. Thus, all the changes in the graphs occur in the vertical measurements of the graph.

New function	How points in graph of $f(x)$ become points of new graph	visual effect
$f(x) + d$	$(a, b) \mapsto (a, b + d)$	shift up by d
$f(x) - d$	$(a, b) \mapsto (a, b - d)$	shift down by d
$cf(x)$	$(a, b) \mapsto (a, cb)$	stretch vertically by c
$\frac{1}{c}f(x)$	$(a, b) \mapsto (a, \frac{1}{c}b)$	shrink vertically by $\frac{1}{c}$
$-f(x)$	$(a, b) \mapsto (a, -b)$	flip over the x-axis

Transformations "before" the original function

We could also make simple algebraic adjustments to $f(x)$ *before* the function f gets a chance to do its job. For example, $f(x+d)$ is the function where you first add d to a number x, and only after that do you feed a number into the function f.

On the next page is a chart that is similar to the chart above. The difference in the next chart is that the algebraic manipulations occur before you feed a number into f, and thus all of the changes occur in the first coordinates of points in the graph. All of the visual changes affect the horizontal measurements of the graph.

One important point of caution to keep in mind is that most of the visual horizontal changes described in the next chart are the exact opposite of the effect that most people anticipate after having seen the chart above. To get an idea for why that's true let's work through one example.

Suppose that $d > 0$. If (a, b) is a point that is contained in the graph of $f(x)$, then $f(a) = b$. Hence, $f((a - d) + d) = f(a) = b$, which is to say that

$(a-d, b)$ is a point in the graph of $f(x+d)$. The visual change between the point (a, b) and the point $(a-d, b)$ is a shift to the left a distance of d.

In the chart below, just as in the previous chart, $d > 0$, $c > 1$, and (a, b) is a point in the graph of $f(x)$.

New function	How points in graph of $f(x)$ become points of new graph	visual effect
$f(x+d)$	$(a, b) \mapsto (a-d, b)$	shift left by d
$f(x-d)$	$(a, b) \mapsto (a+d, b)$	shift right by d
$f(cx)$	$(a, b) \mapsto (\frac{1}{c}a, b)$	shrink horizontally by $\frac{1}{c}$
$f(\frac{1}{c}x)$	$(a, b) \mapsto (ca, b)$	stretch horizontally by c
$f(-x)$	$(a, b) \mapsto (-a, b)$	flip over the y-axis

Transformations before and after the original function

As long as there is only one type of operation involved "inside the function" – either multiplication or addition – and only one type of operation involved "outside of the function" – either multiplication or addition – you can apply the rules from the two charts above to transform the graph of a function.

Examples.

• Let's look at the function $-2f(x+3)$. There is only one kind of operation inside of the parentheses, and that operation is addition – you are adding 3.

There is only one kind of operation outside of the parentheses, and that operation is multiplication – you are multiplying by 2, and you are multiplying by -1.

So to find the graph of $-2f(x+3)$, take the graph of $f(x)$, shift it to the left by a distance of 3, stretch vertically by a factor of 2, and then flip over the x-axis.

• The graph of $2f(3x)$ is obtained by shrinking the horizontal coordinate by 3, and stretching the vertical coordinate by 2.

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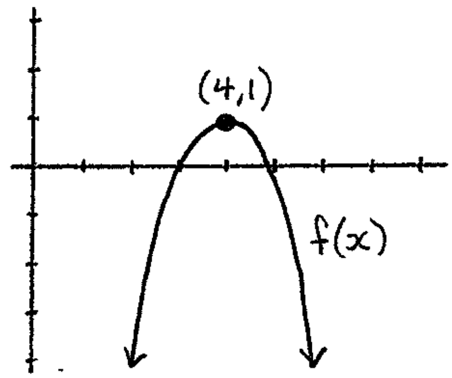
Exercises

For #1-10, suppose $f(x) = x^8$. Match each of the numbered functions on the left with the lettered function on the right that it equals.

1.) $f(x) + 2$	A.) $(-x)^8$
2.) $3f(x)$	B.) $\frac{1}{3}x^8$
3.) $f(-x)$	C.) $x^8 - 2$
4.) $f(x-2)$	D.) $x^8 + 2$
5.) $\frac{1}{3}f(x)$	E.) $(\frac{x}{3})^8$
6.) $f(3x)$	F.) $-x^8$
7.) $f(x) - 2$	G.) $(x-2)^8$
8.) $-f(x)$	H.) $(3x)^8$
9.) $f(x+2)$	I.) $3x^8$
10.) $f(\frac{x}{3})$	J.) $(x+2)^8$

For #11 and #12, suppose $g(x) = \frac{1}{x}$. Match each of the numbered functions on the left with the lettered function on the right that it equals.

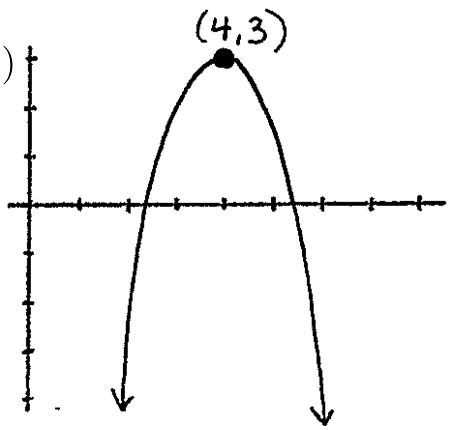
11.) $-4g(3x-7) + 2$	A.) $\frac{6}{-2x+5} - 3$
12.) $6g(-2x+5) - 3$	B.) $\frac{-4}{3x-7} + 2$



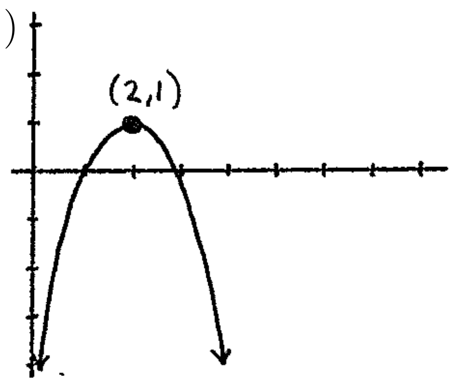
Given the graph of $f(x)$ above, match the following four functions with their graphs.

13.) $f(x)+2$ 14.) $f(x)-2$ 15.) $f(x+2)$ 16.) $f(x-2)$

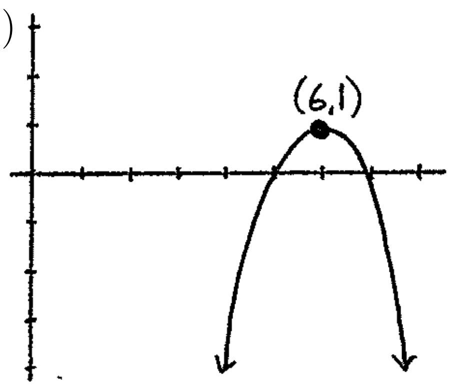
A.)



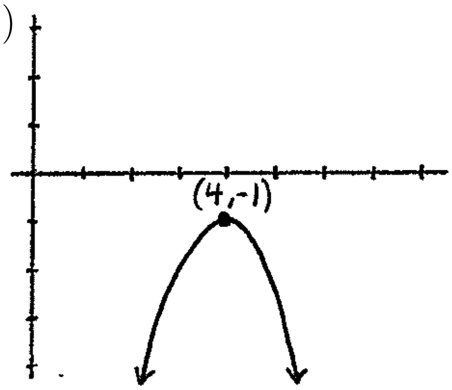
B.)

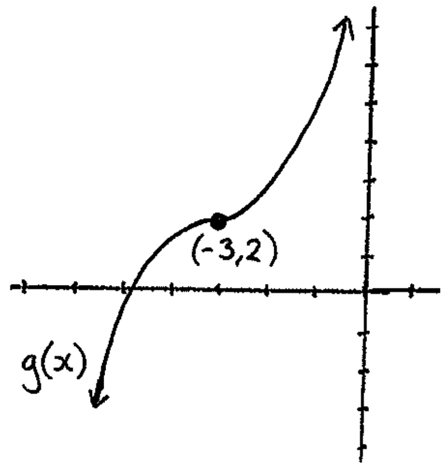


C.)



D.)

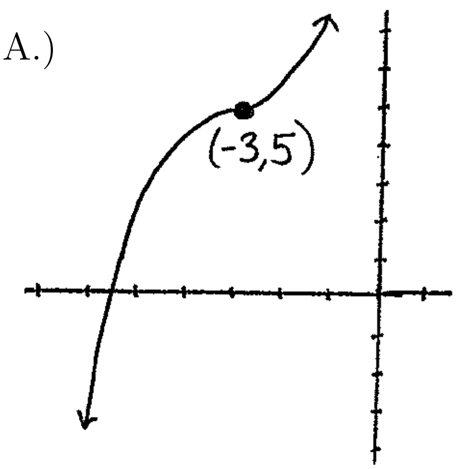




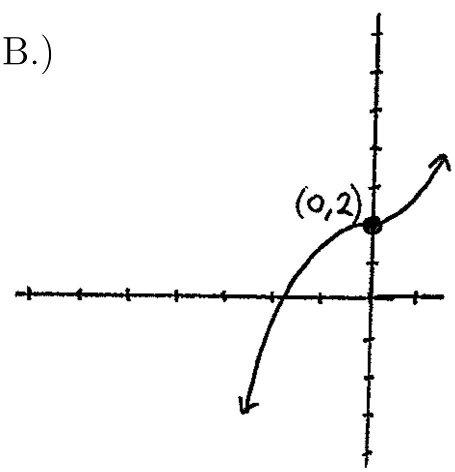
Given the graph of $g(x)$ above, match the following four functions with their graphs.

17.) $g(x) + 3$ 18.) $g(x) - 3$ 19.) $g(x + 3)$ 20.) $g(x - 3)$

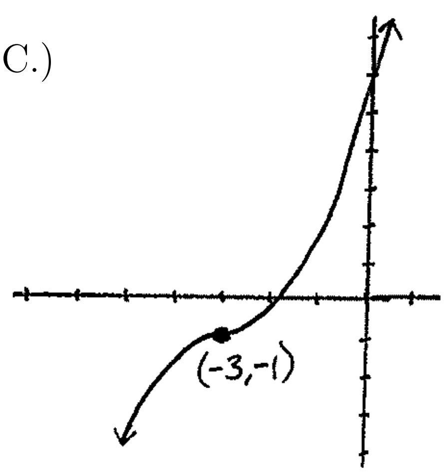
A.)



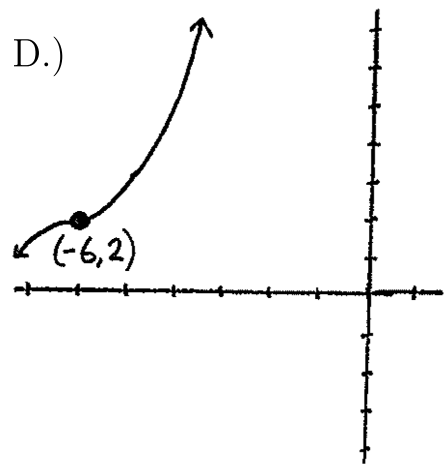
B.)

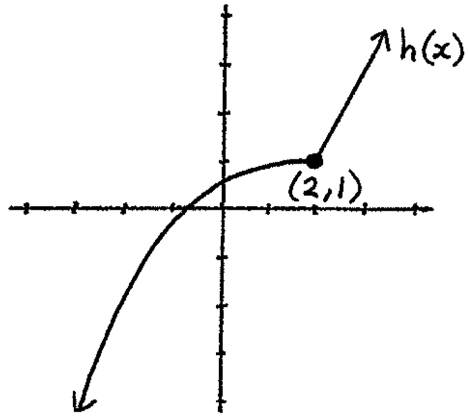


C.)



D.)

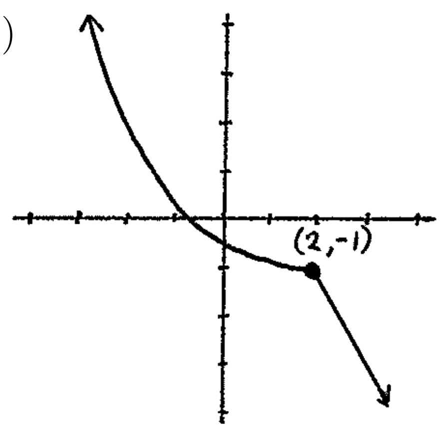




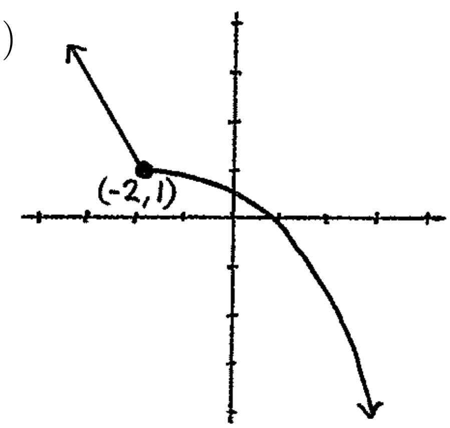
Given the graph of $h(x)$ above, match the following two functions with their graphs.

21.) $-h(x)$ 22.) $h(-x)$

A.)



B.)



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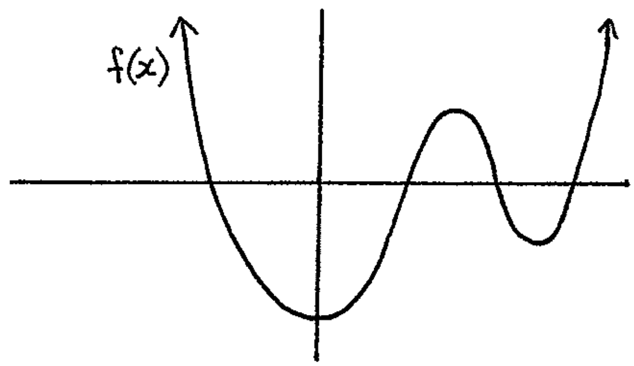
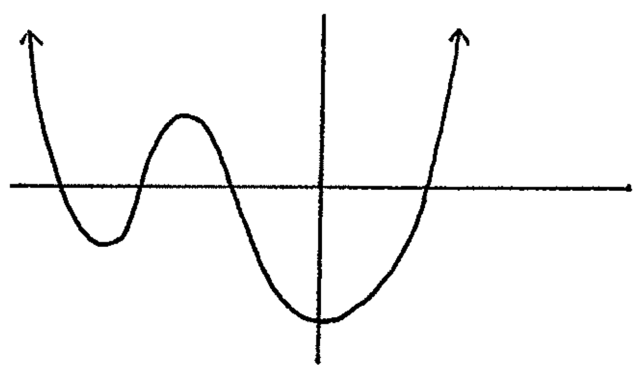


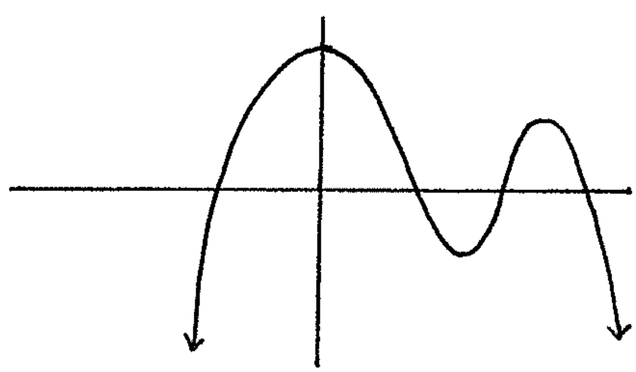
Given the graph of $f(x)$ above, match the following two functions with their graphs.

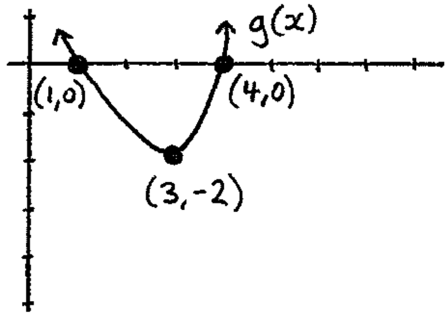
23.) $-f(x)$ 24.) $f(-x)$

A.)



B.)

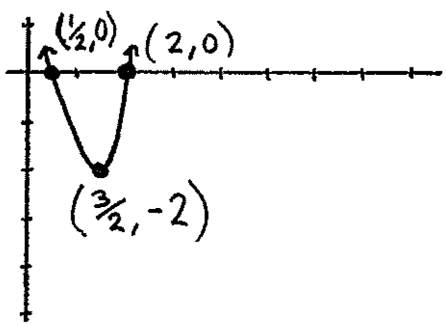




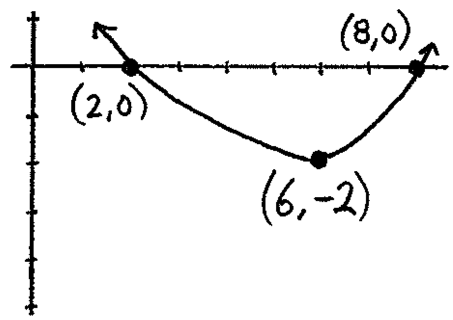
Given the graph of $g(x)$ above, match the following four functions with their graphs.

25.) $2g(x)$ 26.) $\frac{1}{2}g(x)$ 27.) $g(2x)$ 28.) $g(\frac{x}{2})$

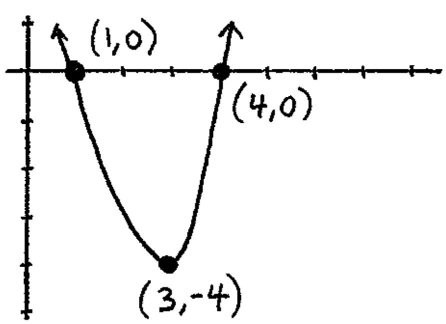
A.)



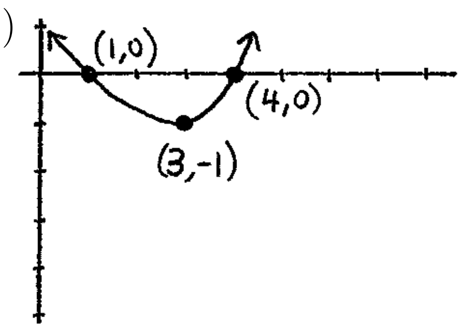
B.)

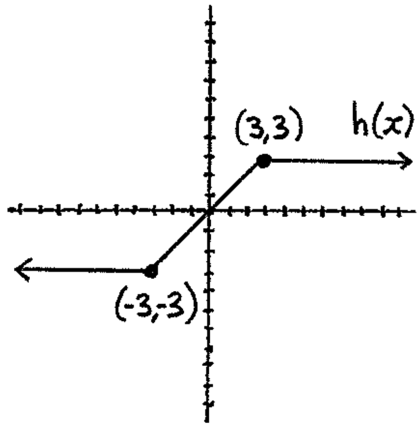


C.)



D.)

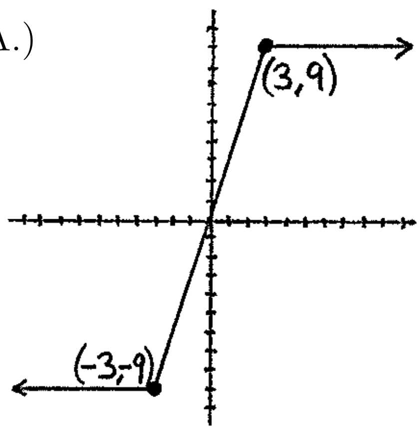




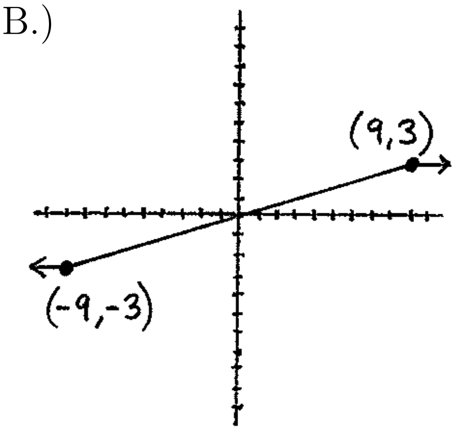
Given the graph of $h(x)$ above, match the following four functions with their graphs.

29.) $3h(x)$ 30.) $\frac{1}{3}h(x)$ 31.) $h(3x)$ 32.) $h(\frac{x}{3})$

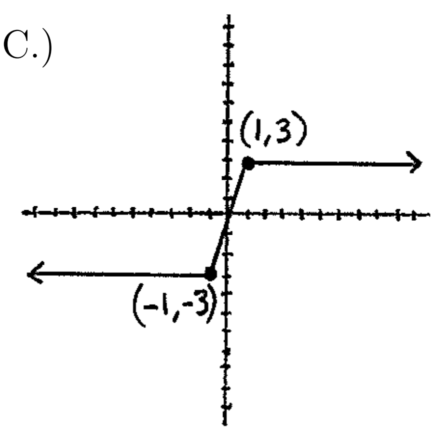
A.)



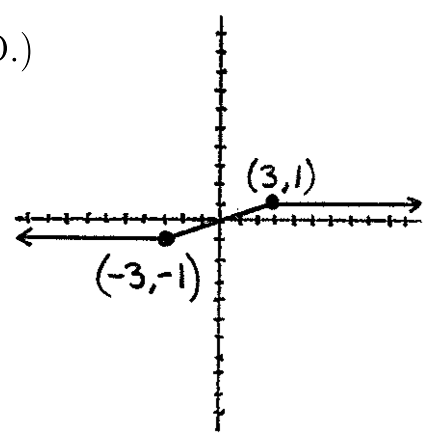
B.)



C.)



D.)



Inverse Functions

One-to-one

Suppose $f : A \to B$ is a function. We call f *one-to-one* if every distinct pair of objects in A is assigned to a distinct pair of objects in B. In other words, each object of the target has at most one object from the domain assigned to it.

There is a way of phrasing the previous definition in a more mathematical language: f is one-to-one if whenever we have two objects $a, c \in A$ with $a \neq c$, we are guaranteed that $f(a) \neq f(c)$.

Example. $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ is not one-to-one because $3 \neq -3$ and yet $f(3) = f(-3)$ since $f(3)$ and $f(-3)$ both equal 9.

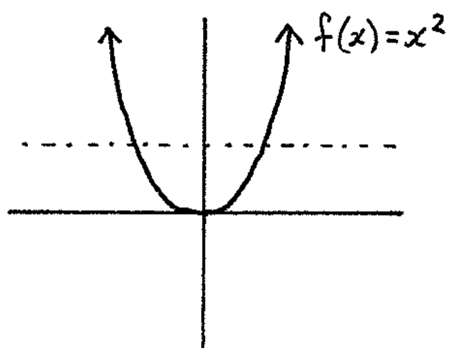
Horizontal line test

If a horizontal line intersects the graph of $f(x)$ in more than one point, then $f(x)$ is not one-to-one.

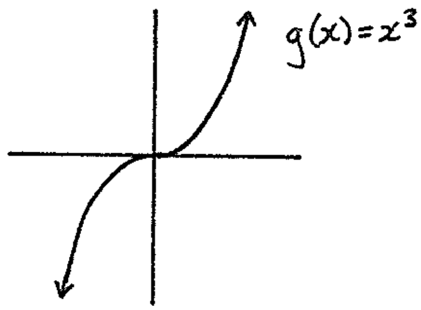
The reason $f(x)$ would not be one-to-one is that the graph would contain two points that have the same second coordinate – for example, $(2, 3)$ and $(4, 3)$. That would mean that $f(2)$ and $f(4)$ both equal 3, and one-to-one functions can't assign two different objects in the domain to the same object of the target.

If *every* horizontal line in $\mathbb{R}^2$ intersects the graph of a function *at most once*, then the function is one-to-one.

Examples. Below is the graph of $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$. There is a horizontal line that intersects this graph in more than one point, so f is not one-to-one.



Below is the graph of $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = x^3$. Any horizontal line that could be drawn would intersect the graph of g in at most one point, so g is one-to-one.

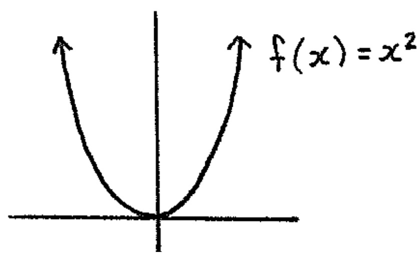


Onto

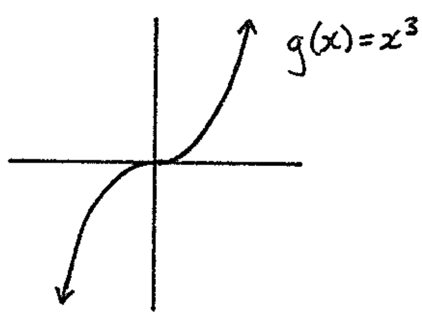
Suppose $f : A \to B$ is a function. We call f *onto* if the range of f equals B.

In other words, f is onto if every object in the target has at least one object from the domain assigned to it by f.

Examples. Below is the graph of $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$. Using techniques learned in the chapter "Intro to Graphs", we can see that the range of f is $[0, \infty)$. The target of f is $\mathbb{R}$, and $[0, \infty) \neq \mathbb{R}$ so f is not onto.



Below is the graph of $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = x^3$. The function g has the set $\mathbb{R}$ for its range. This equals the target of g, so g is onto.



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What an inverse function is

Suppose $f : A \to B$ is a function. A function $g : B \to A$ is called the *inverse function* of f if $f \circ g = id$ and $g \circ f = id$.

If g is the inverse function of f, then we often rename g as f^{-1}.

Examples.

- Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x + 3$, and let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = x - 3$. Then

$$f \circ g(x) = f(g(x)) = f(x-3) = (x-3) + 3 = x$$

Because $f \circ g(x) = x$ and $id(x) = x$, these are the same function. In symbols, $f \circ g = id$.

Similarly

$$g \circ f(x) = g(f(x)) = g(x+3) = (x+3) - 3 = x$$

so $g \circ f = id$. Therefore, g is the inverse function of f, so we can rename g as f^{-1}, which means that $f^{-1}(x) = x - 3$.

- Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 2x + 2$, and let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = \frac{1}{2}x - 1$. Then

$$f \circ g(x) = f(g(x)) = f\Big(\frac{1}{2}x - 1\Big) = 2\Big(\frac{1}{2}x - 1\Big) + 2 = x$$

Similarly

$$g \circ f(x) = g(f(x)) = g(2x+2) = \frac{1}{2}\Big(2x + 2\Big) - 1 = x$$

Therefore, g is the inverse function of f, which means that $f^{-1}(x) = \frac{1}{2}x - 1$.

The Inverse of an inverse is the original

If f^{-1} is the inverse of f, then $f^{-1} \circ f = id$ and $f \circ f^{-1} = id$. We can see from the definition of inverse functions above, that f is the inverse of f^{-1}. That is $(f^{-1})^{-1} = f$.

Inverse functions "reverse the assignment"

The definition of an inverse function is given above, but the essence of an inverse function is that it reverses the assignment dictated by the original function. If f assigns a to b, then f^{-1} will assign b to a. Here's why:

If $f(a) = b$, then we can apply f^{-1} to both sides of the equation to obtain the new equation $f^{-1}(f(a)) = f^{-1}(b)$. The left side of the previous equation involves function composition, $f^{-1}(f(a)) = f^{-1} \circ f(a)$, and $f^{-1} \circ f = id$, so we are left with $f^{-1}(b) = id(a) = a$.

The above paragraph can be summarized as "If $f(a) = b$, then $f^{-1}(b) = a$."

Examples.

- If $f(3) = 4$, then $3 = f^{-1}(4)$.
- If $f(-2) = 16$, then $-2 = f^{-1}(16)$.
- If $f(x+7) = -1$, then $x + 7 = f^{-1}(-1)$.
- If $f^{-1}(0) = -4$, then $0 = f(-4)$.
- If $f^{-1}(x^2 - 3x + 5) = 3$, then $x^2 - 3x + 5 = f(3)$.

In the 5 examples above, we "erased" a function from the left side of the equation by applying its inverse function to the right side of the equation.

When a function has an inverse

A function has an inverse exactly when it is *both* one-to-one and onto. This will be explained in more detail during lecture.

* * * * * * * * * * * * *

Using inverse functions

Inverse functions are useful in that they allow you to "undo" a function. Below are some rather abstract (though important) examples. As the semester continues, we'll see some more concrete examples.

Examples.

- Suppose there is an object in the domain of a function f, and that this object is named a. Suppose that you know $f(a) = 15$.

If f has an inverse function, f^{-1}, and you happen to know that $f^{-1}(15) = 3$, then you can solve for a as follows: $f(a) = 15$ implies that $a = f^{-1}(15)$. Thus, $a = 3$.

- If b is an object of the domain of g, g has an inverse, $g(b) = 6$, and $g^{-1}(6) = -2$, then

$$b = g^{-1}(6) = -2$$

- Suppose $f(x + 3) = 2$. If f has an inverse, and $f^{-1}(2) = 7$, then

$$x + 3 = f^{-1}(2) = 7$$

so

$$x = 7 - 3 = 4$$

* * * * * * * * * * * * *

The Graph of an inverse

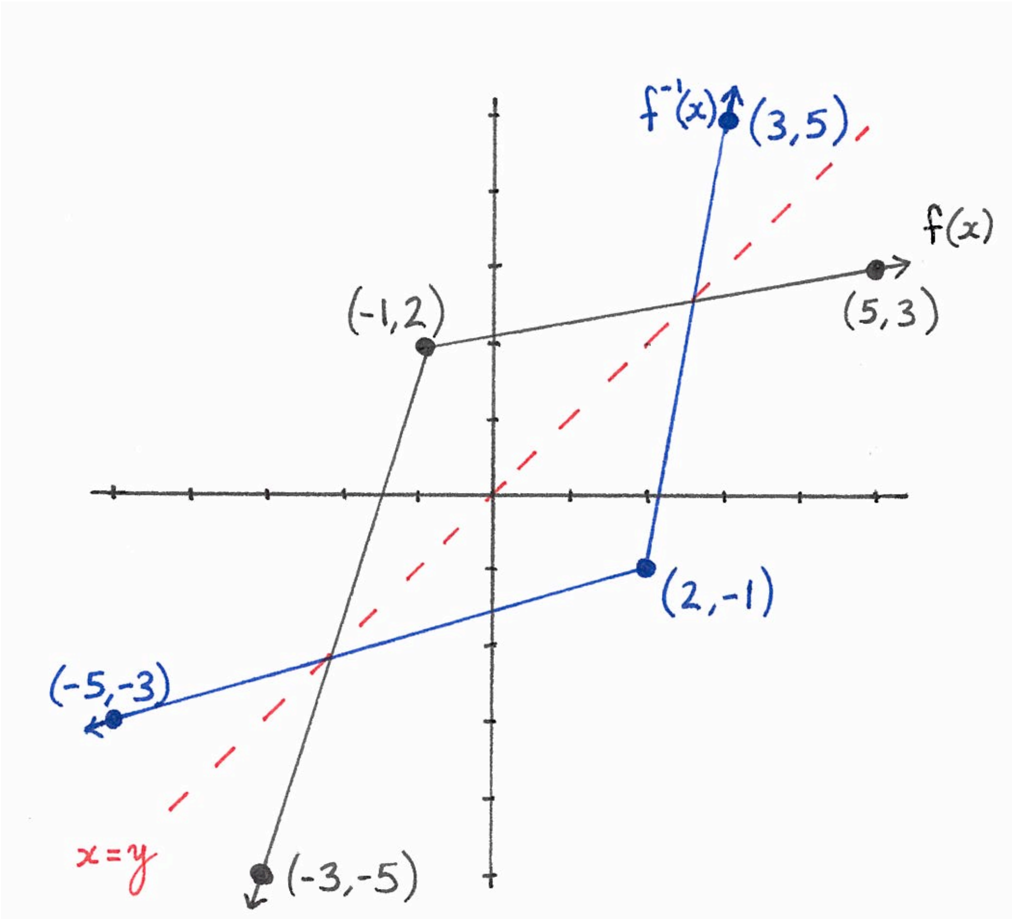
If f is an *invertible* function (that means if f has an inverse function), and if you know what the graph of f looks like, then you can draw the graph of f^{-1}.

If (a, b) is a point in the graph of $f(x)$, then $f(a) = b$. Hence, $f^{-1}(b) = a$. That means f^{-1} assigns b to a, so (b, a) is a point in the graph of $f^{-1}(x)$.

Geometrically, if you switch all the first and second coordinates of points in $\mathbb{R}^2$, the result is to flip $\mathbb{R}^2$ over the "$x = y$ line".

New function	How points in graph of $f(x)$ become points of new graph	visual effect
$f^{-1}(x)$	$(a,b) \mapsto (b,a)$	flip over the "$x = y$ line"

Example.



*   *   *   *   *   *   *   *   *   *   *   *   *

How to find an inverse

If you know that f is an invertible function, and you have an equation for $f(x)$, then you can find the equation for f^{-1} in three steps.

Step 1 is to replace $f(x)$ with the letter y.

Step 2 is to use algebra to solve for x.

Step 3 is to replace x with $f^{-1}(y)$.

After using these three steps, you'll have an equation for the function $f^{-1}(y)$.

Examples.

- Find the inverse of $f(x) = x + 5$.

Step 1. $y = x + 5$

Step 2. $x = y - 5$

Step 3. $f^{-1}(y) = y - 5$

- Find the inverse of $g(x) = \frac{2x}{x-1}$.

Step 1. $y = \frac{2x}{x-1}$

Step 2. $x = \frac{y}{y-2}$

Step 3. $g^{-1}(y) = \frac{y}{y-2}$

Make sure that you are comfortable with the algebra required to carry out step 2 in the above problem. You will be expected to perform similar algebra on future exams.

You should also be able to check that $g \circ g^{-1} = id$ and that $g^{-1} \circ g = id$.

Exercises

In #1-6, g is an invertible function.

1.) If $g(2) = 3$, what is $g^{-1}(3)$?

2.) If $g(7) = -2$, what is $g^{-1}(-2)$?

3.) If $g(-10) = 5$, what is $g^{-1}(5)$?

4.) If $g^{-1}(6) = 8$, what is $g(8)$?

5.) If $g^{-1}(0) = 9$, what is $g(9)$?

6.) If $g^{-1}(4) = 13$, what is $g(13)$?

For #7-12, solve for x. Use that f is an invertible function and that

$$\begin{aligned} f^{-1}(1) &= -2 \\ f^{-1}(2) &= 3 \\ f^{-1}(3) &= 2 \\ f^{-1}(4) &= 5 \\ f^{-1}(5) &= -7 \\ f^{-1}(6) &= 8 \\ f^{-1}(7) &= -3 \\ f^{-1}(8) &= 1 \\ f^{-1}(9) &= 4 \end{aligned}$$

7.) $f(x+2) = 5$

8.) $f(3x-4) = 3$

9.) $f(-5x) = 1$

10.) $f(-2-x) = 2$

11.) $f(\frac{1}{x}) = 8$

12.) $f(\frac{5}{x-1}) = 3$

Each of the functions given in #13-18 is invertible. Find the equations for their inverse functions.

13.) $f(x) = 3x + 2$

14.) $g(x) = -x + 5$

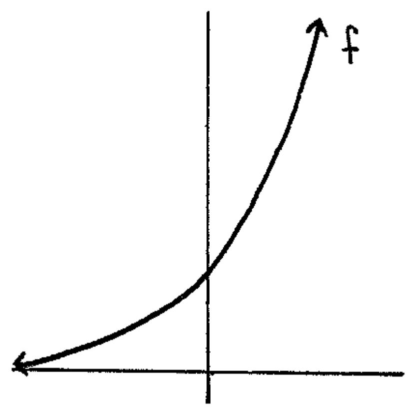
15.) $h(x) = \frac{1}{x}$

16.) $f(x) = \frac{x}{x-1}$

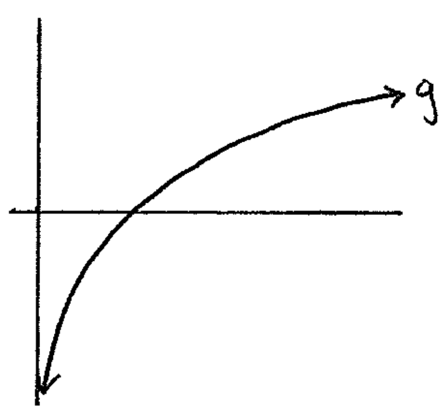
17.) $g(x) = \frac{2x+3}{x}$

18.) $h(x) = \frac{x}{4-x}$

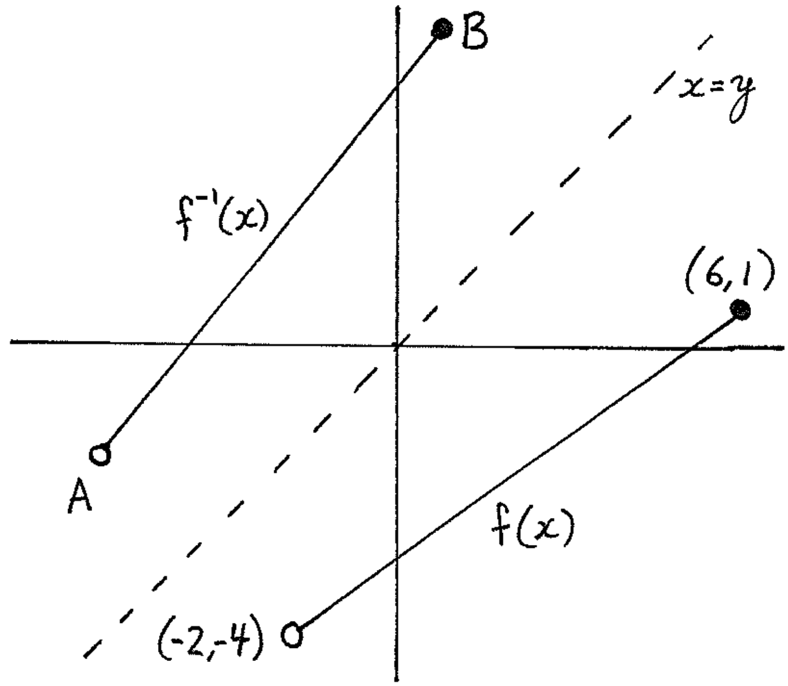
19.) Below is the graph of $f : \mathbb{R} \to (0, \infty)$. Does f have an inverse?



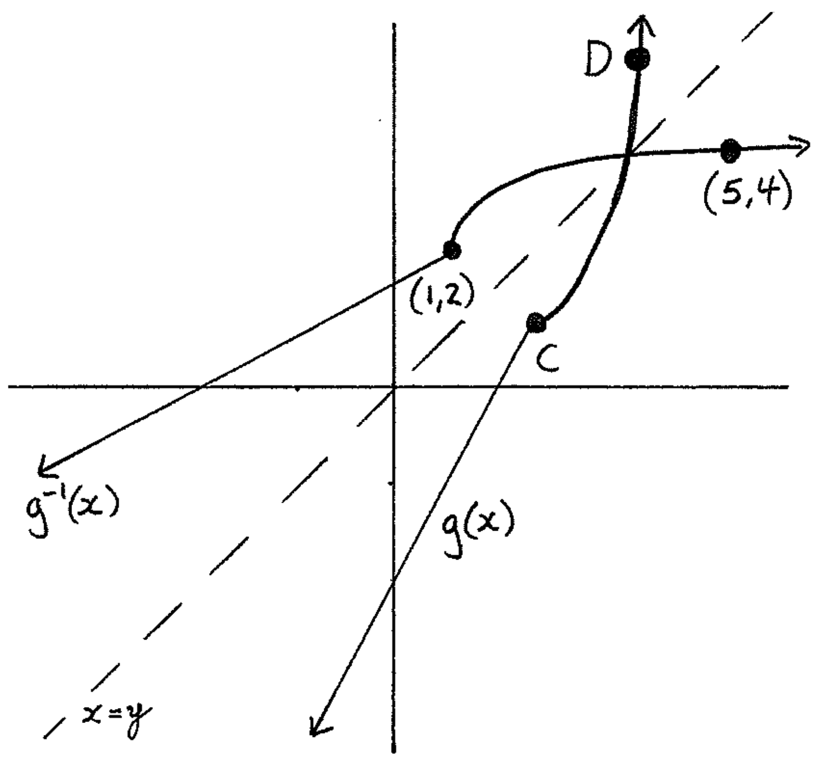
20.) Below is the graph of $g : (0, \infty) \to \mathbb{R}$. Does g have an inverse?



21.) Below are the graphs of $f(x)$ and $f^{-1}(x)$. What are the coordinates of the points A and B on the graph of $f^{-1}(x)$?



22.) Below are the graphs of $g(x)$ and $g^{-1}(x)$. What are the coordinates of the points C and D on the graph of $g(x)$?



n-th Roots

Cube roots

Suppose $g : \mathbb{R} \to \mathbb{R}$ is the cubing function $g(x) = x^3$.

We saw in the previous chapter that g is one-to-one and onto. Therefore, g has an inverse function.

The inverse of g is named the *cube root*, and it's written as $\sqrt[3]{\ }$. In other words, $g^{-1} : \mathbb{R} \to \mathbb{R}$ is the function $g^{-1}(x) = \sqrt[3]{x}$. The definition of inverse functions says that $\sqrt[3]{x^3} = x$ and $(\sqrt[3]{x})^3 = x$.

Inverse functions work backwards of each other:

$$4^3 = 64 \qquad \sqrt[3]{64} = 4$$

$$3^3 = 27 \qquad \sqrt[3]{27} = 3$$

$$2^3 = 8 \qquad \sqrt[3]{8} = 2$$

$$1^3 = 1 \qquad \sqrt[3]{1} = 1$$

$$0^3 = 0 \qquad \sqrt[3]{0} = 0$$

$$(-1)^3 = -1 \qquad \sqrt[3]{-1} = -1$$

$$(-2)^3 = -8 \qquad \sqrt[3]{-8} = -2$$

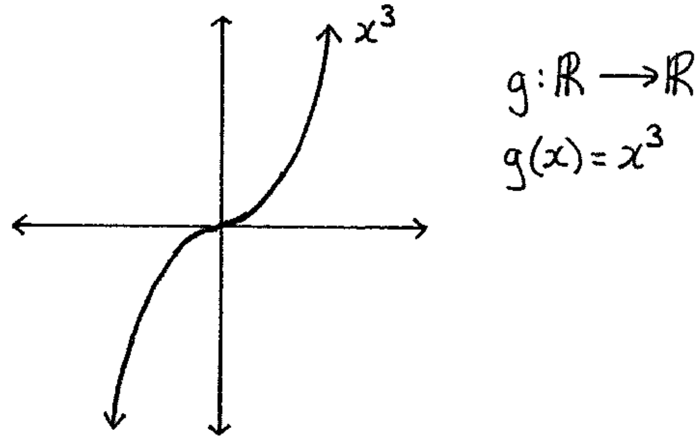
$$(-3)^3 = -27 \qquad \sqrt[3]{-27} = -3$$

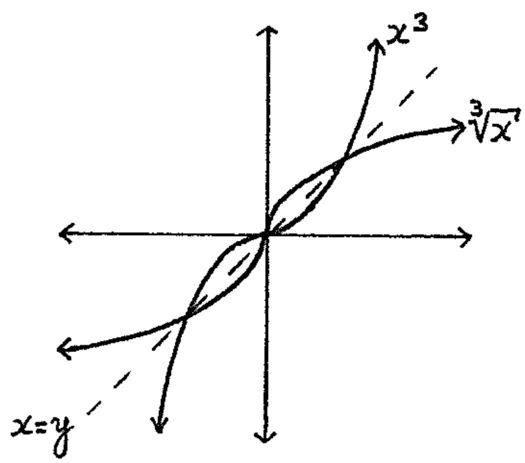
$$(-4)^3 = -64 \qquad \sqrt[3]{-64} = -4$$

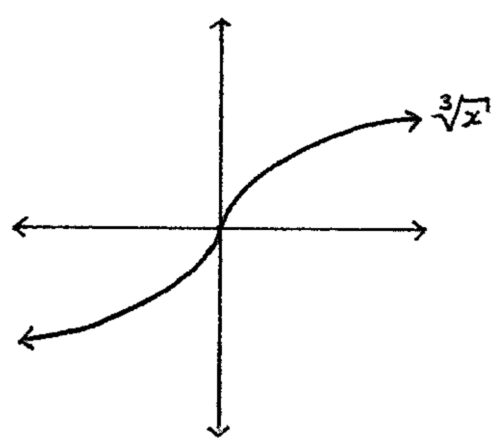
Notice that the domain of the cube root is $\mathbb{R}$. That means you can take the cube root of *any* real number.

To graph $\sqrt[3]{\ }$, first graph x^3, and then flip the graph over the $x = y$ line as was described in the "Inverse Functions" chapter. The graph is drawn on the next page.

Graph of $\sqrt[3]{}$







Square roots

Let $f : \mathbb{R} \to \mathbb{R}$ be the squaring function $f(x) = x^2$.

We saw in the previous chapter that f is neither one-to-one nor onto, so it has no inverse. But, there is a way to change the domain and the target of the squaring function in such a way that squaring becomes both one-to-one and onto.

If $h : [0, \infty) \to [0, \infty)$ is the squaring function $h(x) = x^2$, then we can check that the graph of h passes the horizontal line test and that the range of h is the same as its target, $[0, \infty)$. (The graph of h is drawn on the next page.) Therefore, h is one-to-one and onto and thus h has an inverse function, which is called the *square root* and is written as $h^{-1}(x) = \sqrt[2]{x}$.

Notice that the domain of $\sqrt[2]{\ }$ is $[0, \infty)$, and not $\mathbb{R}$. That means we can't square root a negative number. We cannot, under any circumstances, take the square root of a negative number.

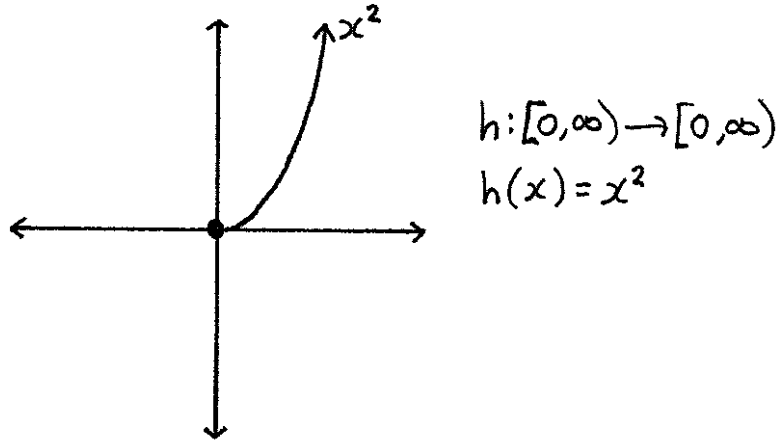
$0^2 = 0$	$\sqrt[2]{0} = 0$
$1^2 = 1$	$\sqrt[2]{1} = 1$
$2^2 = 4$	$\sqrt[2]{4} = 2$
$3^2 = 9$	$\sqrt[2]{9} = 3$
$4^2 = 16$	$\sqrt[2]{16} = 4$
$5^2 = 25$	$\sqrt[2]{25} = 5$
$6^2 = 36$	$\sqrt[2]{36} = 6$

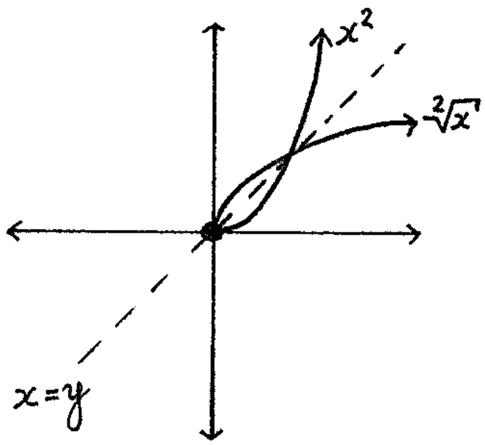
The graph of $\sqrt[2]{\ }$ is drawn on the next page.

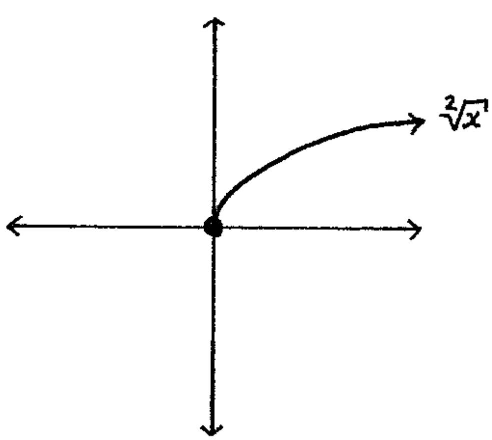
Common shorthand

Often people will write $\sqrt{x}$ to mean $\sqrt[2]{x}$. Be careful, $\sqrt{x}$ can never be used as a shorthand for $\sqrt[3]{x}$.

Graph of $\sqrt[2]{}$







n-th roots

If $n \in \mathbb{N}$ and $n \geq 2$, then x^n describes a function.

The "odd exponent" functions x^3, x^5, x^7, x^9, ... are all different functions, but they behave similarly, and their graphs are similar. As a result of this similarity, if n is odd then x^n has in inverse function named $\sqrt[n]{} : \mathbb{R} \to \mathbb{R}$. In particular, the domain of $\sqrt[n]{}$ is $\mathbb{R}$ whenever n is odd, so we can can take an "odd root" of any real number, even a negative number.

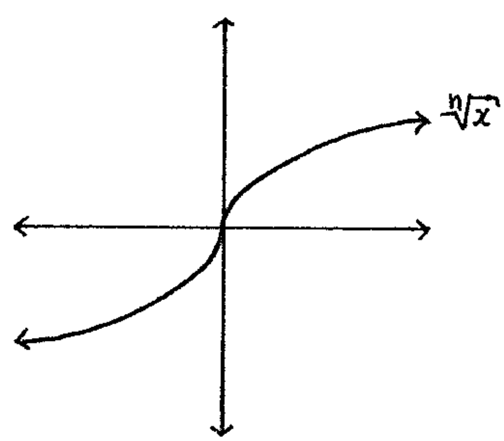
On the other hand, the "even exponent" functions x^4, x^6, x^8, x^{10}, ... all behave like x^2. If n is even, then $\sqrt[n]{} : [0, \infty) \to [0, \infty)$ is the inverse of x^n. That means that you can't ever put a negative number into an even root function, and negative numbers never come out of even root functions.

Once more, you can take an even root of any positive number. You can take an even root of the number 0. But you can never take an even root of a negative number. Ever.

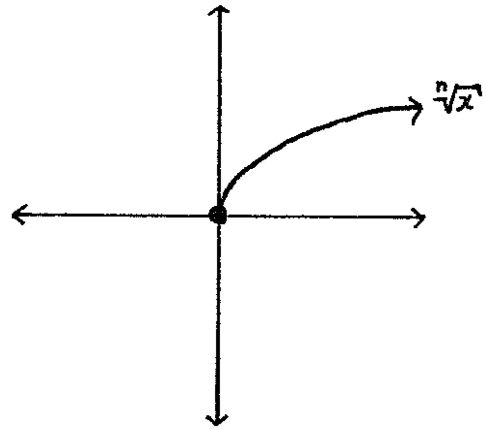
The most important thing to remember about n-th roots is that they are inverses of the functions x^n. That's the content of the following two equations:

$$(\sqrt[n]{x})^n = x \quad \text{and} \quad \sqrt[n]{x^n} = x$$

Graph of $\sqrt[n]{}$ if n is odd ($n \geq 3$)



Graph of $\sqrt[n]{\ }$ if n is even $(n \geq 2)$



Using n-th roots

This section is a special case of the "Using inverse functions" section from the "Inverse Functions" chapter. Also compare to exercises #7-12 from the "Inverse Functions" chapter.

Problem 1. Solve for x where $2(x-5)^3 = 16$.

Solution. First notice the order of the algebra on the left side of the equal sign: In the expression $2(x-5)^3 = 16$, the first thing we do to x is subtract 5. Then we cube, and last we multiply by 2.

We can erase what happened last (multiplication by 2) by applying its inverse (division by 2) to the right side of the equation.

$$(x-5)^3 = \frac{16}{2} = 8$$

Then we can erase the cube by applying its inverse (cube-root) to the right side.

$$x - 5 = \sqrt[3]{8} = 2$$

Then we can erase subtracting 5 by adding 5 to the right side.

$$x = 2 + 5 = 7$$

Problem 2. If $\sqrt[2]{2x} = 6$, what is x?

Solution. Squaring is the inverse of the square root, so $2x = 6^2 = 36$, which means that $x = 18$.

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Inequalities

Here are some rules for inequalities that you have to know.

If $a < b$, then:

$a + d < b + d$	
$a - d < b - d$	
$ca < cb$	if $c > 0$
$cb < ca$	if $c < 0$
$a^n < b^n$	if $0 \leq a < b$
$\sqrt[n]{a} < \sqrt[n]{b}$	if $0 \leq a < b$
$\frac{1}{b} < \frac{1}{a}$	if $0 < a < b$

If $a \leq b$, then:

$a + d \leq b + d$	
$a - d \leq b - d$	
$ca \leq cb$	if $c \geq 0$
$cb \leq ca$	if $c \leq 0$
$a^n \leq b^n$	if $0 \leq a \leq b$
$\sqrt[n]{a} \leq \sqrt[n]{b}$	if $0 \leq a \leq b$
$\frac{1}{b} \leq \frac{1}{a}$	if $0 < a \leq b$

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Implied domains

Problem 1. What is the implied domain of $f(x) = \sqrt[3]{x^2+15}$?

Solution. For any $x \in \mathbb{R}$, x^2 is a real number. Add 15, and then $x^2 + 15$ is a real number. You can take a cube root of any real number, so $\sqrt[3]{x^2+15}$ is a real number.

To recap, any real number that you put into f results in a real number coming out, so the implied domain for f is $\mathbb{R}$.

Problem 2. What is the implied domain of $g(x) = \sqrt[2]{x-2}$?

Solution. We can't take the square root of a negative number. So $g(x)$ only makes sense if the number we are supposed to take the square root of, $x-2$, is positive or 0. That means we need to have that $x - 2 \geq 0$. Therefore, after adding 2 to both sides of the previous inequality, $x \geq 2$.

The implied domain of g – which is all of those numbers that we may safely feed into g – is the set of all x such that $x \geq 2$. This set is $[2, \infty)$.

* * * * * * * * * * * * *

Some algebra rules for n-th roots

If $n \in \mathbb{N}$, $a > 0$, and $b > 0$, then

$$\begin{aligned}(ab)^n &= abababab \cdots abab \\ &= (aaaa \cdots aa)(bbbb \cdots bb) \\ &= a^n b^n\end{aligned}$$

Let's take two other positive numbers: $x > 0$ and $y > 0$. Since $\sqrt[n]{}$ is the inverse of the function x^n, we have $x = (\sqrt[n]{x})^n$ and $y = (\sqrt[n]{y})^n$. Thus, $xy = (\sqrt[n]{x})^n(\sqrt[n]{y})^n$.

If we let $a = (\sqrt[n]{x})$ and $b = (\sqrt[n]{y})$, then the above paragraph showed that $(\sqrt[n]{x})^n(\sqrt[n]{y})^n = (\sqrt[n]{x}\sqrt[n]{y})^n$. So we have that

$$xy = (\sqrt[n]{x})^n(\sqrt[n]{y})^n = (\sqrt[n]{x}\sqrt[n]{y})^n$$

Using that $\sqrt[n]{}$ is an inverse function, the above equation tells us the following equation on the next page.

$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$$

A special case of the rule above is

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

You might be tempted at some point to write that $\sqrt[n]{x+y}$ is the same thing as $\sqrt[n]{x} + \sqrt[n]{y}$, but it is not. For example, $\sqrt[2]{9+16} = \sqrt[2]{25} = 5$, but $\sqrt[2]{9} + \sqrt[2]{16} = 3 + 4 = 7$, and 5 does not equal 7.

$$\sqrt[n]{x+y} \neq \sqrt[n]{x} + \sqrt[n]{y}$$

Simplifying square roots of natural numbers

When writing the square root of a natural number, you'll usually be expected to write a final result that does not include taking the square root of a number that is a square. For example, you should write $\sqrt{4}$ as 2, because $4 = 2^2$ and $\sqrt{2^2} = 2$.

You can use the rule $\sqrt{xy} = \sqrt{x}\sqrt{y}$ to help you remove squares from the inside of a square root. For example, $20 = (2)(2)(5) = 2^2\,5$. Thus,

$$\sqrt{20} = \sqrt{2^2\,5} = \sqrt{2^2}\sqrt{5} = 2\sqrt{5}$$

For one more example, if asked for $\sqrt{360}$, first factor 360 into a product of prime numbers to see that $360 = 2^3\,3^2\,5 = 2^2\,3^2\,(2)(5)$. Then we have

$$\sqrt{360} = \sqrt{2^2}\sqrt{3^2}\sqrt{(2)(5)} = (2)(3)\sqrt{(2)(5)} = 6\sqrt{10}$$

You can be sure that you are done simplifying at this point because 10 written as a product of primes is (2)(5), and this product does not include more than one of the same prime number.

Exercises

1.) What is x if $(x+7)^3 = 8$?

2.) Solve for x where $\sqrt[2]{x+2} = 4$.

3.) If $4(2x+7)^5 = 12$, then what is x?

4.) Find x when $3\sqrt[4]{4-x} = 9$.

5.) What is the inverse function of $f(x) = x^3 + 5$?

6.) What is the inverse function of $g(x) = 4\sqrt[3]{x+7}$?

In #7-13, solve the inequality for x.

7.) $2x - 13 < 4$

8.) $-3x < 16 + x$

9.) $\frac{4}{x} > \frac{1}{9}$

10.) $\sqrt[5]{2x-6} > 2$

11.) $12 \leq -x^3 + 4$

12.) $\sqrt[2]{3x} \geq 1$

13.) $\frac{12}{3-x} \geq 24$

In #14-17, find the implied domains of the given functions.

14.) $f(x) = \sqrt[15]{3x^2 - 14x + 9}$

15.) $g(x) = \sqrt[2]{17-2x}$

16.) $h(x) = 5\sqrt[2]{9x-4}$

17.) $f(x) = 10 - \frac{\sqrt[8]{-2x+4}}{x^2+1}$

Simplify the expression in #18-23.

18.) $\sqrt{27}$

19.) $\sqrt{24}$

20.) $\sqrt{100}$

21.) $\sqrt{52}$

22.) $\sqrt{150}$

23.) $\sqrt{48}$

Graph the functions given in #24-28.

24.) $\sqrt{x-2}$

25.) $-\sqrt{x}$

26.) $\sqrt[3]{x}-1$

27.) $\sqrt{-x}$

28.) $\sqrt[3]{x+1}$

Polynomials

Basics of Polynomials

A *polynomial* is what we call any function that is defined by an equation of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, ...a_1, a_0 \in \mathbb{R}$.

Examples. The following three functions are examples of polynomials.

• $p(x) = -2x^2 - \pi x + \sqrt[2]{2}$ is a polynomial. We could rewrite $p(x)$ as $p(x) = (-2)x^2 + (-\pi)x + (\sqrt[2]{2})$, so $a_2 = -2$, $a_1 = -\pi$, and $a_0 = \sqrt[2]{2}$.

• $p(x) = 3x^4 - \frac{1}{2}x$ is a polynomial. Notice that $p(x) = (3)x^4 + (0)x^3 + (0)x^2 + (-\frac{1}{2})x + (0)$, so $a_4 = 3$, $a_3 = 0$, $a_2 = 0$, $a_1 = -\frac{1}{2}$, and $a_0 = 0$.

• $p(x) = 15$ is a polynomial. Here $a_0 = 15$.

Coefficients, degree, and leading terms

The numbers $a_n, a_{n-1}, ...a_1, a_0 \in \mathbb{R}$ in the definition of a polynomial are called the *coefficients* of the polynomial. A coefficient a_k is called the *degree k coefficient*.

We almost always write a polynomial as $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_n \neq 0$. That means we never really write a polynomial as $0x^2 + 3x$, although it is technically a polynomial. It would just be silly to write $0x^2 + 3x$ instead of $3x$, since they are equal.

If we write a polynomial as $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_n \neq 0$, then n is the *degree* of $p(x)$, a_n is the *leading coefficient* of $p(x)$, and $a_n x^n$ is the *leading term* of $p(x)$.

For example, the leading coefficient of $4x^3 - 5x^2 + 6x - 7$ is 4. Its degree is 3, and its leading term is $4x^3$. Notice that the leading term of $4x^3 - 5x^2 + 6x - 7$ records both the leading coefficient and the degree of $4x^3 - 5x^2 + 6x - 7$.

If the degree of a polynomial is small, there is usually a word to describe it. For example, degree 0 polynomials are called constant polynomials. Degree 1 polynomials are called linear polynomials...

degree	common name	example	leading coefficient	leading term
0	constant	2	2	2
1	linear	$3x-1$	3	$3x$
2	quadratic	x^2+2x-4	1	x^2
3	cubic	$-x^3-x$	-1	$-x^3$
4	quartic	$\frac{1}{2}x^4-x^3+1$	$\frac{1}{2}$	$\frac{1}{2}x^4$
5	quintic	$-3x^5+x^2-12$	-3	$-3x^5$

Watch out for this trick. The leading coefficient of $3+2x^5$ is 2, its leading term is $2x^5$, and its degree is 5. Such a polynomial is most often written as $2x^5+3$, but changing the way it's written does not change its degree, leading coefficient, or leading term.

The degree is the biggest exponent above any of the x's. If the degree of a polynomial equals n, then the leading coefficient is the coefficient in front of x^n, whether it's the first number written on the left of the polynomial or not.

Addition, subtraction, and multiplication

Adding, subtracting, and multiplying polynomials usually boils down to an exercise in using the distributive law.

Examples in addition and subtraction. To add or subtract the polynimials

$$p(x) = 3x^5 - 7x^3 + 4x^2 - 2$$

and

$$q(x) = x^4 - 2x^3 + 3x^2 - 5$$

you just have to pair up the coefficients whose degree is the same, and then add or subtract each pair.

Thus, $p(x) + q(x)$ equals

$$\begin{aligned}
&\Big[\quad (3)x^5 + \quad (0)x^4 + \quad (-7)x^3 + \quad (4)x^2 + \quad (0)x + \quad (-2)\Big] \\
+&\Big[\quad (0)x^5 + \quad (1)x^4 + \quad (-2)x^3 + \quad (3)x^2 + \quad (0)x + \quad (-5)\Big]
\end{aligned}$$

$$=(3+0)x^5 + (0+1)x^4 + (-7+(-2))x^3 + (4+3)x^2 + (0+0)x + (-2+(-5))$$

$$=3x^5 + x^4 - 9x^3 + 7x^2 - 7$$

and $p(x) - q(x)$ equals

$$\begin{aligned}
&\Big[\quad (3)x^5 + \quad\quad + \quad (-7)x^3 + \quad (4)x^2 + \quad\quad + \quad (-2)\Big] \\
-&\Big[\quad\quad\quad (1)x^4 + \quad (-2)x^3 + \quad (3)x^2 + \quad\quad + \quad (-5)\Big]
\end{aligned}$$

$$=(3-0)x^5 + (0-1)x^4 + (-7-(-2))x^3 + (4-3)x^2 + \quad\quad +(-2-(-5))$$

$$=3x^5 - x^4 - 5x^3 + x^2 + 3$$

Below are three examples of polynomial multiplication.

- $2(x-4) = 2x - 2(4) = 2x - 8$

- $$\begin{aligned} 4x^2(x^3 + 7x - 2) &= 4x^2x^3 + 4x^2 7x - 4x^2 2 \\ &= 4x^5 + 28x^3 - 8x^2 \end{aligned}$$

- $$\begin{aligned} (3x^2 + 4x)(x^4 - 2x^3 + 5) &= (3x^2 + 4x)x^4 - (3x^2 + 4x)2x^3 + (3x^2 + 4x)5 \\ &= 3x^2x^4 + 4xx^4 - 3x^2 2x^3 + 4x2x^3 + 3x^2 5 + 4x5 \\ &= 3x^6 + 4x^5 - 6x^5 + 8x^4 + 15x^2 + 20x \\ &= 3x^6 - 2x^5 + 8x^4 + 15x^2 + 20x \end{aligned}$$

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Leading term of a product is the product of leading terms

Notice that in each of the three examples above, the leading term of the product is the product of the leading terms. That is, the leading term of $2(x-4)$ is the product of 2 and x. The leading term of $4x^2(x^3+7x-2)$ is $4x^5 = (4x^2)(x^3)$, and the leading term of $(3x^2+4x)(x^4-2x^3+5)$ is $3x^6 = (3x^2)(x^4)$.

These examples illustrate an important feature of polynomial multiplication: If you multiply some polynomials together, no matter how many polynomials, you can find the leading term of the resulting product by multiplying together the leading terms of the polynomials that you started with.

Examples.

- The leading term of $2x^2-5x$ is $2x^2$. The leading term of $-7x+4$ is $-7x$. So the leading term of $(2x^2-5x)(-7x+4)$ will be $(2x^2)(-7x) = -14x^3$.
- The leading term of $5(x-2)(x+3)(x^2+3x-7)$ will be $(5)(x)(x)(x^2) = 5x^4$.
- The leading term of $(2x^3-7)(x^5-3x+5)(x-1)(5x^7+6x-9)$ equals $(2x^3)(x^5)(x)(5x^7) = 10x^{16}$.

* * * * * * * * * * * * *

Exercises

For #1-5, find the degree, leading coefficient, and leading term of the given polynomials.

1.) $2x^3 - x^2 + 7x - 4$

2.) $-x^2 + \pi x$

3.) $x - 7$

4.) $23x^5 - 100 + 3x^{17}$

5.) $-4x + 2$

For #6-16, add, subtract, or multiply the polynomials as indicated in the problem.

6.) $(2x + 3) + (-x + 5)$

7.) $(3x^2 - x + 6) - (3x^2 + x - 6)$

8.) $(8x^2 - 5x - 2) + (4x^5 - x^2 + 3x + 7)$

9.) $(7x^{100} + x - 3) - (7x^{100} + 17x + 10)$

10.) $(-x^2 + 4x + 2) - (5x^4 - x^2 + 2x - 8)$

11.) $3x^2(x + 7)$

12.) $-5x(2x - 3)$

13.) $6x^2(3x + 1)$

14.) $2x(x^3 + 4x - 6)$

15.) $(x^2 + 6)(x - 5)$

16.) $(5x^3 + 8)(x^2 + 2x - 1)$

For #17-25, determine the leading term of the given polynomial.

17.) $(x - 2)(x - 7)$

18.) $(x^2 + x + 1)(5x^3 - 4x^2 - x + 9)$

19.) $3(7x^4 - x^3 + 5x^2 - 13x + 3)(4x^5 - 6x^2 - 5x)$

20.) $2(x + 1)$

21.) $-5(x + 4)(x - 5)$

22.) $8(x - 3)(x - 5)(x - 6)(x - 9)$

23.) $-3(x + 3)(x^2 - 4x + 2)$

24.) $(x + 1)(x + 1)(x + 1)(x - 4)(x^2 - 7)(x^2 + 2x - 3)$

25.) $5(x - 3)(x - 5)(x - 6)(x^2 + 1)(x^2 + 2x - 7)$

26.) Suppose $p_1(x)$, $p_2(x)$, ..., $p_k(x)$ are polynomials and that none of them equal 0. Assume that n_1 is the degree of $p_1(x)$, and that n_2 is the degree of $p_2(x)$, and so on. What is the degree of the product $p_1(x)p_2(x)\cdots p_k(x)$?

Division

We saw in the last chapter that if you add two polynomials, the result is a polynomial. If you subtract two polynomials, you get a polynomial. And the product of two polynomials is a polynomial.

Division doesn't work as well. Sometimes when we divide polynomials the result is a polynomial as is the case for

$$\frac{x^2+x}{x} = \frac{x(x+1)}{x} = x+1$$

but more often than not, when we didive two polynomials the result is not a polynomial. For example, $\frac{1}{x}$ is not a polynomial even though 1 and x are polynomials.

Dividing by constant polynomials

Dividing a polynomial by a constant – or degree 0 – polynomial turns out to be the same as multiplying a polynomial by a constant:

$$\frac{4x^2+x-8}{2} = \frac{1}{2}(4x^2+x-8)$$

$$= \frac{1}{2}4x^2 + \frac{1}{2}x - \frac{1}{2}8$$

$$= 2x^2 + \frac{1}{2}x - 4$$

Division becomes more complicated when we divide by polynomials whose degree is greater than 0.

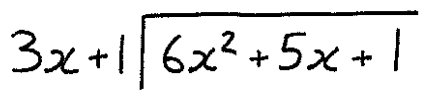
Dividing by non-constant polynomials

The best way to learn polynomial division is to work through a few examples. We'll take a look at a couple of examples here, and we'll work out some examples in class as well.

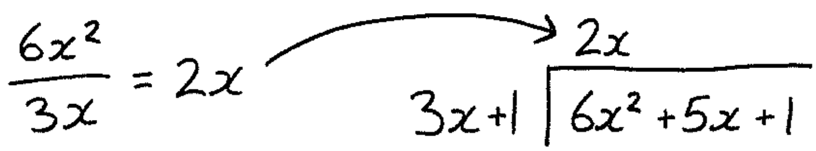
Before we begin, if $p(x)$ and $q(x)$ are polynomials, then $p(x)$ is called the *numerator* of the fraction $\frac{p(x)}{q(x)}$, and $q(x)$ is the *denominator*.

$$\textbf{Example 1:} \quad \frac{6x^2 + 5x + 1}{3x + 1}$$

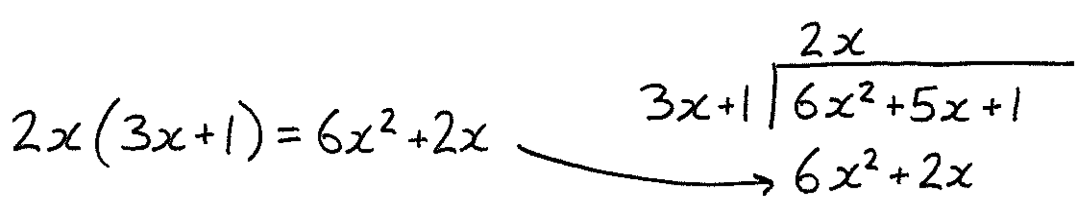
Step 1. Write a division sign. Write the denominator to the left of the division sign. Write the numerator inside of the division sign.



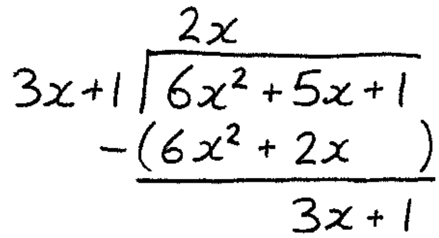
Step 2. Divide the leading term of the numerator by the leading term of the denominator and write the answer on top of the division sign.



Step 3. Multiply the denominator by what you just wrote on top of the division sign. Write this product below the numerator.

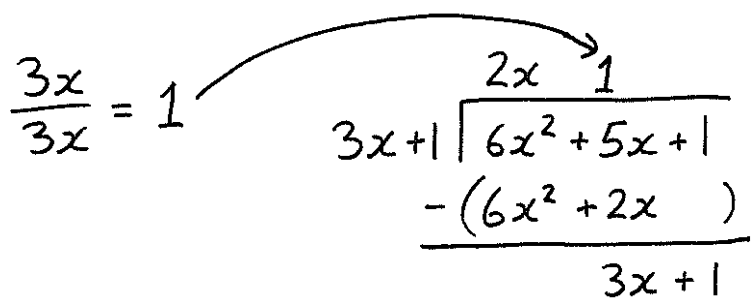


Step 4. Subtract what you just wrote below the numerator from the numerator. Write the answer below.

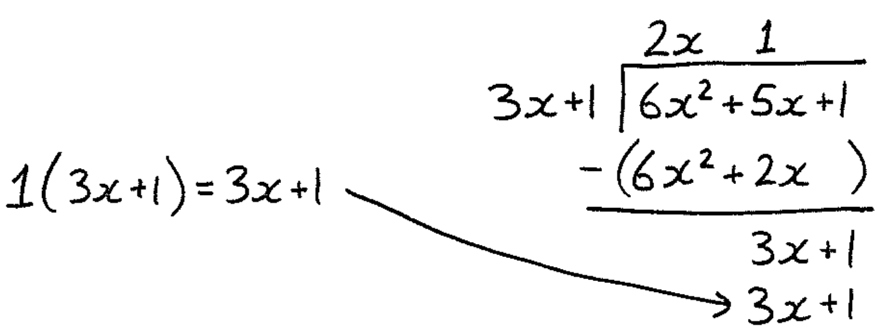


Repeat Steps 2, 3, and 4 with the same denominator, but this time use the difference you found in Step 4 as your *new numerator*.

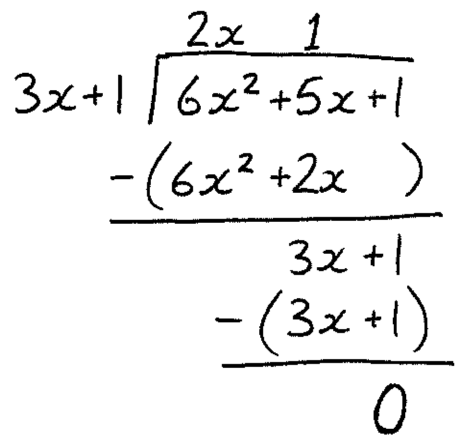
Step 2. Divide the leading term of the new numerator by the leading term of the denominator and write the answer on top of the division sign.



Step 3. Multiply the denominator by what you just wrote on top of the division sign. Write this product below the new numerator.



Step 4. Subtract what you just wrote below the new numerator from the new numerator. Write the answer below.



The division process has ended because we ended Step 4 by writing 0. That means that $\frac{6x^2+5x+1}{3x+1}$ has no remainder. The solution is found by adding together all of the terms that were written on top of the division sign. That is,

$$\frac{6x^2 + 5x + 1}{3x + 1} = 2x + 1$$

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Example 2: $$\frac{10x^4 - 4x^3 + 5x - 4}{2x^3 - 3x}$$

There are two added wrinkles in this example that did not appear in the first example: In this example our division will have a remainder, and we will have to leave spaces where terms of a polynomial are missing – that last part should make sense soon.

Step 1. Write a division sign. Write the denominator to the left of the division sign. Write the numerator inside of the division sign, but in writing the numerator, leave a space wherever a term is "missing".

At this step, our numerator is $10x^4 - 4x^3 + 5x - 4$. There is no x^2 term in $10x^4 - 4x^3 + 5x - 4$, so we'll leave a space where it would have been, between the x^3 and x terms.

$$2x^3-3x\,\overline{\big)\,10x^4-4x^3 \qquad\quad +5x-4}$$

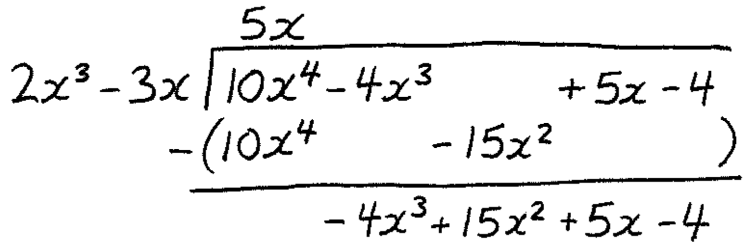
Step 2. Divide the leading term of the numerator by the leading term of the denominator and write the answer on top of the division sign.

$$\frac{10x^4}{2x^3}=5x \longrightarrow \begin{array}{rl} & 5x \\ 2x^3-3x & \overline{\big)\,10x^4-4x^3 \qquad\quad +5x-4} \end{array}$$

Step 3. Multiply the denominator by what you just wrote on top of the division sign. Write this product below the numerator, again, leaving a space wherever a term is missing.

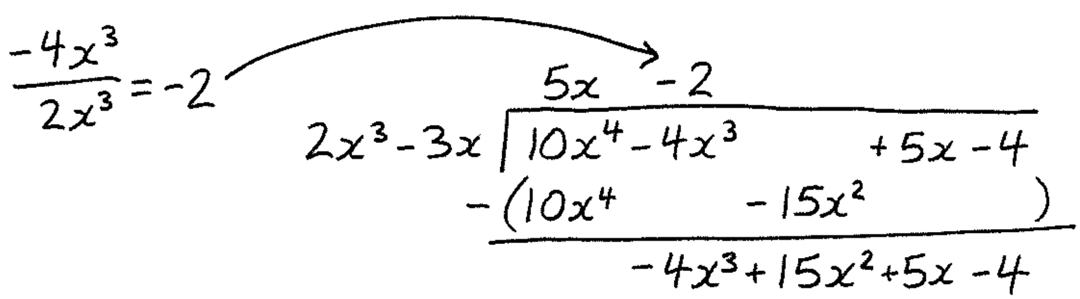
$$5x(2x^3-3x)=10x^4-15x^2 \longrightarrow \begin{array}{rl} & 5x \\ 2x^3-3x & \overline{\big)\,10x^4-4x^3 \qquad\quad +5x-4} \\ & \;\;\,10x^4 \qquad\;\; -15x^2 \end{array}$$

Step 4. Subtract what you just wrote below the numerator from the numerator. Write the answer below.

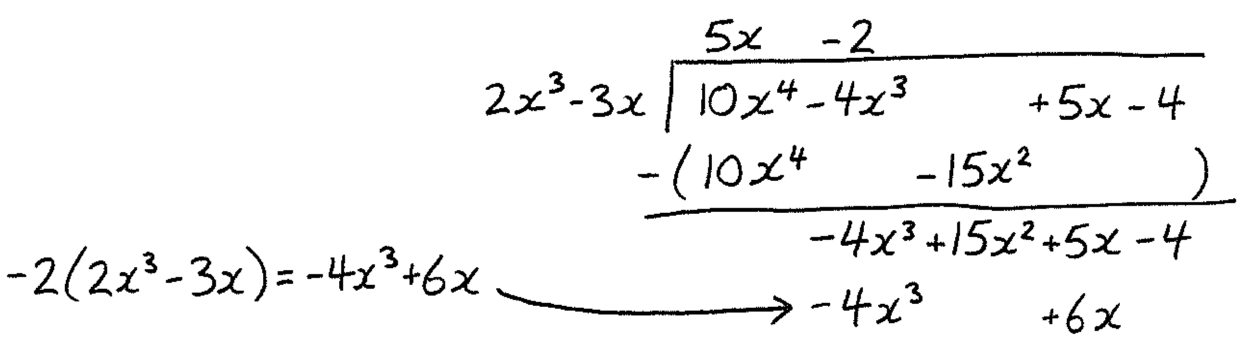


Repeat Steps 2, 3, and 4 with the same denominator, but this time use the difference you found in Step 4 as your *new numerator*.

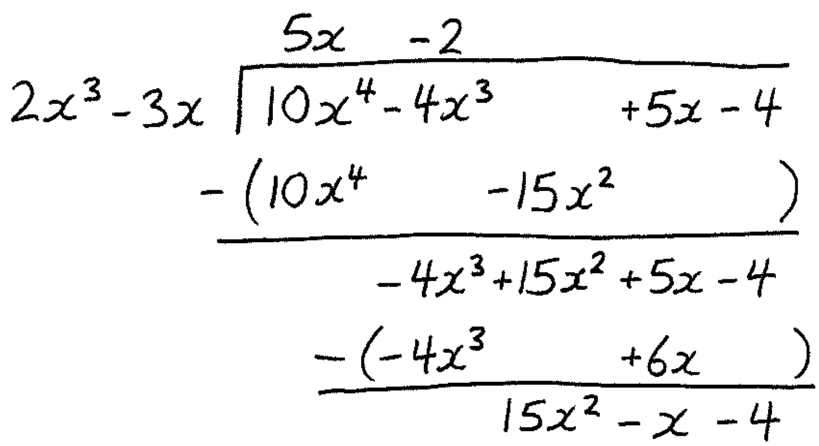
Step 2. Divide the leading term of the new numerator by the leading term of the denominator and write the answer on top of the division sign.



Step 3. Multiply the denominator by what you just wrote on top of the division sign. Write this product below the new numerator, leaving a space where any missing terms would have been.



Step 4. Subtract what you just wrote below the new numerator from the new numerator. Write the answer below.



Important: Once you finish Step 4 with a polynomial whose degree is smaller than the degree of the denominator, you are done. It's the remainder. Otherwise it's the new numerator and you have to repeat Steps 2, 3, and 4 again.

In the first example in this chapter we ended with the polynomial 0 (the polynomial 0 always has smaller degree than the denominator) and that means that the remainder equaled 0, which is sometimes expressed by saying there is no remainder.

In this second example, the remainder is $15x^2 - x - 4$. We know it's the remainder because its degree is smaller than the degree of $2x^3 - 3x$.

We write the solution to our division problem by summing the terms on top of the division sign, and adding the remainder divided by the denominator. In other words,

$$\frac{10x^4 - 4x^3 + 5x - 4}{2x^3 - 3x} = 5x - 2 + \frac{15x^2 - x - 4}{2x^3 - 3x}$$

The way the answer above is written is the way the answer would have to be written on exams.

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Remainders from linear denominators: If you divide a polynomial by a linear polynomial, the remainder is always a constant! That's because the remainder in polynomial division has to have a smaller degree than the degree of the denominator, and the only polynomials with smaller degree than a linear polynomial are the constant polynomials.

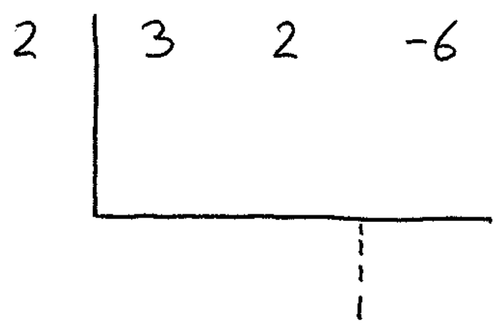
Synthetic division

If $\alpha \in \mathbb{R}$, here's how to do synthetic division to find $\frac{p(x)}{x-\alpha}$:

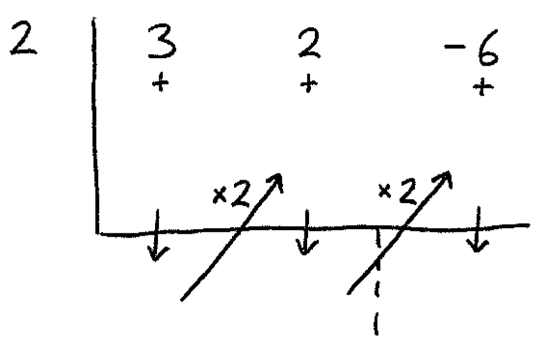
Write α, and to the left of that write the coefficients of $p(x)$ in order, even the coeffcients that equal 0.

$$\frac{3x^2+2x-6}{x-2} \qquad\qquad 2 \quad 3 \quad 2 \quad -6$$

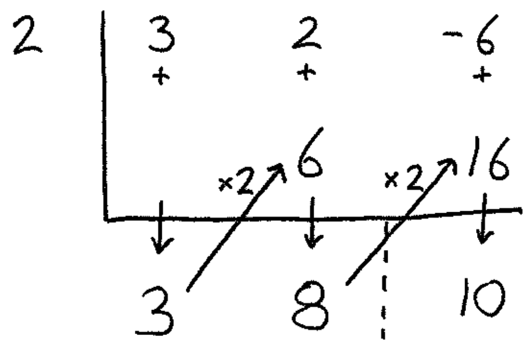
Under the coefficients of $p(x)$, write +-signs, and then leaving space for another row of numbers below that, draw an upside-down long division symbol. (I like to put a dashed horizontal line below all of that to separate the last column form the column preceding it.)



There are two types of moves: going down is addition, moving up and to the right one spot is multiplication by α.



Start with the first coefficient of $p(x)$. Go down, then up and to the right, then down, then up and to the right, then down until you fill in the last space to the right of the dashed horizontal line. (This last spot will be the remainder.)



The numbers to the left of the dashed horizontal line are the coefficients, in order, of the polynomial that is your answer, except you also have to add the remainder (which is the number to the right of the dashed horizontal line) divided by $x - \alpha$.

$$\frac{3x^2+2x-6}{x-2} = 3x+8+\frac{10}{x-2}$$

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Exercises

Divide. You can use synthetic division if the denominator (the polynomial in the bottom of the fraction) is degree 1, and has a leading coefficient of 1.

1.) $$\frac{12x^3 - 13x^2 + 9x - 2}{3x - 1}$$

2.) $$\frac{x^2 + x - 12}{x - 3}$$

3.) $$\frac{15x^2 - 27x + 13}{x + 1}$$

4.) $$\frac{2x^3 + 2x^2 - 48x + 72}{x + 6}$$

5.) $$\frac{12x^4 - 8x^3 - 22x^2 + 4x + 8}{4x^2 - 2}$$

6.) $$\frac{-x^2 + 15x - 1}{x - 1}$$

7.) $$\frac{x^3 + 4x^2 + x - 6}{x - 1}$$

8.) $$\frac{x^4 - 3x^3 + 5x^2 - 2x + 9}{x^2 + 1}$$

9.) $$\frac{4x^3 - x^2 - x - 1}{x + 5}$$

10.) $$\frac{6x^5 + 5x^4 - 2x^2 + 50x - 13}{x - 3}$$

Roots & Factors

Roots of a polynomial

A *root* of a polynomial $p(x)$ is a number $\alpha \in \mathbb{R}$ such that $p(\alpha) = 0$.

Examples.

- 3 is a root of the polynomial $p(x) = 2x - 6$ because
$$p(3) = 2(3) - 6 = 6 - 6 = 0$$
- 1 is a root of the polynomial $q(x) = 15x^2 - 7x - 8$ since
$$q(1) = 15(1)^2 - 7(1) - 8 = 15 - 7 - 8 = 0$$
- $(\sqrt[2]{2})^2 - 2 = 0$, so $\sqrt[2]{2}$ is a root of $x^2 - 2$.

Be aware: What we call a root is what others call a "real root", to emphasize that it is both a root and a real number. Since the only numbers we will consider in this course (and the only numbers considered in a basic calculus course) are real numbers, clarifying that a root is a "real root" won't be necessary.

Factors

A polynomial $q(x)$ is a *factor* of the polynomial $p(x)$ if there is a third polynomial $g(x)$ such that $p(x) = q(x)g(x)$.

Example. $3x^3 - x^2 + 12x - 4 = (3x - 1)(x^2 + 4)$, so $3x - 1$ is a factor of $3x^3 - x^2 + 12x - 4$. The polynomial $x^2 + 4$ is also a factor of $3x^3 - x^2 + 12x - 4$.

Factors and division

If you divide a polynomial $p(x)$ by another polynomial $q(x)$, and there is no remainder, then $q(x)$ is a factor of $p(x)$. That's because if there's no remainder, then $\frac{p(x)}{q(x)}$ is a polynomial, and $p(x) = q(x)\big(\frac{p(x)}{q(x)}\big)$. That's the definition of $q(x)$ being a factor of $p(x)$.

If $\frac{p(x)}{q(x)}$ has a remainder, then $q(x)$ is *not* a factor of $p(x)$.

Example. In the previous chapter we saw that
$$\frac{6x^2 + 5x + 1}{3x + 1} = 2x + 1$$

Multiplying the above equation by $3x+1$ gives

$$6x^2 + 5x + 1 = (3x+1)(2x+1)$$

so $3x+1$ is a factor of $6x^2+5x+1$.

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Most important examples of roots

Notice that the number α is a root of the linear polynomial $x-\alpha$ since $\alpha - \alpha = 0$.

You have to be able to recognize these types of roots when you see them.

polynomial	root
$x-2$	2
$x-3$	3
$x-(-2)$	-2
$x+2$	-2
$x+15$	-15
$x-\alpha$	α

Linear factors give roots

Suppose there is some number α such that $x-\alpha$ is a factor of the polynomial $p(x)$. We'll see that α must be a root of $p(x)$.

That $x-\alpha$ is a factor of $p(x)$ means there is a polynomial $g(x)$ such that

$$p(x) = (x-\alpha)g(x)$$

Then

$$\begin{aligned} p(\alpha) &= (\alpha-\alpha)g(\alpha) \\ &= 0 \cdot g(\alpha) \\ &= 0 \end{aligned}$$

Notice that it didn't matter what polynomial $g(x)$ was, or what number $g(\alpha)$ was; α is a root of $p(x)$.

If $x - \alpha$ is a factor of $p(x)$,
then α is a root of $p(x)$.

Examples.

- 2 is a root of $p(x) = (x-2)(\pi^7 x^{15} - 27x^{11} + \frac{3}{4}x^5 - x^3)$ because

$$\begin{aligned} p(2) &= (2-2)(\pi^7 2^{15} - 27(2)^{11} + \frac{3}{4}2^5 - 2^3) \\ &= 0 \cdot (\pi^7 2^{15} - 27(2)^{11} + \frac{3}{4}2^5 - 2^3) \\ &= 0 \end{aligned}$$

- 4 is a root of $q(x) = (x-4)(x^{101} - x^{57} - 17x^3 + x)$
- -2, 1, and 5 are roots of the polynomial $3(x+2)(x-1)(x-5)$.

Roots give linear factors

Suppose the number α is a root of the polynomial $p(x)$. That means that $p(\alpha) = 0$. We'll see that $x - \alpha$ must be a factor of $p(x)$.

Let's start by dividing $p(x)$ by $(x - \alpha)$. Remember that when you divide a polynomial by a linear polynomial, the remainder is always a constant. So we'll get something that looks like

$$\frac{p(x)}{(x-\alpha)} = g(x) + \frac{c}{(x-\alpha)}$$

where $g(x)$ is a polynomial and $c \in \mathbb{R}$ is a constant.

Next we can multiply the previous equation by $(x - \alpha)$ to get

$$\begin{aligned} p(x) &= (x-\alpha)\Big(g(x) + \frac{c}{(x-\alpha)}\Big) \\ &= (x-\alpha)g(x) + (x-\alpha)\frac{c}{(x-\alpha)} \\ &= (x-\alpha)g(x) + c \end{aligned}$$

That means that

$$\begin{aligned} p(\alpha) &= (\alpha - \alpha)g(\alpha) + c \\ &= 0 \cdot g(\alpha) + c \\ &= 0 + c \\ &= c \end{aligned}$$

Now remember that $p(\alpha) = 0$. We haven't used that information in this problem yet, but we can now: because $p(\alpha) = 0$ and $p(\alpha) = c$, it must be that $c = 0$. Therefore,

$$p(x) = (x - \alpha)g(x) + c = (x - \alpha)g(x)$$

That means that $x - \alpha$ is a factor of $p(x)$, which is what we wanted to check.

If α is a root of $p(x)$,
then $x - \alpha$ is a factor of $p(x)$

Example. It's easy to see that 1 is a root of $p(x) = x^3 - 1$. Therefore, we know that $x - 1$ is a factor of $p(x)$. That means that $p(x) = (x - 1)g(x)$ for some polynomial $g(x)$.

To find $g(x)$, divide $p(x)$ by $x - 1$:

$$g(x) = \frac{p(x)}{x-1} = \frac{x^3 - 1}{x - 1} = x^2 + x + 1$$

Hence, $x^3 - 1 = (x - 1)(x^2 + x + 1)$.

We were able to find two factors of $x^3 - 1$ because we spotted that the number 1 was a root of $x^3 - 1$.

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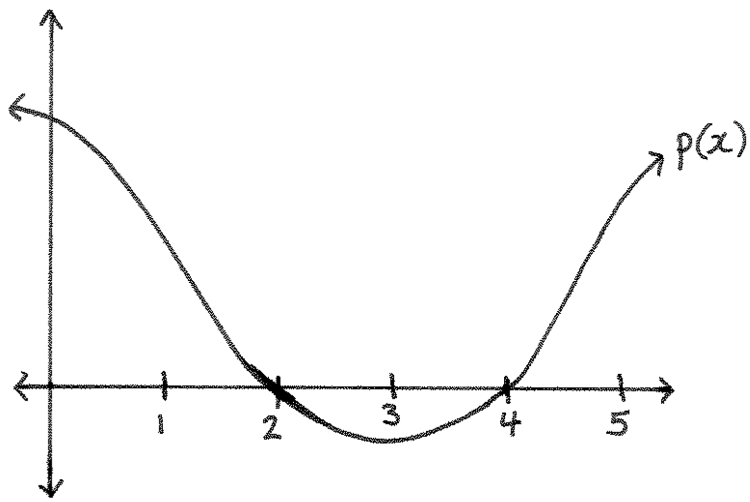
Roots and graphs

If you put a root into a polynomial, 0 comes out. That means that if α is a root of $p(x)$, then $(\alpha, 0) \in \mathbb{R}^2$ is a point in the graph of $p(x)$. These points are exactly the x-intercepts of the graph of $p(x)$.

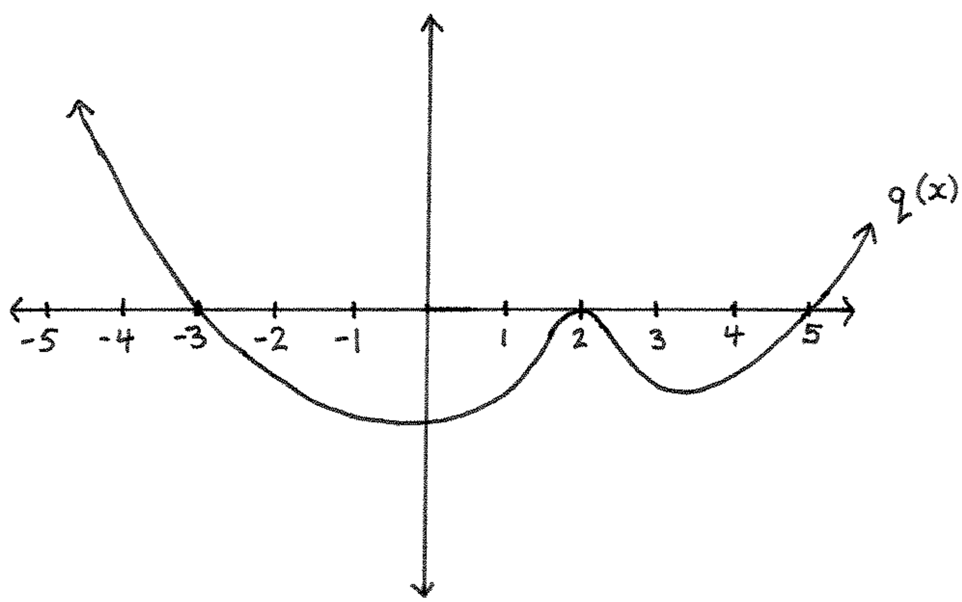
The roots of a polynomial are exactly the x-intercepts of its graph.

Examples.

• Below is the graph of a polynomial $p(x)$. The graph intersects the x-axis at 2 and 4, so 2 and 4 must be roots of $p(x)$. That means that $(x-2)$ and $(x-4)$ are factors of $p(x)$.



• Below is the graph of a polynomial $q(x)$. The graph intersects the x-axis at -3, 2, and 5, so -3, 2, and 5 are roots of $q(x)$, and $(x+3)$, $(x-2)$, and $(x-5)$ are factors of $q(x)$.



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Degree of a product is the sum of degrees of the factors

Let's take a look at some products of polynomials that we saw before in the chapter on "Basics of Polynomials":

The leading term of $(2x^2 - 5x)(-7x + 4)$ is $-14x^3$. This is an example of a degree 2 and a degree 1 polynomial whose product equals 3. Notice that $2 + 1 = 3$

The product $5(x-2)(x+3)(x^2+3x-7)$ is a degree 4 polynomial because its leading term is $5x^4$. The degrees of 5, $(x-2)$, $(x+3)$, and (x^2+3x-7) are 0, 1, 1, and 2, respectively. Notice that $0+1+1+2=4$.

The degrees of $(2x^3-7)$, (x^5-3x+5), $(x-1)$, and $(5x^7+6x-9)$ are 3, 5, 1, and 7, respectively. The degree of their product,

$$(2x^3-7)(x^5-3x+5)(x-1)(5x^7+6x-9),$$

equals 16 since its leading term is $10x^{16}$. Once again, we have that the sum of the degrees of the factors equals the degree of the product: $3+5+1+7=16$.

These three examples suggest a general pattern that always holds for factored polynomials (as long as the factored polynomial does not equal 0):

If a polynomial $p(x)$ is factored into a product of polynomials, then the degree of $p(x)$ equals the sum of the degrees of its factors.

Examples.

- The degree of $(4x^3+27x-3)(3x^6-27x^3+15)$ equals $3+6=9$.
- The degree of $-7(x+4)(x-1)(x-3)(x-3)(x^2+1)$ equals $0+1+1+1+1+2=6$.

Degree of a polynomial bounds the number of roots

Suppose $p(x)$ is a polynomial that has n roots, and that $p(x)$ is not the constant polynomial $p(x)=0$. Let's name the roots of $p(x)$ as $\alpha_1, \alpha_2, ..., \alpha_n$.

Any root of $p(x)$ gives a linear factor of $p(x)$, so

$$p(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)q(x)$$

for some polynomial $q(x)$.

Because the degree of a product is the sum of the degrees, the degree of $p(x)$ is at least n.

The degree of $p(x)$ (if $p(x) \neq 0$) is greater than or equal to the number of roots that $p(x)$ has.

Examples.

- $5x^4 - 3x^3 + 2x - 17$ has at most 4 roots.
- $4x^{723} - 15x^{52} + 37x^{14} - 7$ has at most 723 roots.
- Aside from the constant polynomial $p(x) = 0$, if a function has a graph that has infinitely many x-intercepts, then the function cannot be a polynomial.

  If it were a polynomial, its number of roots (or alternatively, its number of x-intercepts) would be bounded by the degree of the polynomial, and thus there would only be finitely many x-intercepts.

  To illustrate, if you are familiar with the graphs of the functions $\sin(x)$ and $\cos(x)$, then you'll recall that they each have infinitely many x-intercepts. Thus, they cannot be polynomials. (If you are unfamiliar with $\sin(x)$ and $\cos(x)$, then you can ignore this paragraph.)

Exercises

1.) Name two roots of the polynomial $(x-1)(x-2)$.

2.) Name two roots of the polynomial $-(x+7)(x-3)(x^4+x^3+2x^2+x+1)$.

3.) Name four roots of the polynomial $-\frac{2}{5}(x+\frac{7}{3})(x+\frac{1}{2})(x-\frac{4}{3})(x-\frac{9}{2})(x^2+1)$.

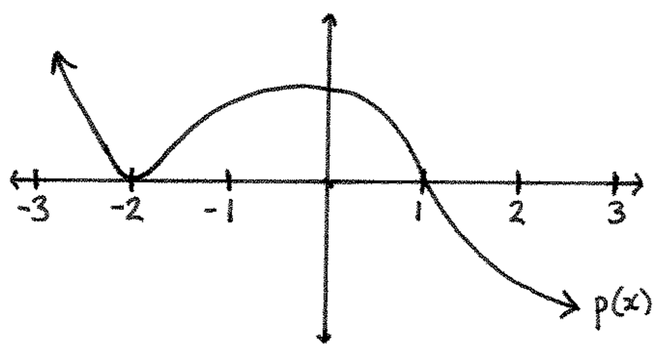
It will help with #4-6 to know that each of the polynomials from those problems has a root that equals either -1, 0, or 1.

4.) Write x^3+4x-5 as a product of a linear and a quadratic polynomial.

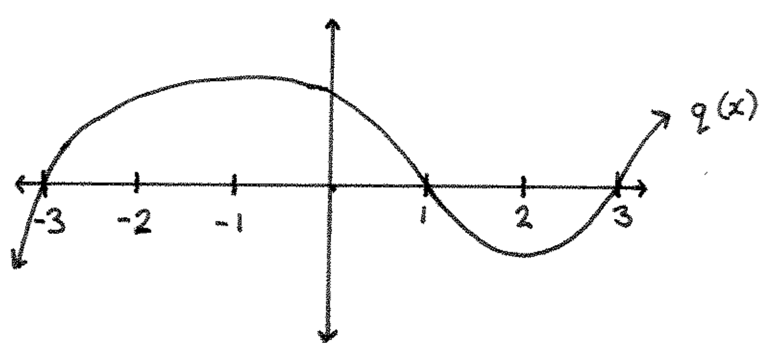
5.) Write x^3+x as a product of a linear and a quadratic polynomial. (Hint: you could use the distributive law here.)

6.) Write $x^5+3x^4+x^3-x^2-x-1$ as a product of a linear and a quartic polynomial.

7.) The graph of a polynomial $p(x)$ is drawn below. Identify as many roots and factors of $p(x)$ as you can.



8.) The graph of a polynomial $q(x)$ is drawn below. Identify as many roots and factors of $q(x)$ as you can.



For #9-13, determine the degree of the given polynomial.

9.) $(x+3)(x-2)$

10.) $(3x+5)(4x^2+2x-3)$

11.) $-17(3x^2+20x-4)$

12.) $4(x-1)(x-1)(x-1)(x-2)(x^2+7)(x^2+3x-4)$

13.) $5(x-3)(x^2+1)$

14.) (True/False) $7x^5+13x^4-3x^3-7x^2+2x-1$ has 8 roots.

Constant & Linear Polynomials

Constant polynomials

A *constant polynomial* is the same thing as a constant function. That is, a constant polynomial is a function of the form

$$p(x) = c$$

for some number c. For example, $p(x) = -\frac{5}{3}$ or $q(x) = -7$.

The output of a constant polynomial does not depend on the input (notice that there is no x on the right side of the equation $p(x) = c$). Constant polynomials are also called degree 0 polynomials.

The graph of a constant polynomial is a horizontal line. A constant polynomial does not have any roots unless it is the polynomial $p(x) = 0$.

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Linear polynomials

A linear polynomial is any polynomial defined by an equation of the form

$$p(x) = ax + b$$

where a and b are real numbers and $a \neq 0$. For example, $p(x) = 3x - 7$ and $q(x) = \frac{-13}{4}x + \frac{5}{3}$ are linear polynomials. A linear polynomial is the same thing as a degree 1 polynomial.

Roots of linear polynomials

Every linear polynomial has exactly one root. Finding the root is just a matter of basic algebra.

Problem: Find the root of $p(x) = 3x - 7$.

Solution: The root of $p(x)$ is the number α such that $p(\alpha) = 0$. In this problem that means that $3\alpha - 7 = 0$. Hence $3\alpha = 7$, so $\alpha = \frac{7}{3}$. Thus, $\frac{7}{3}$ is the root of $3x - 7$.

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Slope

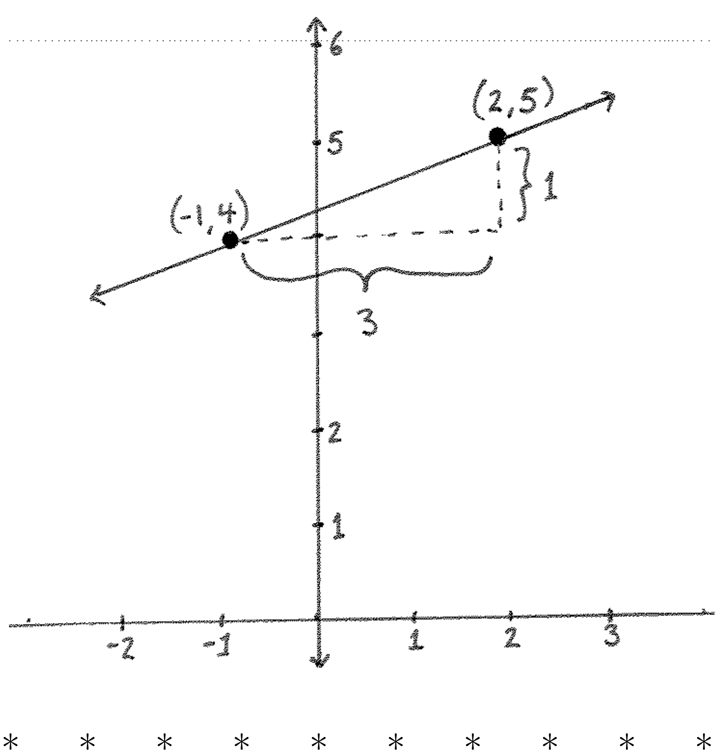
The slope of a line is the ratio of the change in the second coordinate to the change in the first coordinate. In different words, if a line contains the two points (x_1, y_1) and (x_2, y_2), then the slope is the change in the y-coordinate – which equals $y_2 - y_1$ – divided by the change in the x-coordinate – which equals $x_2 - x_1$.

Slope of line containing (x_1, y_1) and (x_2, y_2):

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Example: The slope of the line containing the two points $(-1, 4)$ and $(2, 5)$ equals

$$\frac{5-4}{2-(-1)} = \frac{1}{3}$$



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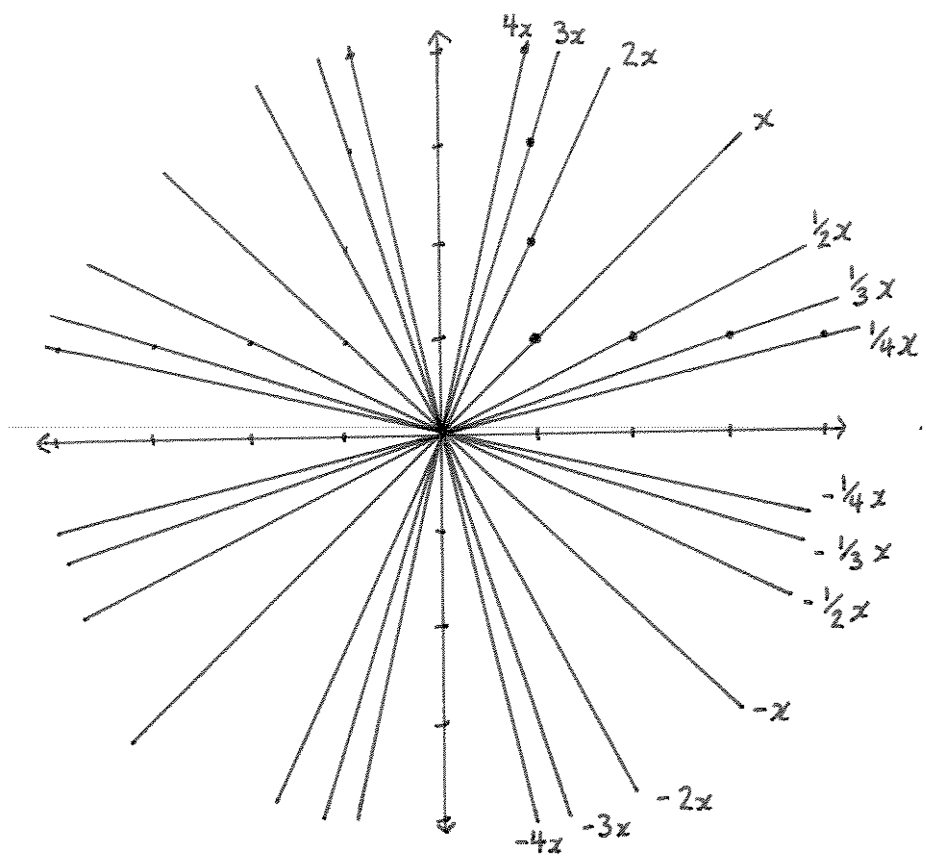
Graphing linear polynomials

Let $p(x) = ax$ where a is a number that does not equal 0. This polynomial is an example of a linear polynomial.

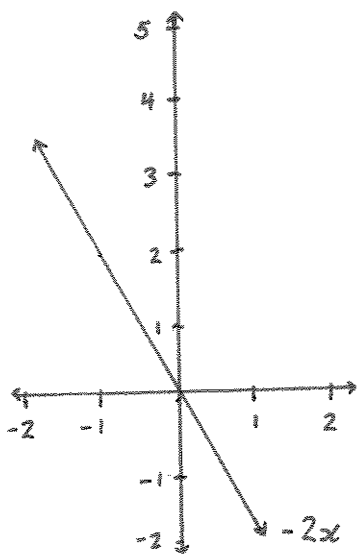
The graph of $p(x) = ax$ is a straight line that passes through $(0, 0) \in \mathbb{R}^2$ and has slope equal to a. We can check this by graphing it. The point $(0, a0) = (0, 0)$ is in the graph, as are the points $(1, a)$, $(2, 2a)$, $(3, 3a)$,... and $(-1, -a)$, $(-2, -2a)$, $(-3, -3a)$,...

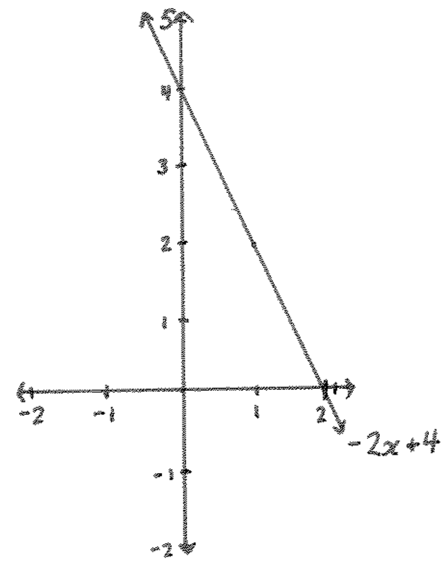


Because the graph of $ax + b$ is the graph of ax shifted up or down by b – depending on whether b is positive or negative – the graph of $ax + b$ is a straight line that passes through $(0, b) \in \mathbb{R}^2$ and has slope equal to a.

Problem: Graph $p(x) = -2x + 4$.

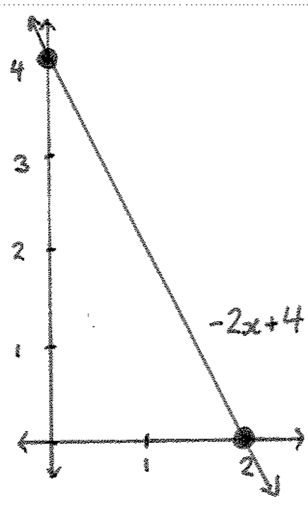
Solution: The graph of $-2x+4$ is the graph of $-2x$ "shifted up" by 4. Draw $-2x$, which is the line of slope -2 that passes through $(0, 0)$, and then shift it up to the line that passes through $(0, 4)$ and is parallel to $-2x$.





Another solution: To graph a linear polynomial, find two points in the graph, and then draw the straight line that passes through them.

Since $p(x) = -2x + 4$ has 2 as a root, it has an x-intercept at 2. The y-intercept is the point in the graph whose first coordinate equals 0, and that's the point $(0, p(0)) = (0, 4)$. To graph $-2x + 4$, draw the line passing through the x- and y-intercepts.



Behind the name. Degree 1 polynomials are called linear polynomials because their graphs are straight lines.

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Exercises

1.) Graph $p(x) = 3$.

2.) Graph $q(x) = -\frac{3}{2}$.

3.) Find the root of $p(x) = -\frac{4}{3}x + \frac{6}{7}$.

4.) Find the root of $q(x) = \frac{2}{9}x - \frac{8}{5}$.

5.) Plot the x- and y-intercepts of $p(x) = 4x - 3$, and then graph $p(x)$.

6.) Plot the x- and y-intercepts of $q(x) = -2x - 3$, and then graph $q(x)$.

7.) Claudia owns a coconut collecting company. She has to pay \$200 for a coconut collecting license to conduct her company, and she earns \$3 for every coconut she collects. If x is the number of coconuts she collects, and $p(x)$ is the number of dollars her company earns, then find an equation for $p(x)$.

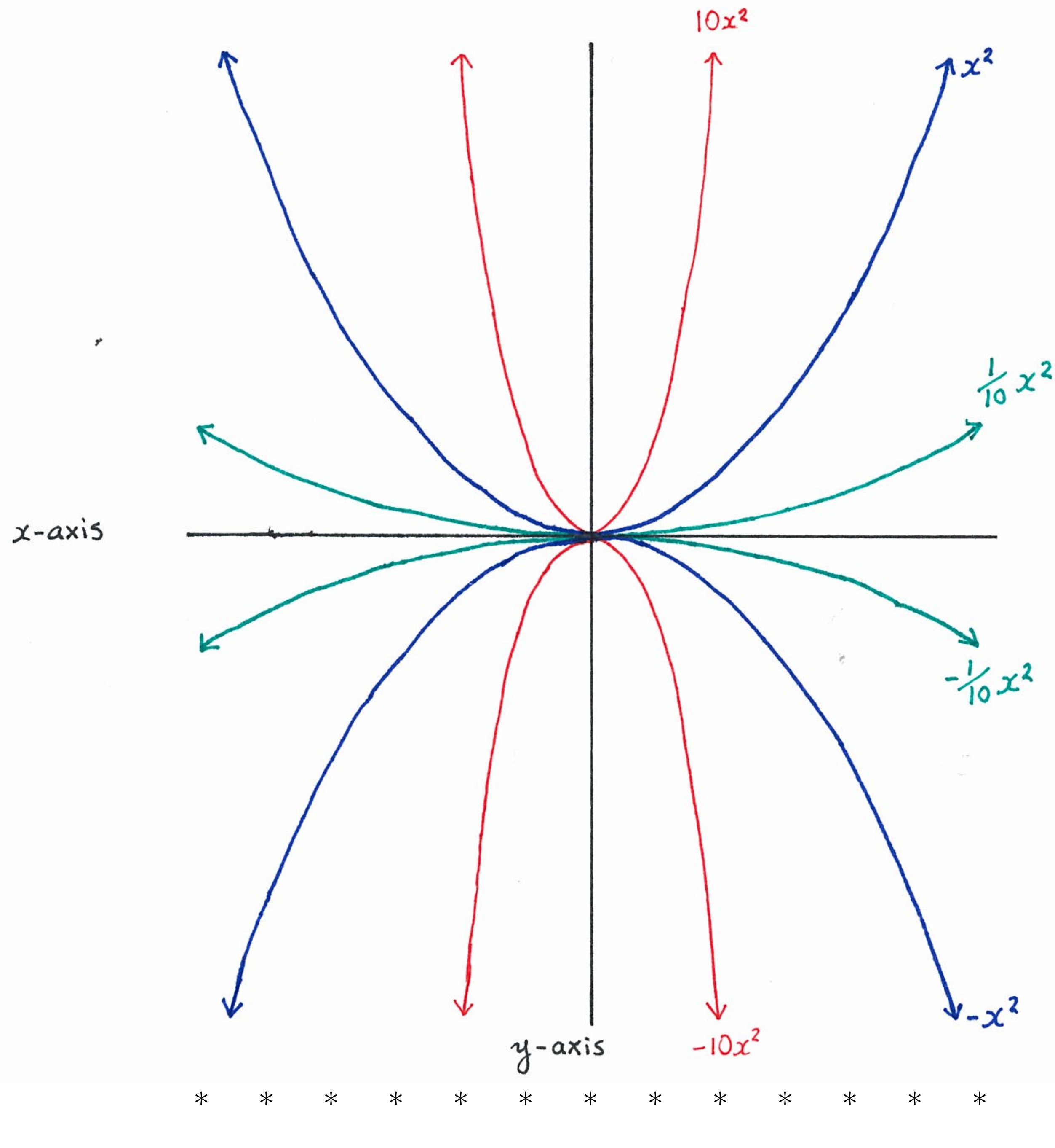
8.) Spencer is payed \$400 to collect coconuts no matter how many coconuts he collects. Because he is collecting coconuts for a flat fee, the local government does not require Spencer to purchase a coconut collecting license. If $q(x)$ is the number of dollars he earns for collecting x coconuts, what is the equation that defines $q(x)$?

9.) If Claudia and Spencer collect the same number of coconuts, then how many coconuts would Claudia have to collect for her company to earn at least as much money as Spencer?

Quadratic Polynomials

If $a > 0$ then the graph of ax^2 is obtained by starting with the graph of x^2, and then stretching or shrinking vertically by a.

If $a < 0$ then the graph of ax^2 is obtained by starting with the graph of x^2, then flipping it over the x-axis, and then stretching or shrinking vertically by the positive number $-a$.



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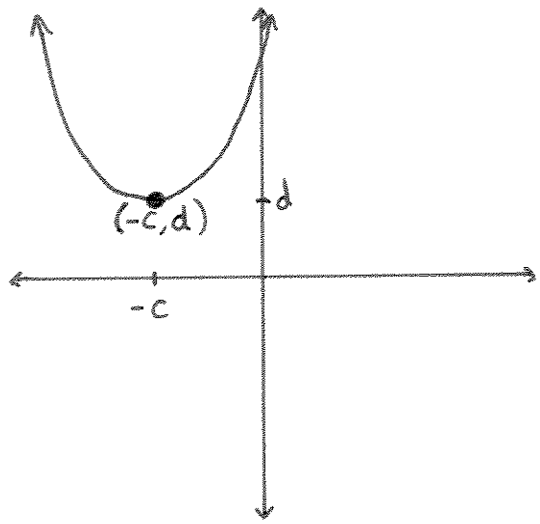
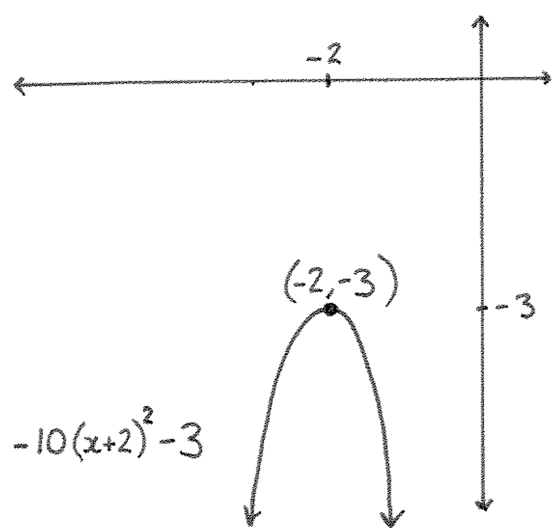
If $a, c, d \in \mathbb{R}$ and $a \neq 0$, then the graph of $a(x+c)^2 + d$ is obtained by shifting the graph of ax^2 horizontally by c, and vertically by d. (Remember that $d > 0$ means moving up, $d < 0$ means moving down, $c > 0$ means moving *left*, and $c < 0$ means moving *right*.)

If $a \neq 0$, the graph of a function $f(x) = a(x+c)^2 + d$ is called a *parabola*. The point $(-c, d) \in \mathbb{R}^2$ is called the *vertex* of the parabola.



Example. Below is the parabola that is the graph of $-10(x+2)^2 - 3$. Its vertex is $(-2, -3)$.



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A *quadratic polynomial* is a degree 2 polynomial. In other words, a quadratic polynomial is any polynomial of the form

$$p(x) = ax^2 + bx + c$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

Completing the square

You should memorize this equation: (it's called *completing the square.*)

$$ax^2 + bx + c \;=\; a\Big(x + \frac{b}{2a}\Big)^2 + c - \frac{b^2}{4a}$$

Let's check that the equation is true:

$$\begin{aligned} a\Big(x + \frac{b}{2a}\Big)^2 + c - \frac{b^2}{4a} &= a\Big(x^2 + 2x\frac{b}{2a} + \Big[\frac{b}{2a}\Big]^2\Big) + c - \frac{b^2}{4a} \\ &= ax^2 + a2x\frac{b}{2a} + a\Big[\frac{b}{2a}\Big]^2 + c - \frac{b^2}{4a} \\ &= ax^2 + bx + a\Big[\frac{b^2}{4a^2}\Big] - \frac{b^2}{4a} + c \\ &= ax^2 + bx + \frac{b^2}{4a} - \frac{b^2}{4a} + c \\ &= ax^2 + bx + c \end{aligned}$$

Graphing quadratics

We can use completing the square to graph quadratic polynomials. If $p(x) = ax^2 + bx + c$, then $p(x) = a\big(x + \frac{b}{2a}\big)^2 + c - \frac{b^2}{4a}$. Therefore, the graph of $p(x) = ax^2 + bx + c$ is obtained by shifting the graph of ax^2 horizontally by $\frac{b}{2a}$, and vertically by $c - \frac{b^2}{4a}$.

Example. To graph $-3x^2 + 5x - 2$, first complete the square to find that $-3x^2 + 5x - 2$ is the same polynomial as $-3(x - \frac{5}{6})^2 + \frac{1}{12}$. To graph this polynomial, we start with the parabola for $-3x^2$.

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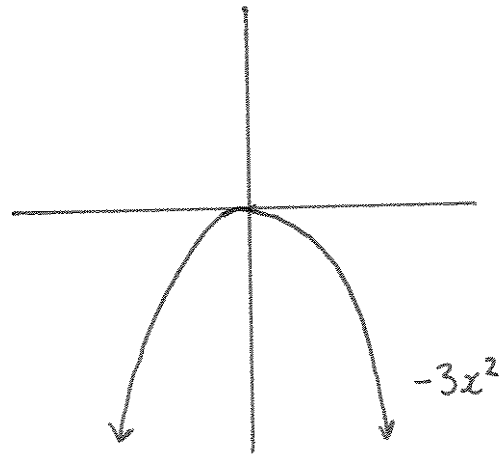
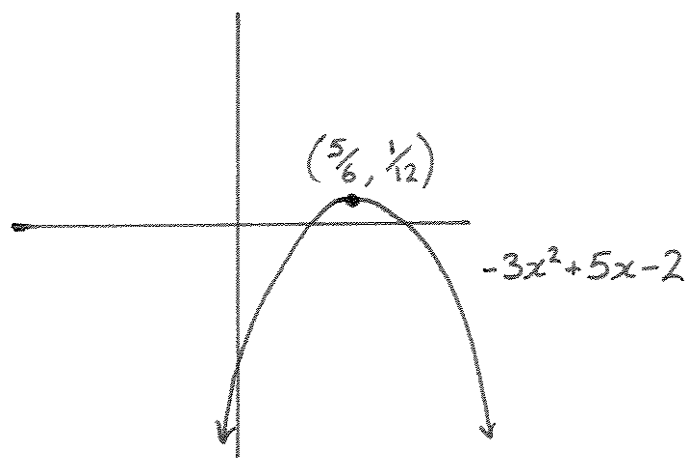


Shift the parabola for $3x^2$ right by $\frac{5}{6}$ and then up by $\frac{1}{12}$. The result is the graph for $-3x^2+5x-2$. Notice that the graph looks like the graph of $-3x^2$, except that its vertex is the point $(\frac{5}{6}, \frac{1}{12})$.



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Discriminant

The *discriminant* of $ax^2 + bx + c$ is defined to be the number $b^2 - 4ac$.

How many roots?

If $p(x) = ax^2 + bx + c$, then the following chart shows how the discriminant of $p(x)$ determines how many roots $p(x)$ has:

$b^2 - 4ac$	number of roots
> 0	2
$= 0$	1
< 0	0

Example. Suppose $p(x) = -2x^2 + 3x - 1$. Because $3^2 - 4(-2)(-1) = 9 - 8 = 1$ is positive, $p(x) = -2x^2 + 3x - 1$ has two roots.

Why the discriminant of a quadratic polynomial tells us about the number of roots of the polynomial, and why the information from the above chart is true, will be explained in lectures.

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Finding roots

If $ax^2 + bx + c$ has at least one root – which is the same as saying that $b^2 - 4ac \geq 0$ – then there is a formula that tells us what those roots are.

Quadratic formula

If $p(x) = ax^2 + bx + c$ with $a \neq 0$ and if $b^2 - 4ac \geq 0$,
then the roots of $p(x)$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Notice in the quadratic formula, that we need $a \neq 0$ to make sure that we are not dividing by 0, and we need $b^2 - 4ac \geq 0$ to make sure that we aren't taking the square root of a negative number.

Also recall that if $ax^2 + bx + c$ has only one root, then $b^2 - 4ac = 0$. That means the two roots from the quadratic formula are really the same root.

It's a good exercise in algebra to check that the quadratic equation is true. To check that it's true, you need to check that

$$p\Big(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\Big) = 0$$

and that

$$p\Big(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\Big) = 0$$

Example. We checked above that $p(x) = -2x^2+3x-1$ has 2 roots, because its discriminant equalled 1. The quadratic formula tells us that those roots equal

$$\frac{-3 + \sqrt{1}}{2(-2)} = \frac{-3+1}{-4} = \frac{-2}{-4} = \frac{1}{2}$$

and

$$\frac{-3 - \sqrt{1}}{2(-2)} = \frac{-4}{-4} = 1$$

Exercises

For each of the quadratic polynomials in problems #1-6:

- Complete the square.
- What's the vertex of the corresponding parabola?
- Is its parabola opening up, or opening down?
- What's its discriminant?
- How many roots does it have?
- What are its roots (if it has any)?
- Graph the polynomial, labeling its vertex and any x-intercepts.

1.) $-2x^2 - 2x + 12$

2.) $x^2 + 2x + 1$

3.) $3x^2 - 9x + 6$

4.) $-4x^2 + 16x - 19$

5.) $x^2 + 2x - 1$

6.) $3x^2 + 6x + 5$

7.) Suppose you shoot a feather straight up into the air, and that t is the time measured in seconds that follow after you shoot the feather into the air. If the height of the feather at time t is given by $-2t^2 + 20t$ feet, then what is the maximum height that the feather reaches? How many seconds does it take for the feather to reach its maximum height?

8.) Let's say you make cogs for a living. After accounting for the cost of building materials, you earn a profit of $x^2 - 10x + 45$ cents on the x-th cog that you make. Which cog do you earn the least amount of profit for making? How much profit do you earn for that cog?

Factoring Polynomials

Any natural number that is greater than 1 can be factored into a product of prime numbers. For example $20 = (2)(2)(5)$ and $30 = (2)(3)(5)$.

In this chapter we'll learn an analogous way to factor polynomials.

Fundamental Theorem of Algebra

A *monic* polynomial is a polynomial whose leading coefficient equals 1. So $x^4 - 2x^3 + 5x - 7$ is monic, and $x - 2$ is monic, but $3x^2 - 4$ is not monic.

Carl Friedrich Gauss was the boy who discovered a really quick way to see that $1 + 2 + 3 + \cdots + 100 = 5050$.

In 1799, a grown-up Gauss proved the following theorem:

Any polynomial is the product of a real number,
and a collection of monic quadratic polynomials that
do not have roots, and of monic linear polynomials.

This result is called the *Fundamental Theorem of Algebra*. It is one of the most important results in all of mathematics, though from the form it's written in above, it's probably difficult to immediately understand its importance.

The explanation for why this theorem is true is somewhat difficult, and it is beyond the scope of this course. We'll have to accept it on faith.

Examples.

- $4x^2 - 12x + 8$ can be factored into a product of a number, 4, and two monic linear polynomials, $x - 1$ and $x - 2$. That is, $4x^2 - 12x + 8 = 4(x - 1)(x - 2)$.

- $-x^5 + 2x^4 - 7x^3 + 14x^2 - 10x + 20$ can be factored into a product of a number, -1, a monic linear polynomial, $x - 2$, and two monic quadratic polynomials that don't have roots, x^2+2 and x^2+5. That is $-x^5 + 2x^4 - 7x^3 + 14x^2 - 10x + 20 = -(x - 2)(x^2 + 2)(x^2 + 5)$. (We can check the discriminants of $x^2 + 2$ and $x^2 + 5$ to see that these two quadratics don't have roots.)

- $2x^4 - 2x^3 + 14x^2 - 6x + 24 = 2(x^2 + 3)(x^2 - x + 4)$. Again, $x^2 + 3$ and $x^2 - x + 4$ do not have roots.

Notice that in each of the above examples, the real number that appears in the product of polynomials – 4 in the first example, -1 in the second, and 2 in the third – is the same as the leading coefficient for the original polynomial. This always happens, so the Fundamental Theorem of Algebra can be more precisely stated as follows:

If $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$,
then $p(x)$ is the product of the real number a_n,
and a collection of monic quadratic polynomials that
do not have roots, and of monic linear polynomials.

Completely factored

A polynomial is *completely factored* if it is written as a product of a real number (which will be the same number as the leading coefficient of the polynomial), and a collection of monic quadratic polynomials that do not have roots, and of monic linear polynomials.

Looking at the examples above, $4(x-1)(x-2)$ and $-(x-2)(x^2+2)(x^2+5)$ and $2(x^2 + 3)(x^2 - x + 4)$ are completely factored.

One reason it's nice to completely factor a polynomial is because if you do, then it's easy to read off what the roots of the polynomial are.

Example. Suppose $p(x) = -2x^5 + 10x^4 + 2x^3 - 38x^2 + 4x - 48$. Written in this form, its difficult to see what the roots of $p(x)$ are. But after being completely factored, $p(x) = -2(x + 2)(x - 3)(x - 4)(x^2 + 1)$. The roots of this polynomial can be read from the monic linear factors. They are -2, 3, and 4.

(Notice that $p(x) = -2(x+2)(x-3)(x-4)(x^2+1)$ is completely factored because x^2+1 has no roots.)

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Factoring linears

To completely factor a linear polynomial, just factor out its leading coefficient:

$$ax + b = a\Big(x + \frac{b}{a}\Big)$$

For example, to completely factor $2x+6$, write it as the product $2(x+3)$.

Factoring quadratics

What a completely factored quadratic polynomial looks like will depend on how many roots it has.

0 Roots. If the quadratic polynomial ax^2+bx+c has 0 roots, then it can be completely factored by factoring out the leading coefficient:

$$ax^2 + bx + c = a\Big(x^2 + \frac{b}{a}x + \frac{c}{a}\Big)$$

(The graphs of ax^2+bx+c and $x^2+\frac{b}{a}x+\frac{c}{a}$ differ by a vertical stretch or shrink that depends on a. A vertical stretch or shrink of a graph won't change the number of x-intercepts, so $x^2+\frac{b}{a}x+\frac{c}{a}$ won't have any roots since ax^2+bx+c doesn't have any roots. Thus, $x^2+\frac{b}{a}x+\frac{c}{a}$ is completely factored.)

Example. The discriminant of $4x^2-2x+2$ equals $(-2)^2-4(4)(2) = 4-32 = -28$, a negative number. Therefore, $4x^2-2x+2$ has no roots, and it is completely factored as $4(x^2-\frac{1}{2}x+\frac{1}{2})$.

2 Roots. If the quadratic polynomial ax^2+bx+c has 2 roots, we can name them α_1 and α_2. Roots give linear factors, so we know that $(x-\alpha_1)$

and $(x - \alpha_2)$ are factors of $ax^2 + bx + c$. That means that there is some polynomial $q(x)$ such that

$$ax^2 + bx + c = q(x)(x - \alpha_1)(x - \alpha_2)$$

The degree of $ax^2 + bx + c$ equals 2. Because the sum of the degrees of the factors equals the degree of the product, we know that the degree of $q(x)$ plus the degree of $(x - \alpha_1)$ plus the degree of $(x - \alpha_2)$ equals 2. In other words, the degree of $q(x)$ plus 1 plus 1 equals 2.

Zero is the only number that you can add to $1 + 1$ to get 2, so $q(x)$ must have degree 0, which means that $q(x)$ is just a constant number.

Because the leading term of $ax^2 + bx + c$ – namely ax^2 – is the product of the leading terms of $q(x)$, $(x - \alpha_1)$, and $(x - \alpha_2)$ – namely the number $q(x)$, x, and x – it must be that $q(x) = a$. Therefore,

$$ax^2 + bx + c = a(x - \alpha_1)(x - \alpha_2)$$

Example. The discriminant of $2x^2 + 4x - 2$ equals $4^2 - 4(2)(-2) = 16 + 16 = 32$, a positive number, so there are two roots.

We can use the quadratic formula to find the two roots, but before we do, it's best to simplify the square root of the discriminant: $\sqrt{32} = \sqrt{(4)(4)(2)} = 4\sqrt{2}$.

Now we use the quadratic formula to find that the roots are

$$\frac{-4 + 4\sqrt{2}}{2(2)} = \frac{-4 + 4\sqrt{2}}{4} = -1 + \sqrt{2}$$

and

$$\frac{-4 - 4\sqrt{2}}{2(2)} = \frac{-4 - 4\sqrt{2}}{4} = -1 - \sqrt{2}$$

Therefore, $2x^2 + 4x - 2$ is completely factored as

$$2\Big(x - (-1 + \sqrt{2})\Big)\Big(x - (-1 - \sqrt{2})\Big) = 2(x + 1 - \sqrt{2})(x + 1 + \sqrt{2})$$

1 Root. If $ax^2 + bx + c$ has exactly 1 root (let's call it α_1) then $(x - \alpha_1)$ is a factor of $ax^2 + bx + c$. Hence,

$$ax^2 + bx + c = g(x)(x - \alpha_1)$$

for some polynomial $g(x)$.

Because the degree of a product is the sum of the degrees of the factors, $g(x)$ must be a degree 1 polynomial, and it can be completely factored into something of the form $\lambda(x-\beta)$ where $\lambda, \beta \in \mathbb{R}$. Therefore,

$$ax^2 + bx + c = \lambda(x-\beta)(x-\alpha_1)$$

Notice that β is a root of $\lambda(x-\beta)(x-\alpha_1)$, so β is a root of ax^2+bx+c since they are the same polynomial. But we know that ax^2+bx+c has only one root, namely α_1, so β must equal α_1. That means that

$$ax^2 + bx + c = \lambda(x-\alpha_1)(x-\alpha_1)$$

The leading term of ax^2+bx+c is ax^2. The leading term of $\lambda(x-\alpha_1)(x-\alpha_1)$ is λx^2. Since ax^2+bx+c equals $\lambda(x-\alpha_1)(x-\alpha_1)$, they must have the same leading term. Therefore, $ax^2 = \lambda x^2$. Hence, $a = \lambda$.

Replace λ with a in the equation above, and we are left with

$$ax^2 + bx + c = a(x-\alpha_1)(x-\alpha_1)$$

Example. The discriminant of $3x^2 - 6x + 3x$ equals $(-6)^2 - 4(3)(3) = 36 - 36 = 0$, so there is exactly one root. We find the root using the quadratic formula:

$$\frac{-(-6)+\sqrt{0}}{2(3)} = \frac{6}{6} = 1$$

Therefore, $3x^2 - 6x + 3x$ is completely factored as $3(x-1)(x-1)$.

Summary. The following chart summarizes the discussion above.

roots of ax^2+bx+c	completely factored form of ax^2+bx+c
no roots	$a(x^2 + \frac{b}{a}x + \frac{c}{a})$
2 roots: α_1 and α_2	$a(x-\alpha_1)(x-\alpha_2)$
1 root: α_1	$a(x-\alpha_1)(x-\alpha_1)$

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Factors in $\mathbb{Z}$

Recall that the factors of an integer n are all of the integers k such that $n = mk$ for some third integer m.

Examples.

- $12 = 3 \cdot 4$, so 4 is a factor of 12.
- $-30 = -2 \cdot 15$, so 15 is a factor of -30.
- 1, -1, n and $-n$ are all factors of an integer n. That's because $n = n \cdot 1$ and $n = (-n)(-1)$.

Important special case. If $\alpha_1, \alpha_2, \ldots \alpha_n \in \mathbb{Z}$, then each of these numbers are factors of the product $\alpha_1\alpha_2 \cdots \alpha_n$. For example, 2, 10, and 7 are each factors of $2 \cdot 10 \cdot 7 = 140$.

Check factors of degree 0 coefficient when searching for roots

If k, α_1, and α_2 are all integers, then the polynomial

$$q(x) = k(x - \alpha_1)(x - \alpha_2) = kx^2 - k(\alpha_1 + \alpha_2)x + k\alpha_1\alpha_2$$

has α_1 and α_2 as roots, and each of these roots are factors of the degree 0 coefficient of $q(x)$. (The degree 0 coefficient is $k\alpha_1\alpha_2$.)

More generally, if $k, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{Z}$, then the degree 0 coefficient of the polynomial

$$g(x) = k(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$$

equals $k\alpha_1\alpha_2 \cdots \alpha_n$. That means that each of the roots of $g(x)$ – which are the α_i – are factors of the degree 0 coefficient of $g(x)$.

Now it's not true that every polynomial has integer roots, but many of the polynomials you will come across do, so the two paragraphs above offer a powerful hint as to what the roots of a polynomial might be.

When searching for roots of a polynomial
whose coefficients are all integers,
check the factors of the degree 0 coefficient.

Example. 3 and -7 are both roots of $2(x-3)(x+7)$.

Notice that $2(x-3)(x+7) = 2x^2 + 8x - 42$, and that 3 and -7 are both factors of -42.

Example. Suppose $p(x) = 3x^4 + 3x^3 - 3x^2 + 3x - 6$. This is a degree 4 polynomial, so it will have at most 4 roots. There isn't a really easy way to find the roots of a degree 4 polynomial, so to find the roots of $p(x)$, we have to start by guessing.

The degree 0 coefficient of $p(x)$ is -6, so a good place to check for roots is in the factors of -6.

The factors of -6 are 1, -1, 2, -2, 3, -3, 6, and -6, so we have eight quick candidates for what the roots of $p(x)$ might be. A quick check shows that of these eight candidates, exactly two are roots of $p(x)$ – namely 1 and -2. That is to say, $p(1) = 0$ and $p(-2) = 0$.

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Factoring cubics

It follows from the Fundamental Theorem of Algebra that a cubic polynomial is either the product of a constant and three linear polynomials, or else it is the product of a constant, one linear polynomial, and one quadratic polynomial that has no roots.

In either case, any cubic polynomial is guaranteed to have a linear factor, and thus is guaranteed to have a root. You're going to have to guess what that root is by looking at the factors of the degree 0 coefficient. (There is a "cubic formula" that like the quadratic formula will tell you the roots of a cubic, but the formula is difficult to remember, and you'd need to know about complex numbers to be able to use it.)

Once you've found a root, factor out the linear factor that the root gives you. You will now be able to write the cubic as a product of a monic linear polynomial and a quadratic polynomial. Completely factor the quadratic and then you will have completely factored the cubic.

Problem. Completely factor $2x^3 - 3x^2 + 4x - 3$.

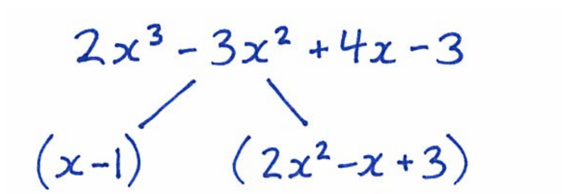
Solution. Start by guessing a root. The degree 0 coefficient is -3, and the factors of -3 are 1, -1, 3, and -3. Check these factors to see if any of them are roots.

After checking, you'll see that 1 is a root. That means that $x-1$ is a factor of $2x^3 - 3x^2 + 4x - 3$. Therefore, we can divide $2x^3 - 3x^2 + 4x - 3$ by $x - 1$ to get another polynomial

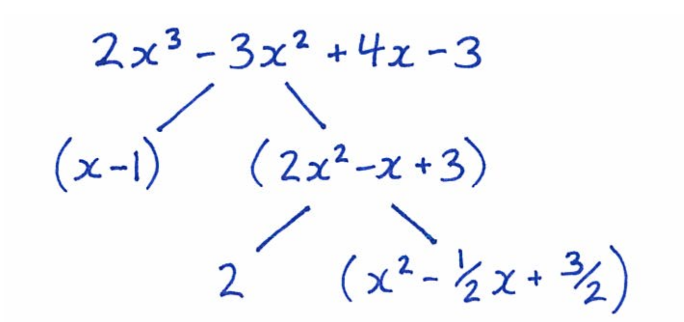
$$\frac{2x^3 - 3x^2 + 4x - 3}{x - 1} = 2x^2 - x + 3$$

Thus,

$$2x^3 - 3x^2 + 4x - 3 \ = \ (x - 1)(2x^2 - x + 3)$$



The discriminant of $2x^2 - x + 3$ equals $(-1)^2 - 4(2)(3) = 1 - 24 = -23$, a negative number. Therefore, $2x^2 - x + 3$ has no roots, so to completely factor $2x^2 - x + 3$ we just have to factor out the leading coefficient as follows: $2x^2 - x + 3 = 2\big(x^2 - \frac{1}{2}x + \frac{3}{2}\big)$.



The final answer is

$$2(x - 1)\Big(x^2 - \frac{1}{2}x + \frac{3}{2}\Big)$$

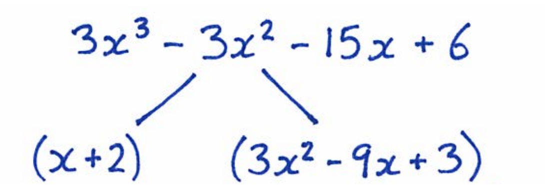
Problem. Completely factor $3x^3 - 3x^2 - 15x + 6$.

Solution. The factors of 6 are $\{1, -1, 2, -2, 3, -3, 6, -6\}$. Check to see that -2 is a root. Then divide by $x + 2$ to find that

$$\frac{3x^3 - 3x^2 - 15x + 6}{x + 2} = 3x^2 - 9x + 3$$

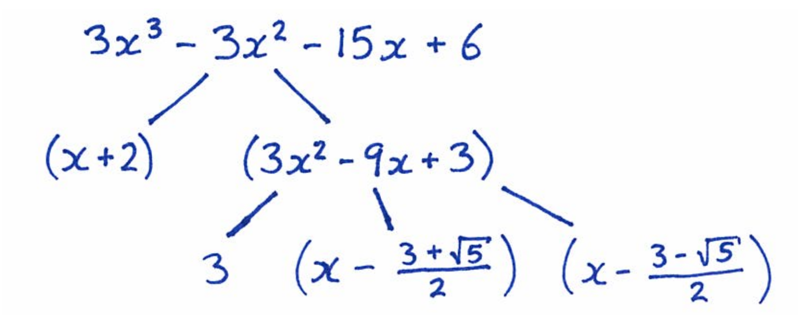
so

$$3x^3 - 3x^2 - 15x + 6 = (x + 2)(3x^2 - 9x + 3)$$



The discriminant of $3x^2 - 9x + 3$ equals 45, and thus $3x^2 - 9x + 3$ has two roots and can be factored further.

The leading coefficient of $3x^2 - 9x + 3$ is 3, and we can use the quadratic formula to check that the roots of $3x^2 - 9x + 3$ are $\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$. From what we learned above about factoring quadratics, we know that $3x^2 - 9x + 3 = 3(x - \frac{3+\sqrt{5}}{2})(x - \frac{3-\sqrt{5}}{2})$.



To summarize,

$$\begin{aligned} 3x^3 - 3x^2 - 15x + 6 &= (x + 2)(3x^2 - 9x + 3) \\ &= (x + 2)3\Big(x - \frac{3 + \sqrt{5}}{2}\Big)\Big(x - \frac{3 - \sqrt{5}}{2}\Big) \\ &= 3(x + 2)\Big(x - \frac{3 + \sqrt{5}}{2}\Big)\Big(x - \frac{3 - \sqrt{5}}{2}\Big) \end{aligned}$$

Factoring quartics

Degree 4 polynomials are tricky. As with cubic polynomials, you should begin by checking whether the factors of the degree 0 coefficient are roots. If one of them is a root, then you can use the same basic steps that we used with cubic polynomials to completely factor the polynomial.

The problem with degree 4 polynomials is that there's no reason that a degree 4 polynomial has to have any roots – take $(x^2+1)(x^2+1)$ for example.

Because a degree 4 polynomial might not have any roots, it might not have any linear factors, and it's very hard to guess which quadratic polynomials it might have as factors.

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Exercises

Completely factor the polynomials given in #1-8

1.) $10x + 20$

2.) $-2x + 5$

3.) $-2x^2 - 12x - 18$

4.) $10x^2 + 3$

5.) $3x^2 - 10x + 5$

6.) $3x^2 - 4x + 5$

7.) $-2x^2 + 6x - 3$

8.) $5x^2 + 3x - 2$

9.) Find a root of $x^3 - 5x^2 + 10x - 8$.

10.) Find a root of $15x^3 + 35x^2 + 30x + 10$.

11.) Find a root of $x^3 - 2x^2 - 2x - 3$.

Completely factor the polynomials in #12-16.

12.) $-x^3 - x^2 + x + 1$

13.) $5x^3 - 9x^2 + 8x - 20$

14.) $-2x^3 + 17x - 3$

15.) $4x^3 - 20x^2 + 25x - 3$

16.) $x^4 - 5x^2 + 4$

17.) How can the Fundamental Theorem of Algebra be used to show that any polynomial of odd degree has at least one root?

Graphing Polynomials

In the previous chapter, we learned how to factor a polynomial. In this chapter, we'll use the completely factored form of a polynomial to help us graph it.

The far right and far left of a polyniomial graph

Suppose $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$ is a polynomial.

If M is a really big number, then M^n is much bigger than M^{n-1}. (For example, if $M = 1000$ then M^n is one thousand times bigger than M^{n-1}.)

In fact, if M is a really, really big number then M^n is much bigger than $a_{n-1}M^{n-1}$, or $a_{n-2}M^{n-2}$, or $a_{n-3}M^{n-3}$, and so on.

Actually, if M is a really really big number, then $a_n M^n$ is much bigger than the numbers in the previous paragraph, and it even dwarfs their sum:

$$a_{n-1}M^{n-1} + a_{n-2}M^{n-2} + \cdots + a_0$$

That means that for really, really big numbers M in the domain of the polynomial $p(x)$, the size of

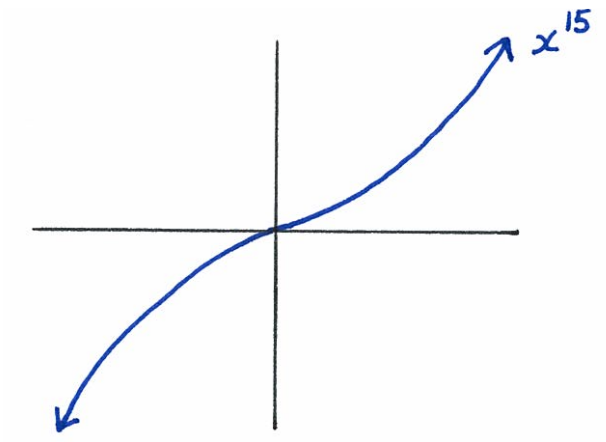
$$p(M) = a_n M^n + a_{n-1}M^{n-1} + a_{n-2}M^{n-2} + \cdots + a_0$$

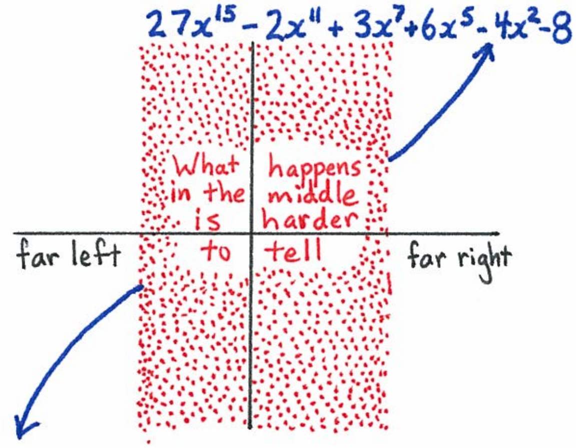
is basically determined by $a_n M^n$. That's because while the rest of $p(M)$ — which is $a_{n-1}M^{n-1} + a_{n-2}M^{n-2} \cdots + a_0$ — might be large, it is so small in comparison to $a_n M^n$ that it's hard to notice it. In other words, while $p(M)$ does not equal $a_n M^n$, it's hard to tell the difference between the two in the same way that it would be hard to tell the difference between a bag of $10,000,576$ pennies and a bag of $10,002,073$ pennies; both bags have about 10 million pennies.

The end result is that the graph of $p(x)$ looks an awful lot like the graph of $a_n x^n$ over the part of the x-axis that has the really, really big numbers: the extreme right portion of the x-axis.

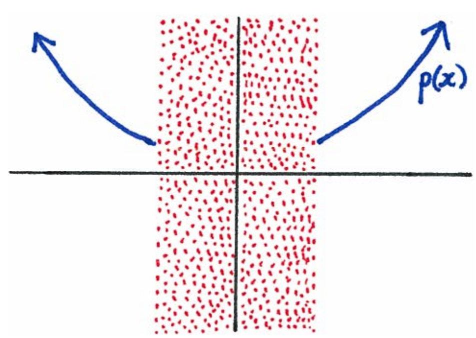
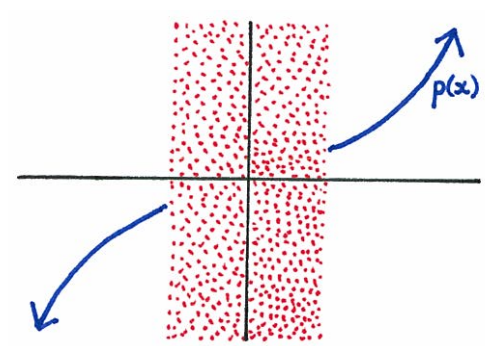
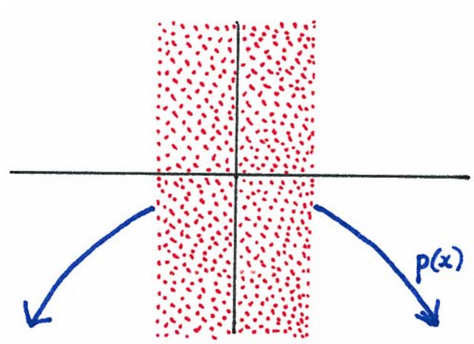
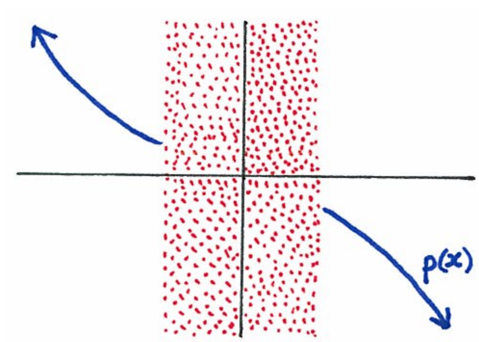
The graph of $p(x)$ also looks an awful lot like the graph of $a_n x^n$ over the extreme left portion of the x-axis.

Example. Over the far left portion of the x-axis, and over the far right portion of the x-axis, the graph of $q(x) = 27x^{15} - 2x^{11} + 3x^7 + 6x^5 - 4x^2 - 8$ basically looks like the graph of its leading term: $27x^{15}$. And the graph of $27x^{15}$ is the graph of x^{15} stretched by 27, which is pretty much the same graph as the graph for x^{15}.





The chart below gives a rough sketch of what the far right and far left portion of the graph of a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ looks like (if $n \geq 2$). It just depends on the leading term, $a_n x^n$. What the middle portion of the graph looks like is harder to say.

	n even	n odd
$a_n > 0$		
$a_n < 0$		

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Steps for graphing a completely factored polynomial $p(x)$.

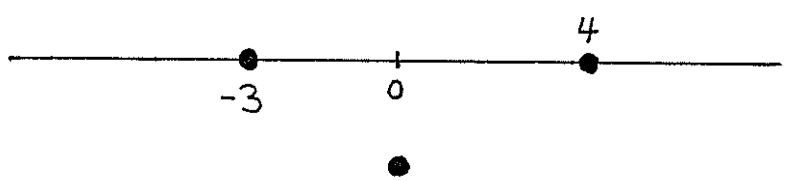
1: The roots of $p(x)$ are the x-intercepts. You can read them off from the monic linear factors of a completely factored polynomial.

The polynomial $p(x) = -5(x+3)(x-4)(x-4)(x^2+2x+6)$ is completely factored. Its roots are -3 and 4.

2: Pick any number on the x-axis between consecutive pairs of x-intercepts.

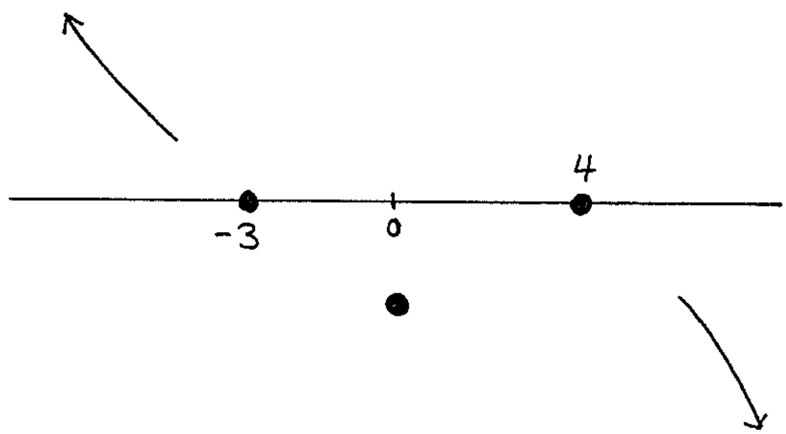
Let's say the number you picked was b. If $p(b) > 0$, but a giant dot directly above the b. Put a giant dot directly below b if $p(b) < 0$.

0 is a number in between -3 and 4, and for $p(x) = -5(x+3)(x-4)(x-4)(x^2+2x+6)$ we have $p(0) = -5(3)(-4)(-4)(6) < 0$, so we can put a giant dot directly below 0.

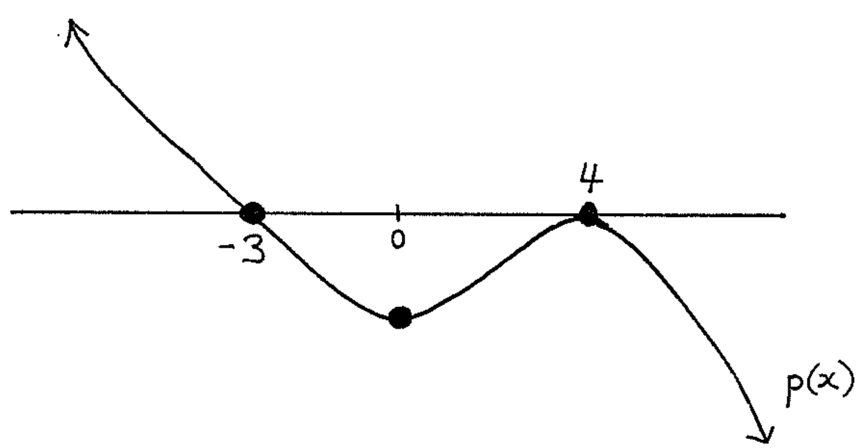


3: The far right and left portion of the graph of $p(x)$ looks like the graph of its leading term. Draw what the graph of the leading term looks like on the far right and left sides of your picture.

The leading term of $p(x) = -5(x+3)(x-4)(x-4)(x^2+2x+6)$ is $-5x^5$, and that looks like the graph of x^5 turned upside down.



4: Draw any type of smooth, curvy, and continuous line that passes through all of the points in $\mathbb{R}^2$ that you labeled from Steps 1 and 2, that does not touch the x-axis at any points not listed in Step 1, and that meets up with the pieces of the graph you drew in Step 3.



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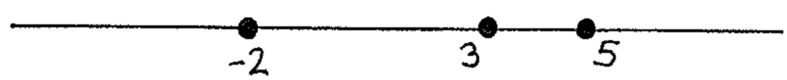
Problem. The polynomial $q(x) = 7(x+2)(x-3)(x-5)(x^2+1)$ is completely factored. Graph it.

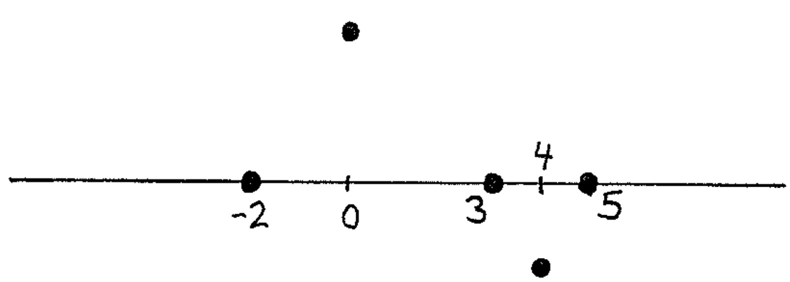
Solution.

1: The monic linear factors of $q(x)$ are $(x+2)$, $(x-3)$, and $(x-5)$. So the roots of $q(x)$ are -2, 3, and 5. Draw these three points on the x-axis.

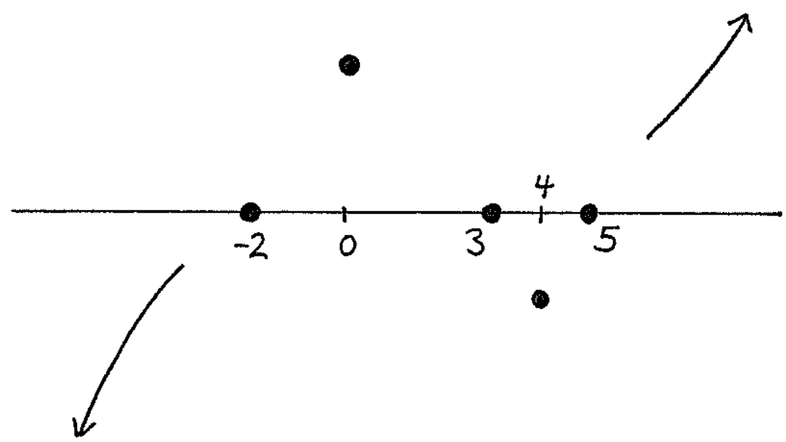


2: Choose any number between -2 and 3, for example, the number 0 is between -2 and 3. Then check to see if $q(0)$ is positive or negative: $q(0) = 7(0+2)(0-3)(0-5)(0^2+1) = 7(2)(-3)(-5)(1)$ is a positive number, so draw a dot above 0.

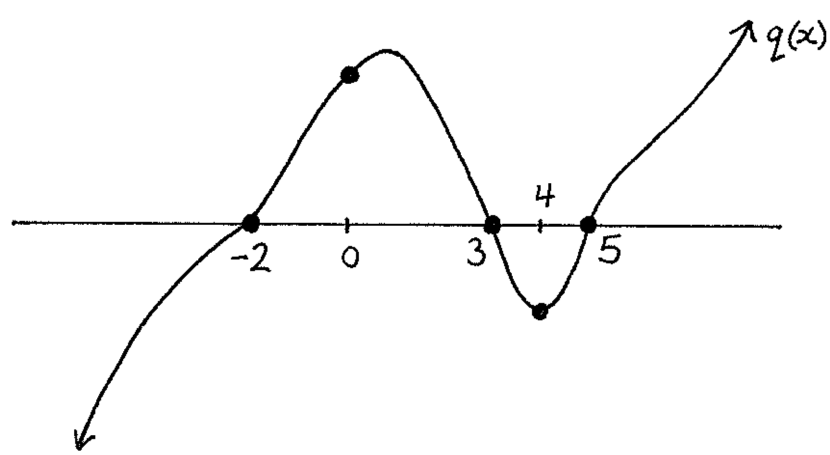
Similarly, choose a point between 3 and 5, say 4. Check that $q(4) = 7(4+2)(4-3)(4-5)(4^2+1) = 7(6)(1)(-1)(17)$ is negative, so we draw a dot below 4.



3: The leading term of $q(x)$ is $7x^5$. Draw the part of $7x^5$ that is to the left of everything you've drawn in your picture so far. Draw the part of the graph of $7x^5$ that is to the right of everything you've drawn so far.



4: Draw the graph of a function that connects everything you've drawn, but make sure it only touches the x-axis at the x-intercepts that you've already labelled. That is more or less what the graph of $q(x)$ looks like. That's our answer.



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Exercises

For #1-6, graph the given completely factored polynomials.

1.) $4(x-3)(x-5)$

2.) $-(x-1)(x^2+x+5)$

3.) $-6(x+4)(x+4)(x-2)(x-3)(x^2+1)(x^2+3)$

4.) $2(x-3)(x-3)(x-3)(x-6)(x-6)(x^2+2x+7)$

5.) $-(x-1)(x-2)(x-2)(x-3)(x^2+x+5)$

6.) $5(x-4)(x^2-2x+4)(x^2+3x+5)(x^2+4)$

For #7-10, first completely factor, and then graph, the given polynomials.

7.) $2x^2+2x-24$

8.) $7x^2-3x+4$

9.) $-x^3+6x^2+7x$

10.) $3x^4-9x^2-12$

11.) How can what we know about the far right and left of the graph of an odd degree polynomial be used to show that any polynomial of odd degree has at least one root?

Exponential Functions

Exponential Functions

In this chapter, a will always be a positive number.

For any positive number $a > 0$, there is a function $f : \mathbb{R} \to (0, \infty)$ called an *exponential function* that is defined as $f(x) = a^x$.

For example, $f(x) = 3^x$ is an exponential function, and $g(x) = (\frac{4}{17})^x$ is an exponential function.

There is a big difference between an exponential function and a polynomial. The function $p(x) = x^3$ is a polynomial. Here the "variable", x, is being raised to some constant power. The function $f(x) = 3^x$ is an exponential function; the variable is the exponent.

Rules for exponential functions

Here are some algebra rules for exponential functions that will be explained in class.

If $n \in \mathbb{N}$, then a^n is the product of n a's. For example, $3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$

$$a^0 = 1$$

If $n, m \in \mathbb{N}$, then

$$a^{\frac{n}{m}} = \sqrt[m]{a^n} = (\sqrt[m]{a})^n$$

$$a^{-x} = \frac{1}{a^x}$$

The rules above were designed so that the following most important rule of exponential functions holds:

$$a^x a^y = a^{x+y}$$

Another variant of the important rule above is

$$\frac{a^x}{a^y} = a^{x-y}$$

And there is also the following slightly related rule

$$(a^x)^y = a^{xy}$$

Examples.

- $4^{\frac{1}{2}} = \sqrt[2]{4} = 2$
- $7^{-2} \cdot 7^6 \cdot 7^{-4} = 7^{-2+6-4} = 7^0 = 1$
- $10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$
- $\frac{15^6}{15^5} = 15^{6-5} = 15^1 = 15$
- $(2^5)^2 = 2^{10} = 1024$
- $(3^{20})^{\frac{1}{10}} = 3^2 = 9$
- $8^{-\frac{2}{3}} = \frac{1}{(8)^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2} = \frac{1}{4}$

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The base of an exponential function

If $f(x) = a^x$, then we call a the *base* of the exponential function. The base must always be positive.

Base 1

If $f(x)$ is an exponential function whose base equals 1 – that is if $f(x) = 1^x$ – then for $n, m \in \mathbb{N}$ we have

$$f\left(\frac{n}{m}\right) = 1^{\frac{n}{m}} = \sqrt[m]{1^n} = \sqrt[m]{1} = 1$$

In fact, for any real number x, $1^x = 1$, so $f(x) = 1^x$ is the same function as the constant function $f(x) = 1$. For this reason, we usually don't talk much about the exponential function whose base equals 1.

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Graphs of exponential functions

It's really important that you know the general shape of the graph of an exponential function. There are two options: either the base is greater than 1, or the base is less than 1 (but still positive).

Base greater than 1. If a is greater than 1, then the graph of $f(x) = a^x$ grows taller as it moves to the right. To see this, let $n \in \mathbb{Z}$. We know that $1 < a$, and we know from our rules of inequalities that we can multiply both sides of this inequality by a positive number. The positive number we'll multiply by is a^n, so that we'll have

$$a^n(1) < a^n a$$

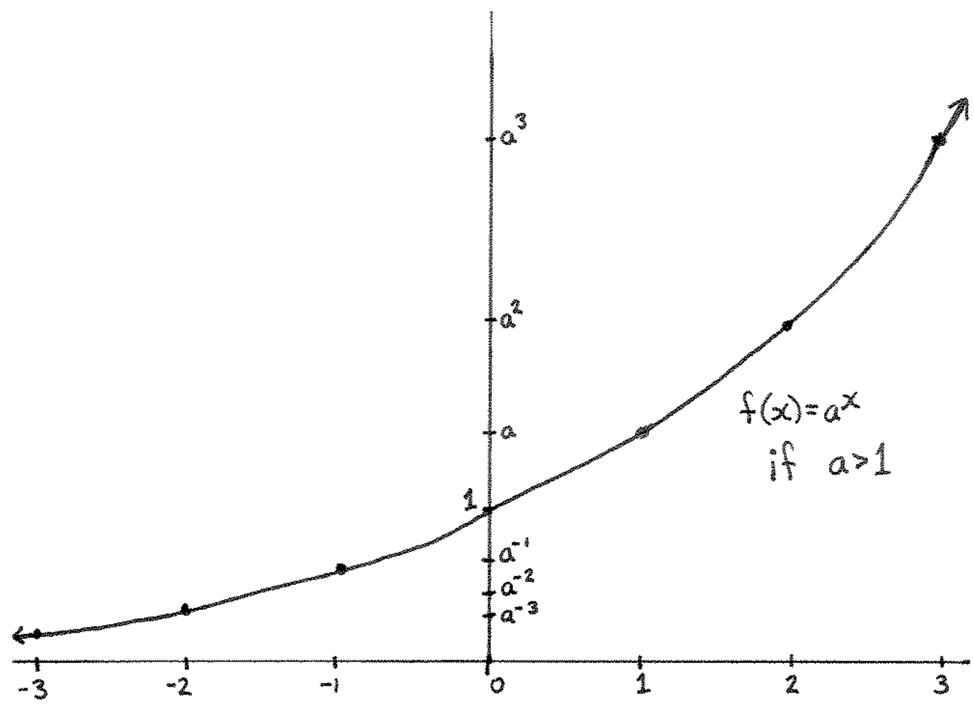
Because $a^n(1) = a^n$ and $a^n a = a^{n+1}$, the inequality above is the same as

$$a^n < a^{n+1}$$

Because the last inequality we found is true for any $n \in \mathbb{Z}$, we actually have an entire string of inequalities:

$$\cdots < a^{-3} < a^{-2} < a^{-1} < a^0 < a^1 < a^2 < a^3 < \cdots$$

Keeping in mind that a^x is positive for any number x, and that $a^0 = 1$, we now have a pretty good idea of what the graph of $f(x) = a^x$ looks like if $a > 1$. The y-intercept is at 1; when moving to the right, the graph grows taller and taller; and when moving to the left, the graph becomes shorter and shorter, shrinking towards, but never touching, the x-axis.



Not only does the graph grow bigger as it moves to the right, but it gets big in a hurry. For example, if we look at the exponential function whose base is 2, then

$$f(64) = 2^{64} = 18,446,744,073,709,525,000$$

And 2 isn't even a very big number to be using for a base (any positive number can be a base, and plenty of numbers are much, much bigger than 2). The bigger the base of an exponential function, the faster its graph grows as it moves to the right.

Moving to the left, the graph of $f(x) = a^x$ grows small very quickly if $a > 1$. Again if we look at the exponential function whose base is 2, then

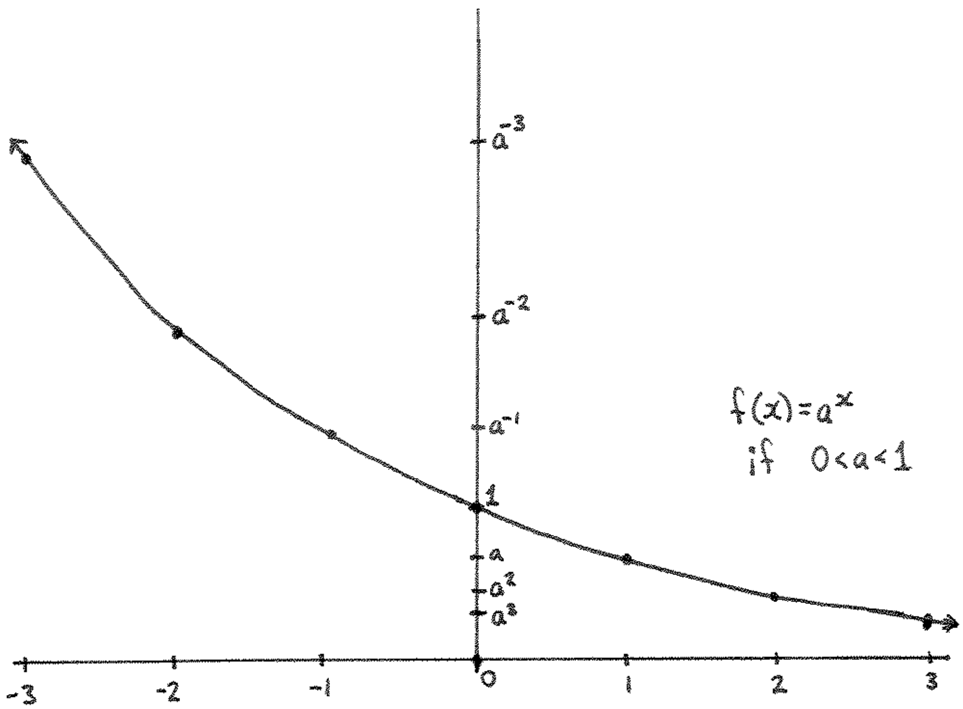
$$f(-10) = 2^{-10} = \frac{1}{2^{10}} = \frac{1}{1024}$$

The bigger the base, the faster the graph of an exponential function shrinks as it moves to the left.

Base less than 1 (but still positive). If a is positive and less than 1, then we we can show from our rules of inequalities that $a^{n+1} < a^n$ for any $n \in \mathbb{Z}$. That means that

$$\cdots > a^{-3} > a^{-2} > a^{-1} > a^0 > a^1 > a^2 > a^3 > \cdots$$

So the graph of $f(x) = a^x$ when the base is smaller than 1 slopes down as it moves to the right, but it is always positive. As it moves to the left, the graph grows tall very quickly.



One-to-one and onto

Recall that an exponential function $f : \mathbb{R} \to (0, \infty)$ has as its domain the set $\mathbb{R}$ and has as its target the set $(0, \infty)$.

We see from the graph of $f(x) = a^x$, if either $a > 1$ or $0 < a < 1$, that $f(x)$ is one-to-one and onto. Remember that to check if $f(x)$ is one-to-one, we can use the horizontal line test (which $f(x)$ passes). To check what the range of $f(x)$ is, we think of compressing the graph of $f(x)$ onto the y-axis. If we did that, we would see that the range of $f(x)$ is the set of positive numbers, $(0, \infty)$. Since the range and target of $f(x)$ are the same set, $f(x)$ is onto.

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Where exponential functions appear

Exponential functions are closely related to geometric sequences. They appear whenever you are multiplying by the same number over and over and over again.

The most common example is in population growth. If a population of a group increases by say 5% every year, then every year the total population is multiplied by 105%. That is, after one year the population is 1.05 times what it originally was. After the second year, the population will be $(1.05)^2$ times what it originally was. After 100 years, the population will be $(1.05)^{100}$ times what it originally was. After x years, the population will be $(1.05)^x$ times what it originally was.

Interest rates on credit cards measure a population growth of sorts. If your credit card charges you 20% interest every year, then after 5 years of not making payments, you will owe $(1.20)^5 = 2.48832$ times what you originally charged on your credit card. After x years of not making payments, you will owe $(1.20)^x$ times what you originally charged.

Sometimes a quantity decreases exponentially over time. This process is called *exponential decay.*

If a tree dies to become wood, the amount of carbon in it decreases by 0.0121% every year. Scientists measure how much carbon is in something that died, and use the exponential function $f(x) = (0.999879)^x$ to figure out when it must have died. (The number 0.999879 is the base of this exponential function because $0.999879 = 1 - 0.000121$.) This technique is called carbon dating and it can tell us about history. For example, if scientists discover that the wood used to build a fort came from trees that died 600 years ago, then the fort was probably built 600 years ago.

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e

Some numbers are so important in math that they get their own name. One such number is e. It is a real number, but it is not a rational number. It's very near to – but not equal to – the rational number $\frac{27}{10} = 2.7$. The importance of the number e becomes more apparent after studying calculus, but it can be explained without calculus.

Let's say you just bought a new car. You're driving it off the lot, and the odometer says that it's been driven exactly 1 mile. You are pulling out of the lot slowly at 1 mile per hour, and for fun you decide to keep the odometer and the speedometer so that they always read the same number.

After something like an hour, you've driven one mile, and the odometer says 2, so you accelerate to 2 miles per hour. After driving for something like a half hour, the odometer says 3, so you speed up to 3 miles an hour. And you continue in this fashion.

After some amount of time, you've driven 100 miles, so you are moving at a speed of 100 miles per hour. The odometer will say 101 after a little while, and then you'll have to speed up. After you've driven 1000 miles (and here's where the story starts to slide away from reality) you'll have to speed up to 1000 miles per hour. Now it will be just around 3 seconds before you have to speed up to 1001 miles per hour.

You're traveling faster and faster, and as you travel faster, it makes you travel faster, which makes you travel faster still, and things get out of hand very quickly, even though you started out driving at a very reasonable speed of 1 mile per hour.

If x is the number of hours you had been driving for, and $f(x)$ was the distance the car had travelled at time x, then $f(x)$ is the exponential function with base e. In symbols, $f(x) = e^x$.

Calculus studies the relationship between a function and the slope of the graph of the function. In the previous example, the function was distance travelled, and the slope of the distance travelled is the speed the car is moving at. The exponential function $f(x) = e^x$ has at every number x the same "slope" as the value of $f(x)$. That makes it a very important function for calculus.

For example, at $x = 0$, the slope of $f(x) = e^x$ is $f(0) = e^0 = 1$. That means when you first drove off the lot ($x = 0$) the odometer read 1 mile, and your speed was 1 mile per hour. After 10 hours of driving, the car will have travelled e^{10} miles, and you will be moving at a speed of e^{10} miles per hour. (By the way, e^{10} is about $22,003$.)

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Exercises

For #1-11, write each number in simplest form without using a calculator, as was done in the "Examples" in this chapter. (On exams you will be asked to simplify problems like these without a calculator.)

1.) 8^{-1}

2.) $(\frac{17}{43})^3(\frac{17}{43})^{-3}$

3.) $125^{-\frac{1}{3}}$

4.) $100^{-5} \cdot 100^{457} \cdot 100^{-50} \cdot 100^{-400}$

5.) $4^{-\frac{3}{2}}$

6.) $(3^{200})^{\frac{1}{100}}$

7.) $1000^{\frac{2}{3}}$

8.) $97^{-16}\, 97^{15}$

9.) $36^{\frac{3}{6}}$

10.) $\frac{3^{297}}{3^{300}}$

11.) $(5^{\frac{4}{7}})^{\frac{14}{4}}$

12.) Graph $f(x) = 100(\frac{1}{2})^x$

13.) Graph $f(x) = 3^{(x-4)} + 7$

14.) Graph $f(x) = -2^{(x+5)}$

15.) Graph $f(x) = (\frac{4}{17})^{-x} - 10$

For #16-24, decide which is the only number x that satisfies the given equation.

16.) $4^x = 16$

17.) $2^x = 8$

18.) $10^x = 10,000$

19.) $3^x = 9$

20.) $5^x = 125$

21.) $(\frac{1}{2})^x = 16$

22.) $(\frac{1}{4})^x = 64$

23.) $8^x = \frac{1}{4}$

24.) $27^x = \frac{1}{9}$

25.) Suppose you accidentally open a canister of plutonium in your living room and 160 units of radiation leaks out. If every year, there is half as much radiation as there was the year before, will your living room ever be free of radiation? How many units of radiation will there be after 4 years?

26.) Your uncle has an investment scheme. He guarantees that if you invest in the stock of his company, then you'll earn 10% on your money every year. If you invest \$100, and you uncle is right, how much money will you have after 20 years?

Rational Functions

Rational Functions

In this chapter, you'll learn what a rational function is, and you'll learn how to sketch the graph of a rational function.

Rational functions

A *rational function* is a fraction of polynomials. That is, if $p(x)$ and $q(x)$ are polynomials, then

$$\frac{p(x)}{q(x)}$$

is a rational function. The *numerator* is $p(x)$ and the *denominator* is $q(x)$.

Examples.

- $\frac{3(x-5)}{(x-1)}$
- $\frac{1}{x}$
- $\frac{2x^3}{1} = 2x^3$

The last example is both a polynomial and a rational function. In a similar way, any polynomial is a rational function.

In this class, from this point on, most of the rational functions that we'll see will have both their numerators and their denominators completely factored.
We will also only see examples where the numerator and the denominator have no common factors. (If they did have a common factor, we could just cancel them.)

*   *   *   *   *   *   *   *   *   *   *   *   *

Implied domains

The implied domain of a rational function is the set of all real numbers *except* for the roots of the denominator. That's because it doesn't make sense to divide by 0.

Example. The implied domain of

$$\frac{-7(x-2)(x^2+1)}{8(x-4)(x-6)}$$

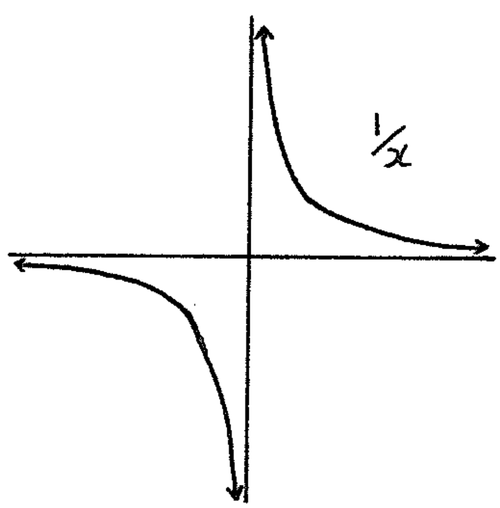
is the set $\mathbb{R} - \{4, 6\}$.

Vertical asymptotes

To graph a rational function, begin by marking every number on the x-axis that is a root of the denominator. (The denominator might not have any roots.)

Draw a vertical dashed line through these points. These vertical lines are called *vertical asymptotes*. The graph of the rational function will "climb up" or "slide down" the sides of a vertical asymptote.

Examples. For the rational function $\frac{1}{x}$, 0 is the only root of the denominator, so the y-axis is the vertical asymptote. Notice that the graph of $\frac{1}{x}$ climbs up the right side of the y-axis and slides down the left side of the y-axis.



The rational function

$$\frac{-7(x-2)(x^2+1)}{8(x-4)(x-6)}$$

has vertical asymptotes at $x = 4$ and at $x = 6$.

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x-intercepts

The x-intercepts of a rational function $\frac{p(x)}{q(x)}$ (if there are any) are the numbers $\alpha \in \mathbb{R}$ where

$$\frac{p(\alpha)}{q(\alpha)} = 0$$

If α is such a number, then we can multiply by $q(\alpha)$ to find that

$$p(\alpha) = 0 \cdot q(\alpha) = 0$$

In other words, α is a root of $p(x)$. Thus, the roots of the numerator are exactly the x-intercepts.

Example. 2 is the only x-intercept of the rational function

$$\frac{-7(x-2)(x^2+1)}{8(x-4)(x-6)}$$

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In between x-intercepts and vertical asymptotes

When graphing a rational polynomial, first mark the vertical asymptotes and the x-intercepts. Then choose a number $c \in \mathbb{R}$ between any consecutive pairs of these marked points on the x-axis and see if the rational function is positive or negative when $x = c$. If it's positive, draw a dot above the x-axis whose first coordinate is c. If it's negative, draw a dot below the x-axis whose first coordinate is c.

Example. Let's look at the function

$$r(x) = \frac{-7(x-2)(x^2+1)}{8(x-4)(x-6)}$$

again. The x-intercept of its graph is at $x = 2$ and it has vertical asymptotes at $x = 4$ and $x = 6$. We need to decide whether $r(x)$ is positive or negative between 2 and 4 on the x-axis, and between 4 and 6 on the x-axis.

Let's start by choosing a number between 2 and 4, say 3. Then

$$r(3) = \frac{-7(3-2)(3^2+1)}{8(3-4)(3-6)}$$

Notice that -7, $(3-4)$, and $(3-6)$ are negative, while 8, $(3-2)$, and (3^2+1) are positive.

If you are multiplying and dividing a collection of numbers that aren't equal to 0, just count how many negative numbers there are. If there is an even number of negatives, the result will be positive. If there is an odd number of negatives, the result will be negative. In the previous paragraph, there are three negative numbers — -7, $(3-4)$, and $(3-6)$ — so $r(3) < 0$.

The number 5 is a number that is in between 4 and 6, and

$$r(5) = \frac{-7(5-2)(3^2+1)}{8(5-4)(5-6)} > 0$$

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Far right and far left

Let ax^n be the leading term of $p(x)$ and let bx^m be the leading term of $q(x)$.

Recall that far to the right and left, $p(x)$ looks like its leading term, ax^n. And far to the right and left, $q(x)$ looks like its leading term, bx^m. It follows that the far right and left portion of the graph of,

$$\frac{p(x)}{q(x)}$$

looks like

$$\frac{ax^n}{bx^m}$$

and this is a function that we know how to graph.

Example. The leading term of $-7(x-2)(x^2+1)$ is $-7x^3$, and the leading term of $8(x-4)(x-6)$ is $8x^2$. Therefore, the graph of

$$r(x) = \frac{-7(x-2)(x^2+1)}{8(x-4)(x-6)}$$

looks like the graph of

$$\frac{-7x^3}{8x^2} = \frac{-7}{8}x$$

on the far left and far right part of its graph.

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Putting the graph together

To graph a rational function

$$\frac{p(x)}{q(x)}$$

mark its vertical asymptotes (if any). Mark its x-intercepts (if any). Determine whether the function is positive or negative in between x-intercepts and vertical asymptotes.

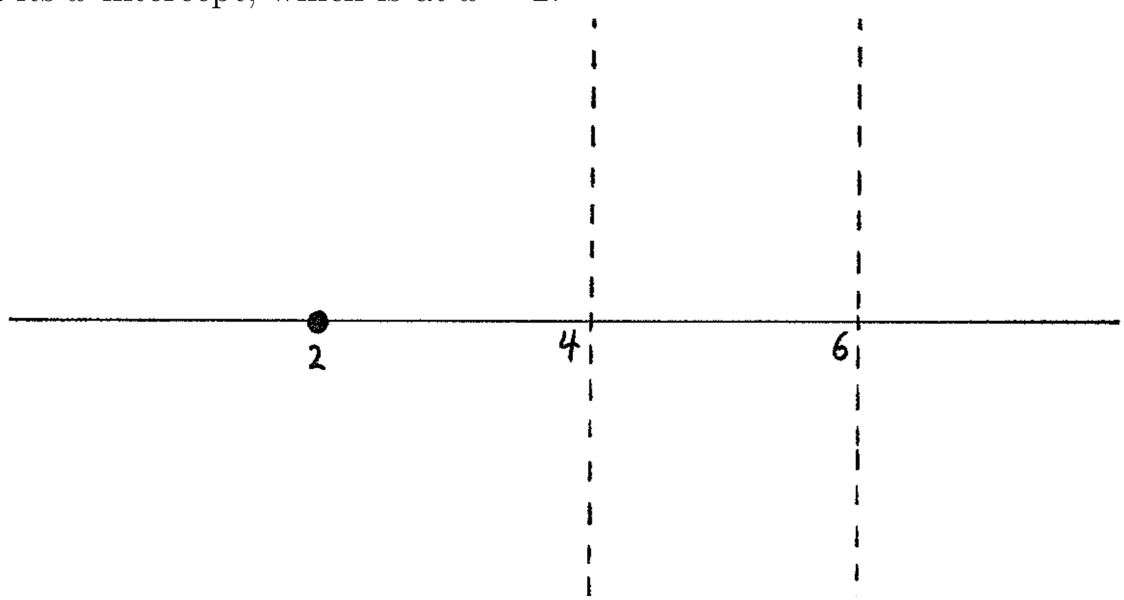
Replace $p(x)$ with its leading term, replace $q(x)$ with its leading term, and then graph the resulting fraction of leading terms to the right and left of everything you've drawn so far in your graph.

Now draw a reasonable looking graph that fits with everything you've drawn so far, remembering that the graph has to climb up or slide down the sides of vertical asymptotes, and that the graph can only touch the x-axis at the x-intercepts that you already marked.

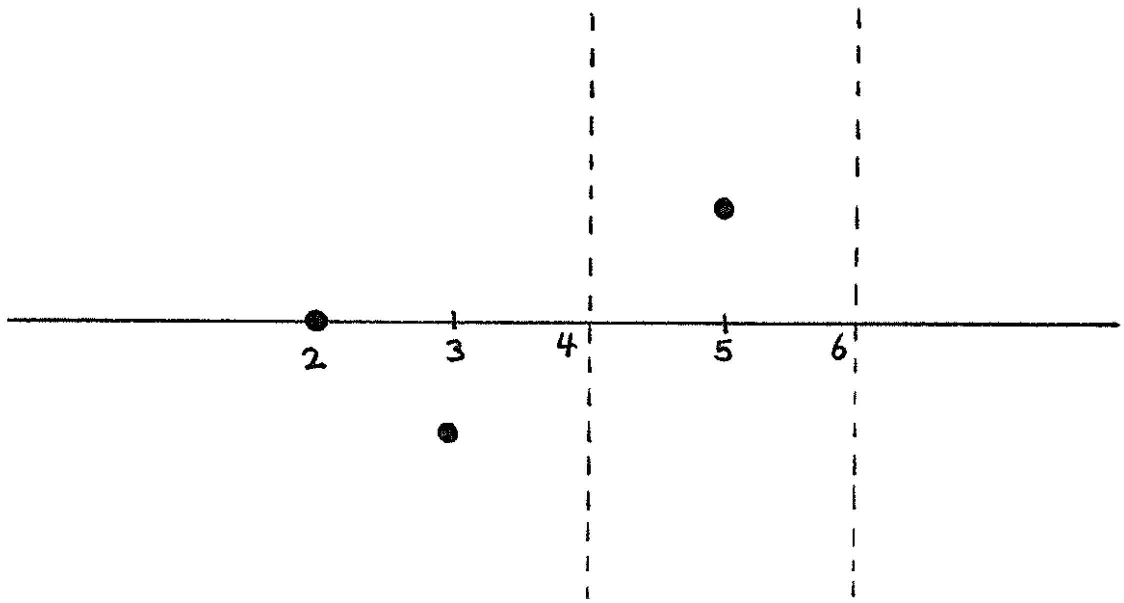
Example. Let's graph

$$r(x) = \frac{-7(x-2)(x^2+1)}{8(x-4)(x-6)}$$

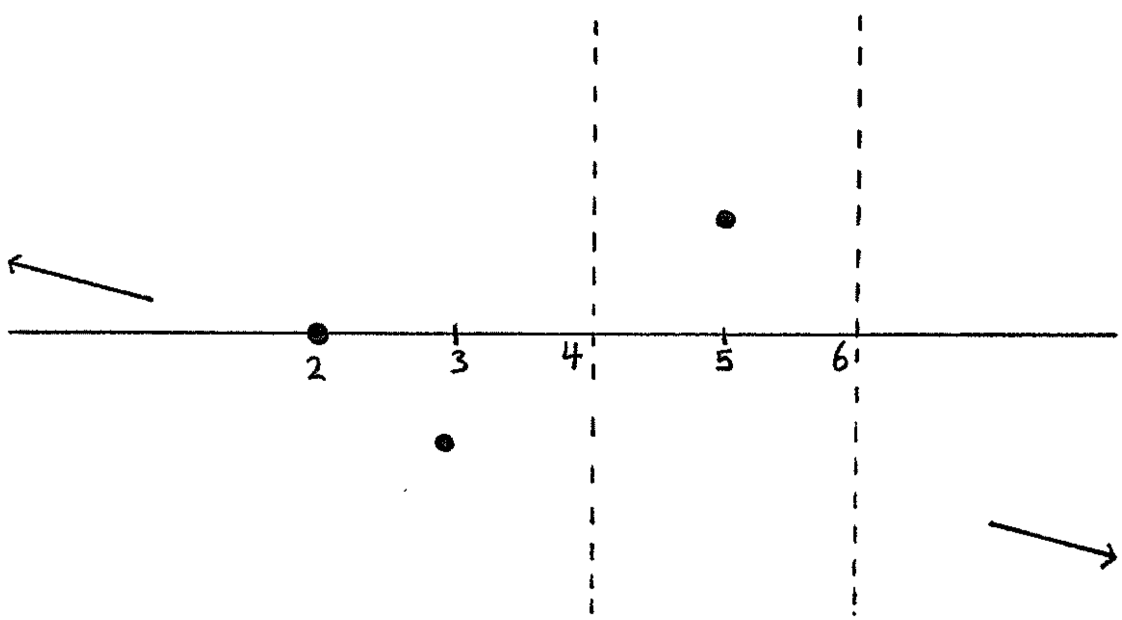
First we mark its its vertical asymptotes, which are at $x = 4$ and $x = 6$, and its x-intercept, which is at $x = 2$.



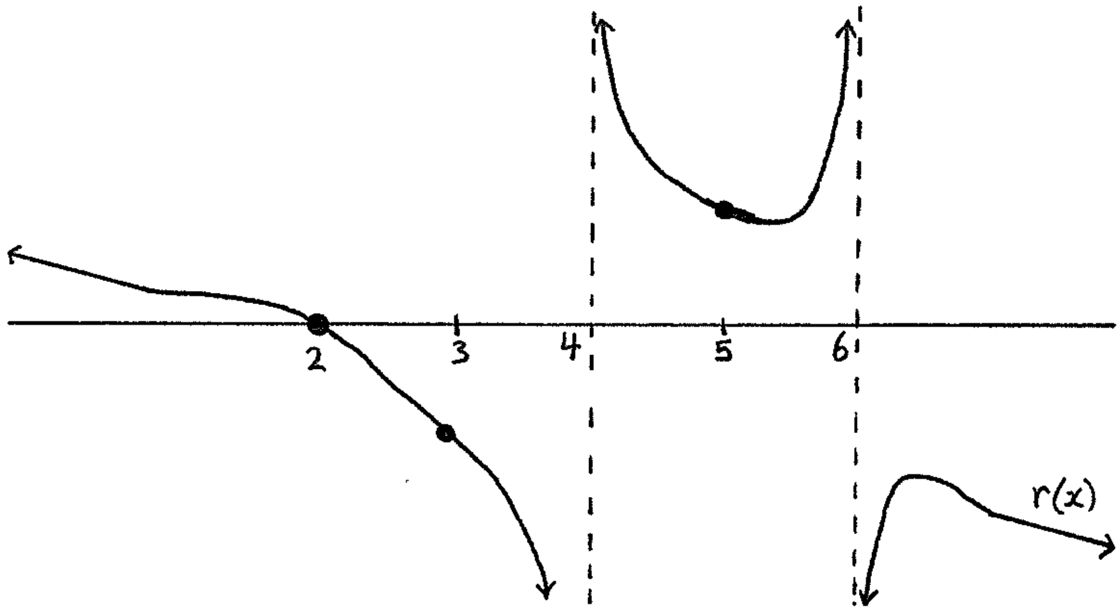
Then we plot points that represent what we had checked earlier for what happens in between consecutive pairs of x-intercepts and vertical asymptotes: that $r(3) < 0$ and $r(5) > 0$.



To the left and right of what we've graphed so far, we draw the graph of $\frac{-7}{8}x$.



Now we connect what we've drawn so far, making sure our graph climbs up or slides down the vertical asymptotes, and that it only touches the x-axis at the previously labelled x-intercept.



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Exercises

For #1-3, use that $4x^2-4=4(x-1)(x+1)$, $x^3-3x^2+4=(x+1)(x-2)^2$, and $2x-4=2(x-2)$ to match each of the three numbered rational functions on the left with its simplified lettered form on the right.

1.) $\frac{4x^2-4}{x^3-3x^2+4}$

2.) $\frac{x^3-3x^2+4}{2x-4}$

3.) $\frac{2x-4}{4x^2-4}$

A.) $\frac{1}{2}(x+1)(x-2)$

B.) $\frac{4(x-1)}{(x-2)^2}$

C.) $\frac{(x-2)}{2(x-1)(x+1)}$

Graph the rational functions given in #4-10. (Their numerators and denominators have been completely factored.)

4.) $\frac{3(x^2+1)}{(x^2+5)}$

5.) $\frac{4(x+1)^2}{2(x+2)(x-2)}$

6.) $\frac{-(x+1)(x^2+1)(x^2+8)}{(x-7)}$

7.) $7(x+2)^3(x-3)^2$

8.) $\frac{(x-4)(x-6)}{(x^2+3)(x^2+4)(x^2+8)}$

9.) $\frac{3(x^2+7)}{5(x-2)^2(x-6)}$

10.) $\frac{2(x+10)^2(x+30)}{-3(x-5)}$

11.) Completely factor the numerator and the denominator of the rational function below, and then graph it.

$$\frac{3x^3-6x^2+x-2}{x^2+3x+2}$$

Logarithms

Logarithms

If $a > 1$ or $0 < a < 1$, then the exponential function $f : \mathbb{R} \to (0, \infty)$ defined as $f(x) = a^x$ is one-to-one and onto. That means it has an inverse function.

If either $a > 1$ or $0 < a < 1$, then the inverse of the function a^x is

$$\log_a : (0, \infty) \to \mathbb{R}$$

and it's called a *logarithm* of base a.

That a^x and $\log_a(x)$ are inverse functions means that

$$a^{\log_a(x)} = x$$

and

$$\log_a(a^x) = x$$

Problem. Find x if $2^x = 15$.

Solution. The inverse of an exponential function with base 2 is $\log_2$. That means that we can erase the exponential base 2 from the left side of $2^x = 15$ as long as we apply $\log_2$ to the right side of the equation. That would leave us with $x = \log_2(15)$.

The final answer is $x = \log_2(15)$. You stop there. $\log_2(15)$ is a number. It is a perfectly good number, just like 5, -7, or $\sqrt[2]{15}$ are. With some more experience, you will become comfortable with the fact that $\log_2(15)$ cannot be simplified anymore than it already is, just like $\sqrt[2]{15}$ cannot be simplified anymore than it already is. But they are both perfectly good numbers.

Problem. Solve for x where $\log_4(x) = 3$.

Solution. We can erase $\log_4$ from the left side of the equation by applying its inverse, exponential base 4, to the right side of the equation. That would give us $x = 4^3$. Now 4^3 can be simplified; it's 64. So the final answer is $x = 64$.

Problem. Write $\log_3(81)$ as an integer in standard form.

Solution. The trick to solving a problem like this is to rewrite the number being put into the logarithm — in this problem, 81 — as an exponential whose base is the same as the base of the logarithm — in this problem, the base is 3.

Being able to write 81 as an exponential in base 3 will either come from your comfort with exponentials, or from guess-and-check methods. Whether it's immediately obvious to you or not, you can check that $81 = 3^4$. (Notice that 3^4 is an exponential of base 3.) Therefore, $\log_3(81) = \log_3(3^4)$.

Now we use that exponential base 3 and logarithm base 3 are inverse functions to see that $\log_3(3^4) = 4$.

To summarize this process in one line,

$$\log_3(81) = \log_3(3^4) = 4$$

Problem. Write $\log_4(16)$ as an integer in standard form.

Solution. This is a logarithm of base 4, so we write 16 as an exponential of base 4: $16 = 4^2$. Then,

$$\log_4(16) = \log_4(4^2) = 2$$

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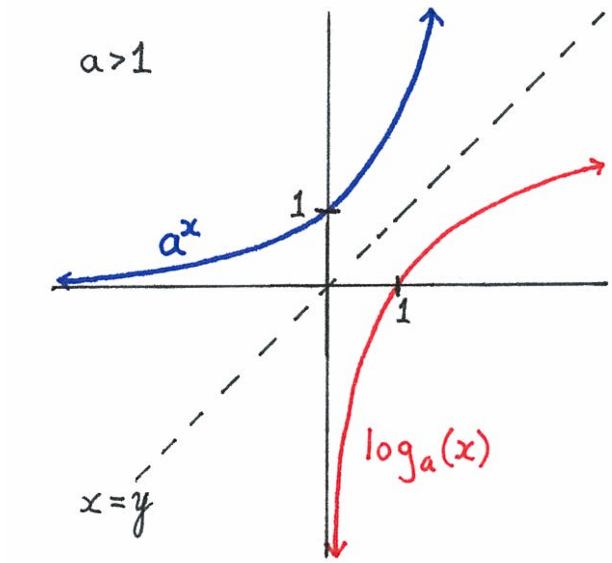
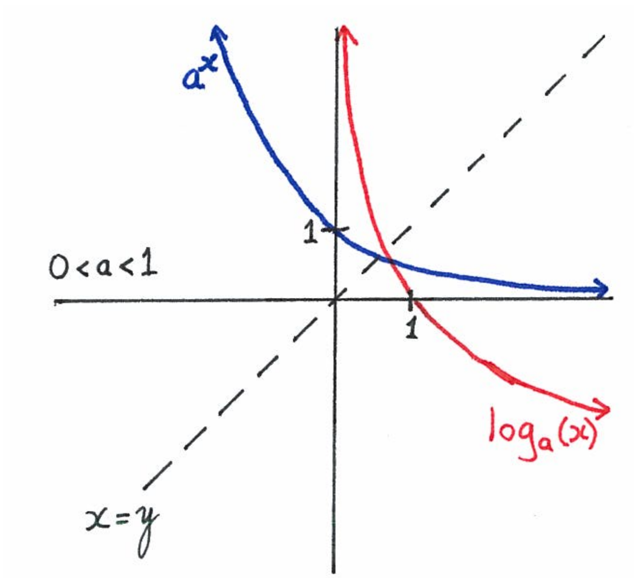
Graphing logarithms

Recall that if you know the graph of a function, you can find the graph of its inverse function by flipping the graph over the line $x = y$.

Below is the graph of a logarithm of base $a > 1$. Notice that the graph grows taller, but very slowly, as it moves to the right.



Below is the graph of a logarithm when the base is between 0 and 1.



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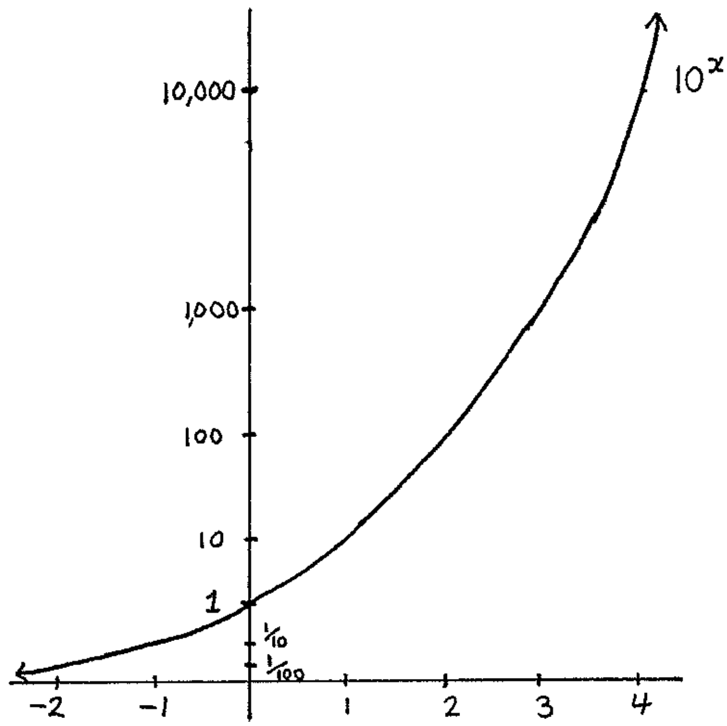
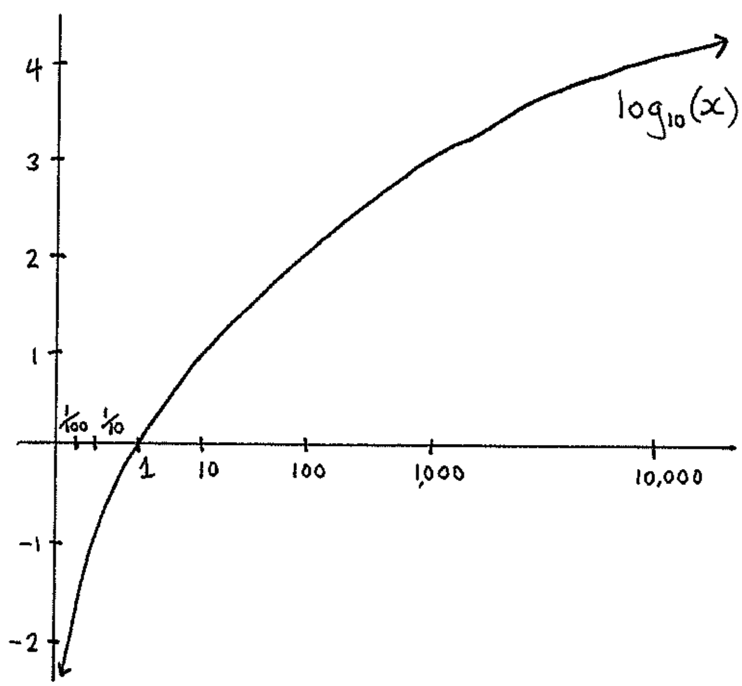
Two base examples

If $a^x = y$, then $x = \log_a(y)$. Below are some examples in base 10.

10^x	$\log_{10}(x)$
$10^{-3} = \frac{1}{1{,}000}$	$-3 = \log_{10}(\frac{1}{1{,}000})$
$10^{-2} = \frac{1}{100}$	$-2 = \log_{10}(\frac{1}{100})$
$10^{-1} = \frac{1}{10}$	$-1 = \log_{10}(\frac{1}{10})$
$10^0 = 1$	$0 = \log_{10}(1)$
$10^1 = 10$	$1 = \log_{10}(10)$
$10^2 = 100$	$2 = \log_{10}(100)$
$10^3 = 1,000$	$3 = \log_{10}(1,000)$
$10^4 = 10,000$	$4 = \log_{10}(10,000)$
$10^5 = 100,000$	$5 = \log_{10}(100,000)$

Below are the graphs of the functions 10^x and $\log_{10}(x)$. The graphs are another way to display the information from the previous chart.

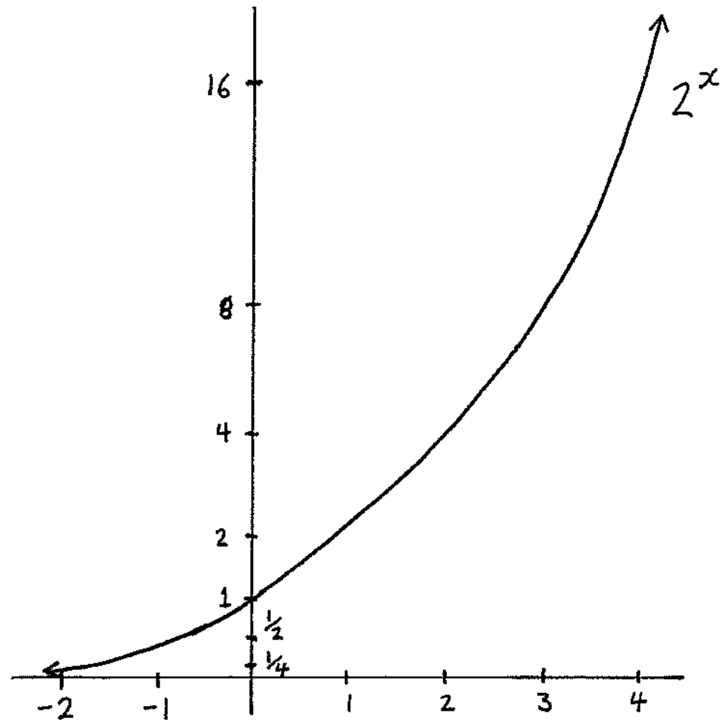


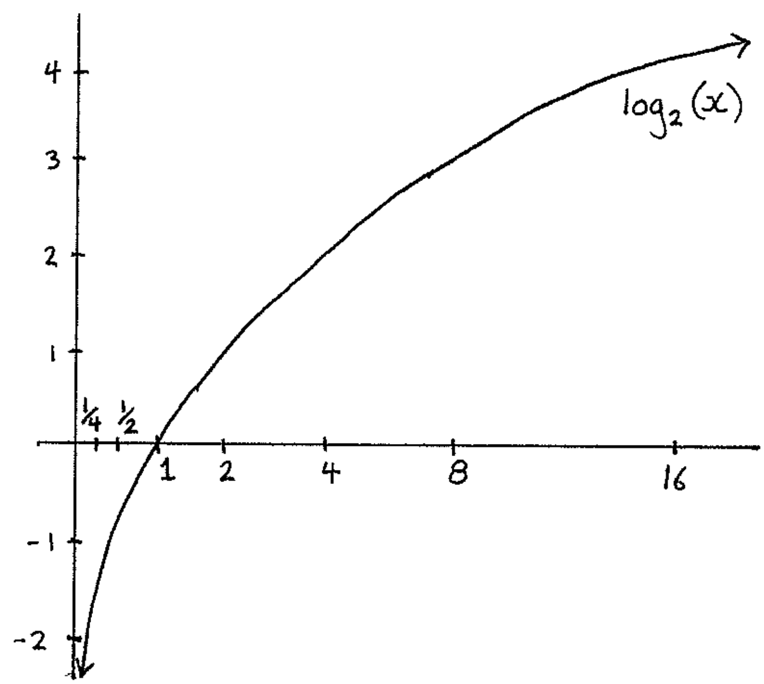


This chart contains examples of exponentials and logarithms in base 2.

2^x	$\log_2(x)$
$2^{-4} = \frac{1}{16}$	$-4 = \log_2(\frac{1}{16})$
$2^{-3} = \frac{1}{8}$	$-3 = \log_2(\frac{1}{8})$
$2^{-2} = \frac{1}{4}$	$-2 = \log_2(\frac{1}{4})$
$2^{-1} = \frac{1}{2}$	$-1 = \log_2(\frac{1}{2})$
$2^0 = 1$	$0 = \log_2(1)$
$2^1 = 2$	$1 = \log_2(2)$
$2^2 = 4$	$2 = \log_2(4)$
$2^3 = 8$	$3 = \log_2(8)$
$2^4 = 16$	$4 = \log_2(16)$
$2^5 = 32$	$5 = \log_2(32)$
$2^6 = 64$	$6 = \log_2(64)$

The information from the previous page is used to draw the graphs of 2^x and $\log_2(x)$.





Rules for logarithms

The most important rule for exponential functions is $a^x a^y = a^{x+y}$. Because $\log_a(x)$ is the inverse of a^x, it satisfies the "opposite" of this rule:

$$\log_a(z) + \log_a(w) = \log_a(zw)$$

Here's why the above equation is true:

$$\begin{aligned}\log_a(z) + \log_a(w) &= \log_a(a^{\log_a(z)+\log_a(w)}) \\ &= \log_a(a^{\log_a(z)}a^{\log_a(w)}) \\ &= \log_a(zw)\end{aligned}$$

The next two rules are different versions of the rule above:

$$\log_a(z) - \log_a(w) = \log_a\left(\frac{z}{w}\right)$$

$$\log_a(z^w) = w\log_a(z)$$

Because $a^0 = 1$, it's also true that

$$\log_a(1) = 0$$

Change of base formula

Let's say that you wanted to know a decimal number that is close to $\log_3(7)$, and you have a calculator that can only compute logarithms in base 10. Your calculator can still help you with $\log_3(7)$ because the change of base formula tells us how to use logarithms in one base to compute logarithms in another base.

The change of base formula is:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

In our example, you could use your calculator to find that 0.845 is a decimal number that is close to $\log_{10}(7)$, and that 0.477 is a decimal number that is close to $\log_{10}(3)$. Then according to the change of base formula

$$\log_3(7) = \frac{\log_{10}(7)}{\log_{10}(3)}$$

is close to the decimal number

$$\frac{0.845}{0.477}$$

which itself is close to 1.771.

We can see why the change of base formula is true. First notice that

$$\log_a(x)\log_b(a) = \log_b(a^{\log_a(x)}) = \log_b(x)$$

The first equal sign above uses the third rule from the section on rules for logarithms. The second equal sign uses that a^x and $\log_a(x)$ are inverse functions.

Now divide the equation above by $\log_b(a)$, and we're left with the change of base formula.

Base confusion

To a mathematician, $\log(x)$ means $\log_e(x)$. Most calculators use $\log(x)$ to mean $\log_{10}(x)$. Sometimes in computer science, $\log(x)$ means $\log_2(x)$. A lot of people use $\ln(x)$ to mean $\log_e(x)$. ($\ln(x)$ is called the "natural logarithm".)

In this class, we'll never write the expression $\log(x)$ or $\ln(x)$. We'll always be explicit with our bases and write logarithms of base 10 as $\log_{10}(x)$, logarithms of base 2 as $\log_2(x)$, and logarithms of base e as $\log_e(x)$. To be safe, when doing math in the future, always ask what base a logarithm is if it's not clear to you.

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Exercises

For #1-8, match each of the numbered functions on the left with the lettered function on the right that is its inverse.

1.) $x+7$	A.) x^7
2.) $3x$	B.) $\frac{x}{3}$
3.) $\sqrt[7]{x}$	C.) 3^x
4.) 7^x	D.) $\sqrt[3]{x}$
5.) $\frac{x}{7}$	E.) $x-7$
6.) $\log_3(x)$	F.) $x+3$
7.) $x-3$	G.) $7x$
8.) x^3	H.) $\log_7(x)$

Graph the functions in #9-12.

9.) $\log_{10}(x-3)$

10.) $\log_2(x+5)$

11.) $\log_{\frac{1}{3}}(x)+4$

12.) $-3\log_e(x)$

For #13-21, write the given number as a rational number in standard form, for example, 2, -3, $\frac{3}{4}$, and $\frac{-1}{5}$ are rational numbers in standard form. These are the exact same questions, in the same order, as those from #16-24 in the chapter on Exponential Functions. They're just written in the language of logarithms instead.

13.) $\log_4(16)$

14.) $\log_2(8)$

15.) $\log_{10}(10,000)$

16.) $\log_3(9)$

17.) $\log_5(125)$

18.) $\log_{\frac{1}{2}}(16)$

19.) $\log_{\frac{1}{4}}(64)$

20.) $\log_8(\frac{1}{4})$

21.) $\log_{27}(\frac{1}{9})$

For #22-29, decide which is the greatest integer that is less than the given number. For example, if you're given the number $\log_2(9)$ then the answer would be 3. You can see that this is the answer by marking 9 on the x-axis of the graph of $\log_2(x)$ that's drawn earlier in this chapter. You can use the graph and the point you marked to see that $\log_2(9)$ is between 3 and 4, so 3 is the greatest of all of the integers that are less than (or below) $\log_2(9)$.

22.) $\log_{10}(15)$

23.) $\log_{10}(950)$

24.) $\log_2(50)$

25.) $\log_2(3)$

26.) $\log_3(18)$

27.) $\log_{10}(\frac{1}{19})$

28.) $\log_2(\frac{1}{10})$

29.) $\log_3(\frac{1}{10})$

In the remaining exercises, use that $\log_a(x)$ and a^x are inverse functions to solve for x.

30.) $\log_4(x) = -2$

31.) $\log_6(x) = 2$

32.) $\log_3(x) = -3$

33.) $\log_{\frac{1}{10}}(x) = -5$

34.) $e^x = 17$

35.) $e^x = 53$

36.) $\log_e(x) = 5$

37.) $\log_e(x) = -\frac{1}{3}$

Exponential & Logarithmic Equations

This chapter is about using the inverses of exponentials or logarithms to solve equations involving exponentials or logarithms.

Solving exponential equations

An exponential equation is an equation that has an unknown quantity, usually called x, written somewhere in the exponent of some positive number. Here are some examples of exponential equations: $e^x = 5$, or $2^{3x-5} = 2$, or $3^{5x-1} = 3^x$. In all of these examples, there is an unknown quantity, x, that appears as an exponent, or as some part of an exponent.

To solve an exponential equation whose unknown quantity is x, the first step is to make the equation look like $a^{f(x)} = c$ where $f(x)$ is some function, and a and c are numbers. Sometimes the equation will already be set up to look like this, as in the examples of $e^x = 5$ or $2^{3x-5} = 2$. Sometimes, you'll have to use the rules of exponentials to make your equation look like $a^{f(x)} = c$. In the third example, we could divide the equation $3^{5x-1} = 3^x$ by 3^x to obtain $3^{5x-1-x} = 1$, which is the same thing as $3^{4x-1} = 1$. (In this last sentence we used the rule of exponentials that a^x divided by a^y equals a^{x-y}.) Now all three of our exponential equations have the form $a^{f(x)} = c$.

Once your equation looks like $a^{f(x)} = c$, use the inverse function $\log_a$. Then your equation will become $f(x) = \log_a(c)$. Sometimes you'll be able to write the number $\log_a(c)$ as a more familiar number. Sometimes you won't. But either way, it's just a number.

If $e^x = 5$, then $x = \log_e(5)$. If $2^{3x-5} = 2$, then $3x - 5 = \log_2(2) = 1$. If $3^{4x-1} = 1$, then $4x - 1 = \log_3(1) = 0$.

At this point in the problem, you might already be finished. If not, you should be able to solve for x using techniques that we've learned or reviewed earlier in the semester. In the three examples above, the answers would be $x = \log_e(5)$, $x = 2$, and $x = \frac{1}{4}$ respectively.

Keep in mind that $\log_e(5)$ is a perfectly good number. Just as good of a number as say 17, or $-\frac{2}{3}$. There's no way to simplify it. You should be comfortable and happy with it as an answer.

Steps for solving exponential equations

Step 1: Make the equation look like $a^{f(x)} = c$ where $a, c \in \mathbb{R}$ and $f(x)$ is a function.

Step 2: Rewrite the equation as $f(x) = \log_a(c)$.

Step 3: Solve for x.

Example. Let's solve for x if

$$e^{3x-7} = 5e^{x-1}$$

To perform Step 1, we can divide both sides of the equation by e^{x-1}. We'd be left with

$$\frac{e^{3x-7}}{e^{x-1}} = 5$$

But $\frac{e^{3x-7}}{e^{x-1}} = e^{3x-7-(x-1)} = e^{2x-6}$. So we're really left with

$$e^{2x-6} = 5$$

and that completes Step 1.

Step 2 is to erase the exponential function in base e from the left side of the equation $e^{2x-6} = 5$ by applying its inverse, the logarithm base e, to the right side of the equation. To put it more simply, we rewrite $e^{2x-6} = 5$ as

$$2x - 6 = \log_e(5)$$

Step 3 is to solve the equation $2x - 6 = \log_e(5)$ using algebra. We can do this by adding 6 to both sides of the equation and then dividing both sides of the equation by 2. We'll be left with the answer

$$x = \frac{\log_e(5) + 6}{2}$$

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Solving logarithmic equations

A logarithmic equation is an equation that contains an unknown quantity, usually called x, inside of a logarithm. For example, $\log_2(5x) = 3$, and $\log_{10}(\sqrt{x}) = 1$, and $\log_e(x^2) = 7 - \log_e(2x)$ are all logarithmic equations.

To solve a logarithmic equation for an unknown quantity x, you'll want to put your equation into the form $\log_a(\, f(x)\,) = c$ where $f(x)$ is a function of x and c is a number. The logarithmic equations $\log_2(5x) = 3$ and $\log_{10}(\sqrt{x}) = 1$ are already written in the form $\log_a(\, f(x)\,) = c$, but $\log_e(x^2) = 7 - \log_e(2x)$ isn't. To arrange the latter equality into our desired form, we can use rules of logarithms. More precisely, add $\log_e(2x)$ to the equation and use the logarithm rule that $\log_e(x^2) + \log_e(2x) = \log_e(x^2 2x)$. Then the equation becomes $\log_e(2x^3) = 7$, and that's the form we want our logarithmic equations to be in.

Once your equation looks like $\log_a(\, f(x)\,) = c$, use that the base a exponential is the inverse of $\log_a$ to rewrite your equation as $f(x) = a^c$. You might want to simplify the number that appears as a^c in your new equation, but other than that, you're done with exponentials and logarithms at this point in the problem. It's time to solve the equation using techniques we used earlier in the semester.

Let's look at the three examples above. We would rewrite $\log_2(5x) = 3$ as $5x = 2^3 = 8$. Then we solve our new equation to find that $x = \frac{8}{5}$.

We would rewrite $\log_{10}(\sqrt{x}) = 1$ as $\sqrt{x} = 10^1 = 10$. Since squaring is the inverse of the square root, we are left with $x = 10^2 = 100$.

For the third equation, we had $\log_e(2x^3) = 7$. Rewrite it as $2x^3 = e^7$, and then solve for x to find that $x = \sqrt[3]{\frac{e^7}{2}}$.

Steps for solving logarithmic equations

Step 1: Make the equation look like $\log_a(f(x)) = c$ where $a, c \in \mathbb{R}$ and $f(x)$ is a function.

Step 2: Rewrite the equation as $f(x) = a^c$.

Step 3: Solve for x.

Example. Let's solve for x if

$$\log_e(-x^2 + 2x) = \log_e(x) + 4$$

To perform Step 1, we can subtract $\log_e(x)$ from both sides of the equation to get

$$\log_e(-x^2 + 2x) - \log_e(x) = 4$$

Recall that $\log_e(-x^2 + 2x) - \log_e(x) = \log_e(\frac{-x^2+2x}{x}) = \log_e(-x + 2)$. That means that

$$\log_e(-x + 2) = 4$$

That's the end of Step 1.

Step 2 is to erase the logarithm base e from the left side of the equation $\log_e(-x + 2) = 4$ by applying the exponential function of base e to the right side of the equation. That is, we rewrite $\log_e(-x + 2) = 4$ as

$$-x + 2 = e^4$$

Step 3 is to solve the equation $-x + 2 = e^4$ using algebra. Subtracting 2 and multiplying by -1 leaves us with the answer

$$x = 2 - e^4$$

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Exercises

Solve the following exponential equations for x.

1.) $10^{3x} = 1000$

2.) $6(14^x) = 30$

3.) $2e^x = 8$

4.) $e^x + 10 = 17$

5.) $(3^x)^5 = 27$

6.) $5^{-\frac{x}{2}} = \frac{1}{5}$

7.) $5^{3x-4} = 125$

8.) $e^{2x} = \frac{e^{x^2}}{e^2}$

9.) $e^{-x^2} = e^{x+5}e^{-11}$

Solve the following logarithmic equations for x.

10.) $\log_3(x-5) = 2$

11.) $\log_e(x) = -6$

12.) $\log_e(2x) = 24$

13.) $\log_e(\sqrt{x-4}) = 5$

14.) $\log_2(x^7) = 28$

15.) $\log_{10}((x+1)^{-5}) = -15$

16.) $5 + \log_e(x^3) = 11$

17.) $\log_2(x) - \log_2(x+4) = 3$

18.) $\log_2(x-2) = -3$

Piecewise Defined Functions

Piecewise Defined Functions

Most of the functions that we've looked at this semester can be expressed as a single equation. For example, $f(x) = 3x^2 - 5x + 2$, or $g(x) = \sqrt{x-1}$, or $h(x) = e^{3x} - 1$.

Sometimes an equation can't be described by a single equation, and instead we have to describe it using a combination of equations. Such functions are called *piecewise defined functions*, and probably the easiest way to describe them is to look at a couple of examples.

First example. The function $g : \mathbb{R} \to \mathbb{R}$ is defined by

$$g(x) = \begin{cases} x^2 - 1 & \text{if } x \in (-\infty, 0]; \\ x - 1 & \text{if } x \in [0, 4]; \\ 3 & \text{if } x \in [4, \infty). \end{cases}$$

The function g is a piecewise defined function. It is defined using three functions that we're more comfortable with: $x^2 - 1$, $x - 1$, and the constant function 3. Each of these three functions is paired with an interval that appears on the right side of the same line as the function: $(-\infty, 0]$, and $[0, 4]$, and $[4, \infty)$ respectively.

If you want to find $g(x)$ for a specific number x, first locate which of the three intervals that particular number x is in. Once you've decided on the correct interval, use the function that interval is paired with to determine $g(x)$.

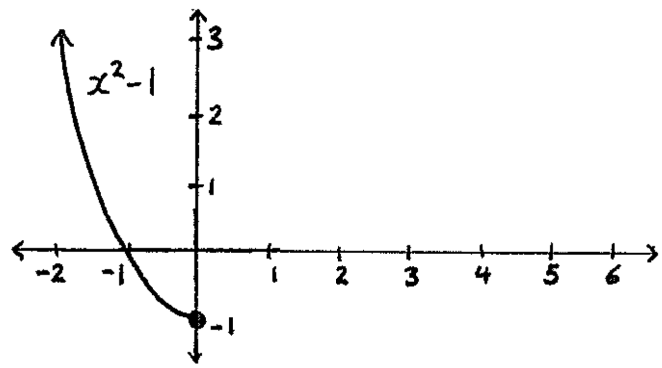
If you want to find $g(2)$, first check that $2 \in [0, 4]$. Therefore, we should use the equation $g(x) = x - 1$, because $x - 1$ is the function that appears on the same line as the interval $[0, 4]$. That means that $g(2) = 2 - 1 = 1$.

To find $g(5)$, notice that $5 \in [4, \infty)$. That means we should be looking at the third interval used in the definition of $g(x)$, and the function coupled with that interval is the constant function 3. Therefore, $g(5) = 3$.

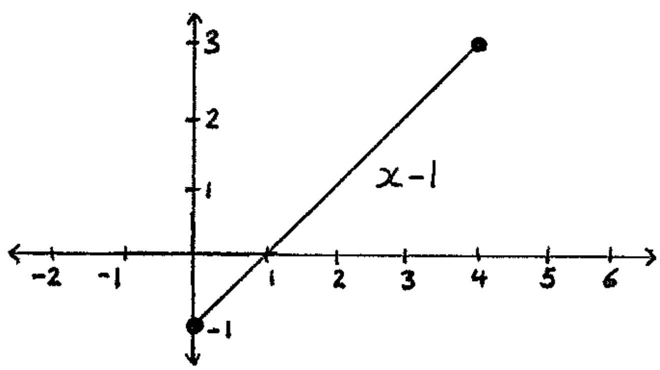
Let's look at one more number. Let's find $g(0)$. First we have to decide which of the three intervals used in the definition of $g(x)$ contains the number 0. Notice that there's some ambiguity here because 0 is contained in both the interval $(-\infty, 0]$ and in the interval $[0, 4]$. Whenever there's ambiguity, choose either of the intervals that are options. Either of the functions that these intervals are paired with will give you the same result. That is, $0^2 - 1 = -1$ is the same number as $0 - 1 = -1$, so $g(0) = -1$.

To graph $g(x)$, graph each of the pieces of g. That is, graph $g : (-\infty, 0] \to \mathbb{R}$ where $g(x) = x^2 - 1$, and graph $g : [0, 4] \to \mathbb{R}$ where $g(x) = x - 1$, and graph $g : [4, \infty) \to \mathbb{R}$ where $g(x) = 3$. Together, these three pieces make up the graph of $g(x)$.

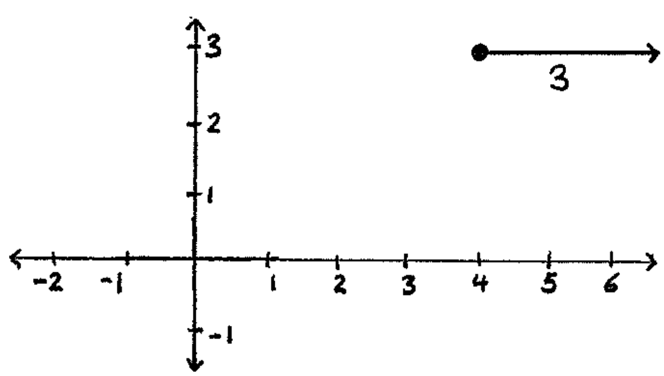
Graph of $g : (-\infty, 0] \to \mathbb{R}$ where $g(x) = x^2 - 1$.



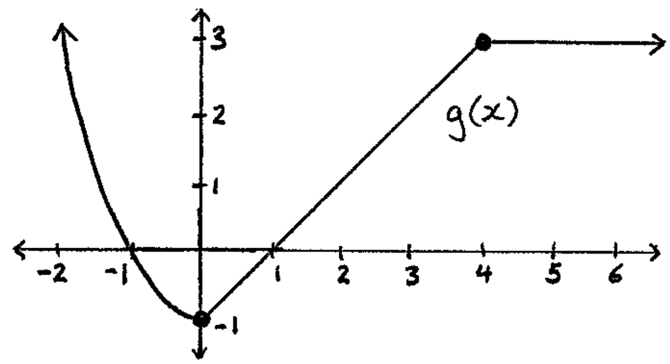
Graph of $g : [0, 4] \to \mathbb{R}$ where $g(x) = x - 1$.



Graph of $g : [4, \infty) \to \mathbb{R}$ where $g(x) = 3$.



To graph $g(x)$, draw the graphs of all three of its pieces.

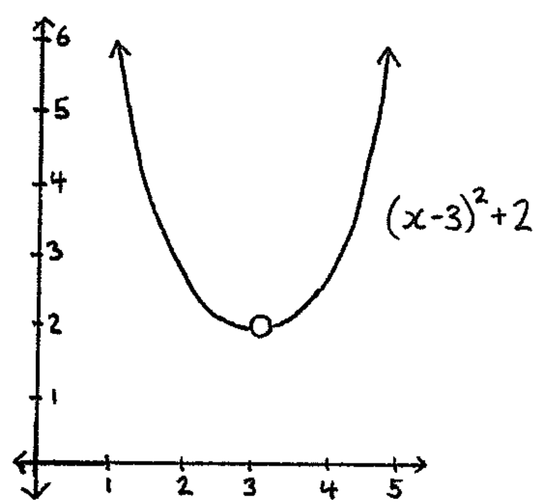


Second example. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

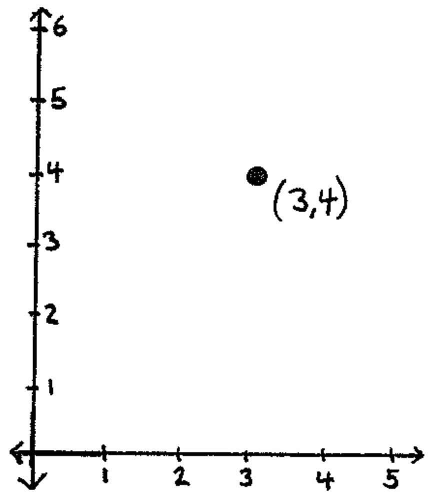
$$f(x) = \begin{cases} (x-3)^2 + 2 & \text{if } x \neq 3; \\ 4 & \text{if } x = 3. \end{cases}$$

This function is made up of two pieces. Either $x \neq 3$, in which case $f(x) = (x-3)^2 + 2$. Or $x = 3$, and then $f(3) = 4$.

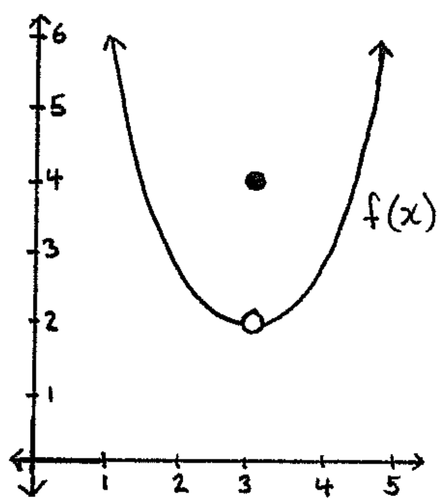
Graph of the first piece of $f(x)$: the graph of x^2 shifted right 3 and up 2 with the vertex removed.
(Remember that a little circle means that point is *not* a point of the graph.)



Graph of the second piece of $f(x)$: a single giant dot.



Graph of both pieces, and hence the entire graph, of $f(x)$.



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Absolute value

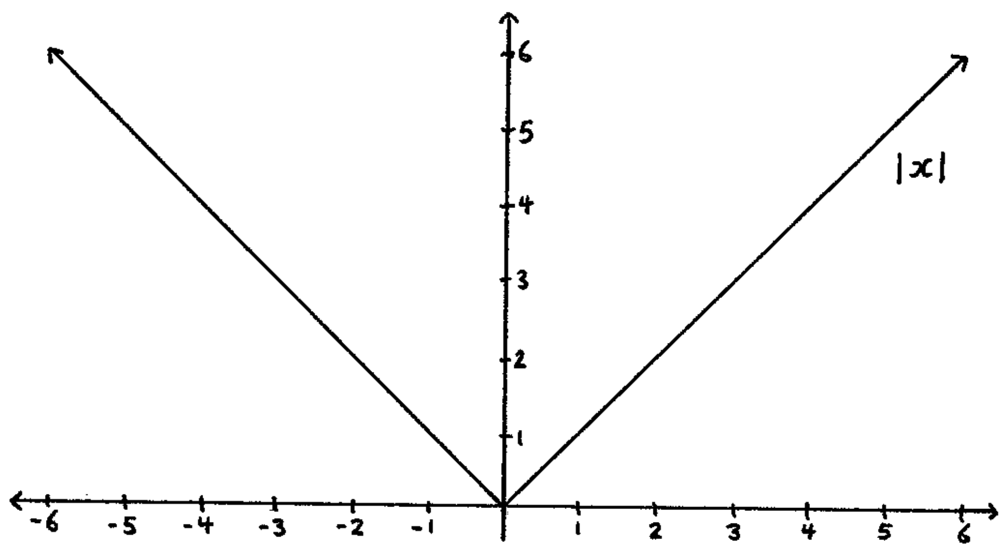
The most important piecewise defined function in calculus is the absolute value function that is defined by

$$|x| = \begin{cases} -x & \text{if } x \in (-\infty, 0]; \\ x & \text{if } x \in [0, \infty). \end{cases}$$

The domain of the absolute value function is $\mathbb{R}$. The range of the absolute value function is the set of non-negative numbers. The number $|x|$ is called the *absolute value* of x.

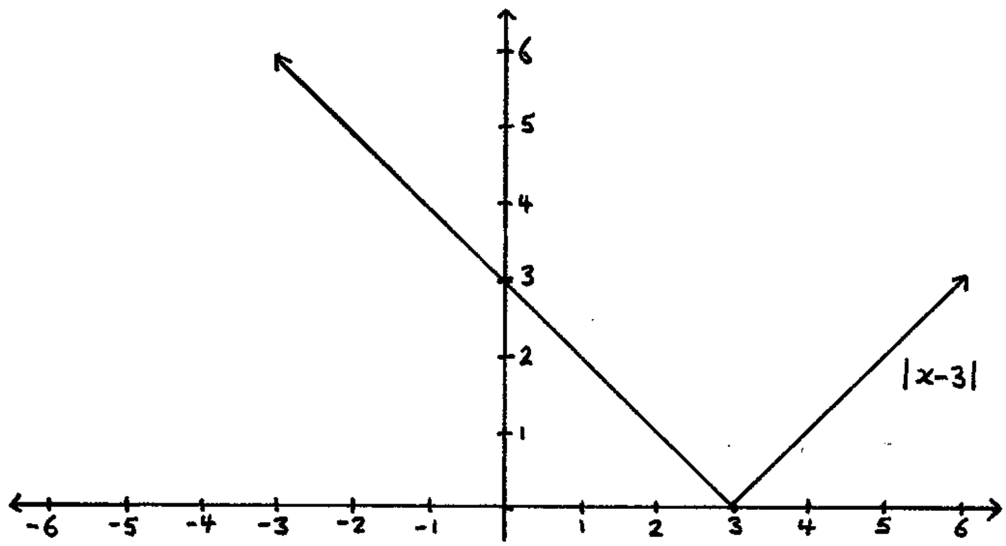
For examples of how this function works, notice that $|4| = 4$, $|0| = 0$, and $|-3| = 3$. If x is positive or 0, then the absolute value of x is x itself. If x is negative, then $|x|$ is the positive number that you'd get from "erasing" the negative sign: $|-10| = 10$ and $|-\frac{1}{2}| = \frac{1}{2}$.

Graph of the absolute value function.



Another interpretation of the absolute value function, and the one that's most important for calculus, is that the absolute value of a number is the same as its distance from 0. That is, the distance between 0 and 5 is $|5| = 5$, the distance between 0 and -7 is $|-7| = 7$, and the distance between 0 and 0 is $|0| = 0$.

Let's look at the graph of say $|x-3|$. It's the graph of $|x|$ shifted right by 3.



You might guess from the graph of $|x-3|$, that $|x-3|$ is the function that measures the distance between x and 3, and that's true. Similarly, $|x-6|$ is the distance between x and 6, $|x+2|$ is the distance between x and -2, and more generally, $|x-y|$ is the distance between x and y.

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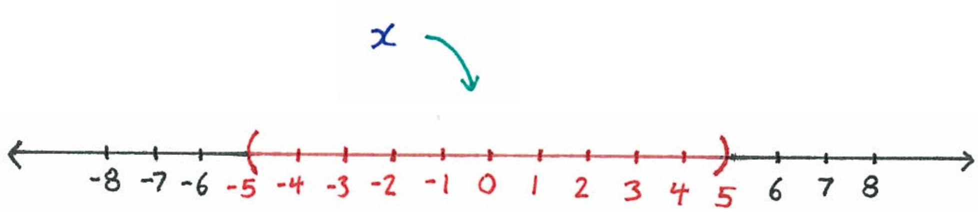
Solving inequalities involving absolute values

The inequality $|x| < 5$ means that the distance between x and 0 is less than 5. Therefore, x is between -5 and 5. Another way to write the previous sentence is $-5 < x < 5$.



Notice in the above paragraph that the precise number 5 wasn't really important for the problem. We could have replaced 5 with any positive number c to obtain the following translation.

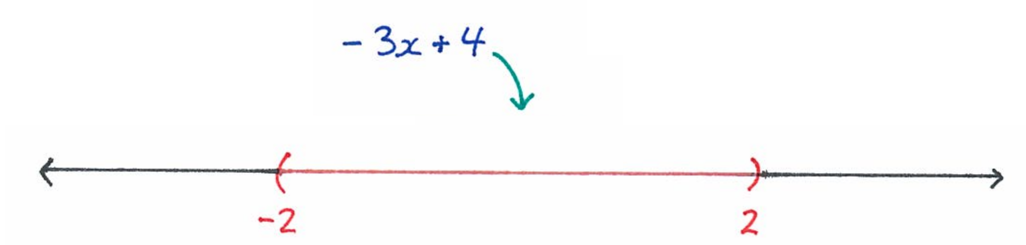
$$|x| < c \text{ means } -c < x < c$$

For example, writing $|x| < 2$ means the same as writing $-2 < x < 2$, and $|2x - 3| < \frac{1}{3}$ means the same as $-\frac{1}{3} < 2x - 3 < \frac{1}{3}$.

We can use the above rule to help us solve some inequalities that involve absolute values.

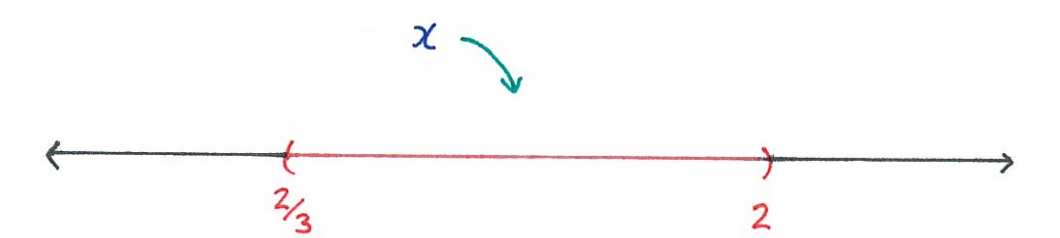
Problem. Solve for x if $|-3x + 4| < 2$.

Solution. We know from the explanation above that $-2 < -3x + 4 < 2$.



Subtracting 4 from all three of the quantities in the previous inequality yields $-2-4 < -3x < 2-4$, and that can be simplified as $-6 < -3x < -2$.

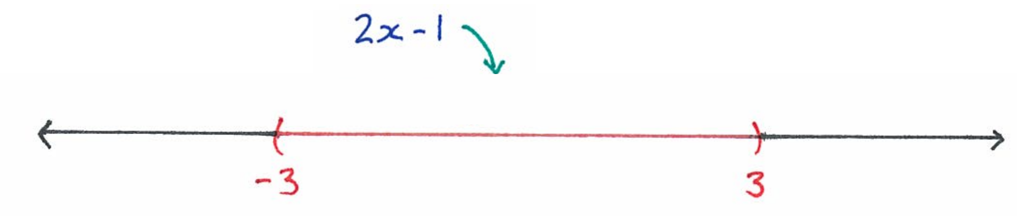
Next divide by -3, keeping in mind that dividing an inequality by a negative number "flips" the inequalities. The result will be $\frac{-6}{-3} > x > \frac{-2}{-3}$, which can be simplified as $2 > x > \frac{2}{3}$. That's the answer.



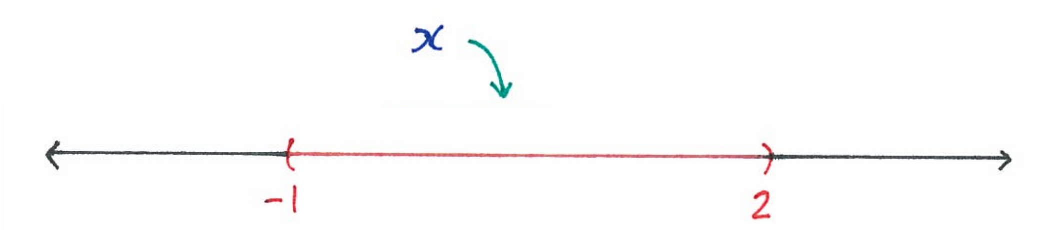
The inequality $2 > x > \frac{2}{3}$ could also be written as $\frac{2}{3} < x < 2$, or as $x \in (\frac{2}{3}, 2)$.

Problem. Solve for x if $|2x - 1| < 3$.

Solution. Write the inequality from the problem as $-3 < 2x - 1 < 3$.



Add 1 to get $-2 < 2x < 4$, and divide by 2 to get $-1 < x < 2$.



Two important rules for absolute values

For the two rules below, $a, b, c \in \mathbb{R}$. Each rule is important for calculus. They'll be explained in class.

1. $|ab| = |a||b|$

2. $|a - c| \leq |a - b| + |b - c|$ (*triangle inequality*)

Exercises

1.) Suppose $f(x)$ is the piecewise defined function given by

$$f(x) = \begin{cases} x+1 & \text{if } x \in (-\infty, 2); \\ x+3 & \text{if } x \in [2, \infty). \end{cases}$$

What is $f(0)$? What is $f(10)$? What is $f(2)$? Graph $f(x)$.

2.) Suppose $g(x)$ is the piecewise defined function given by

$$g(x) = \begin{cases} 3 & \text{if } x \in [1, 5]; \\ 1 & \text{if } x \in (5, \infty). \end{cases}$$

What is $g(1)$? What is $g(100)$? What is $g(5)$? Graph $g(x)$.

3.) Suppose $h(x)$ is the piecewise defined function given by

$$h(x) = \begin{cases} 5 & \text{if } x \in (1, 3]; \\ x+2 & \text{if } x \in [3, 8). \end{cases}$$

What is $h(2)$? What is $h(7)$? What is $h(3)$? Graph $h(x)$.

4.) Suppose $f(x)$ is the piecewise defined function given by

$$f(x) = \begin{cases} 2 & \text{if } x \in [-3, 0); \\ e^x & \text{if } x \in [0, 2]; \\ 3x-2 & \text{if } x \in (2, \infty). \end{cases}$$

What is $f(-2)$? What is $f(0)$? What is $f(2)$? What is $f(15)$? Graph $f(x)$.

5.) Suppose $g(x)$ is the piecewise defined function given by

$$g(x) = \begin{cases} (x-1)^2 & \text{if } x \in (-\infty, 1]; \\ \log_e(x) & \text{if } x \in [1, 5]; \\ \log_e(5) & \text{if } x \in [5, \infty). \end{cases}$$

What is $g(0)$? What is $g(1)$? What is $g(5)$? What is $g(20)$? Graph $g(x)$.

6.) Suppose $h(x)$ is the piecewise defined function given by

$$h(x) = \begin{cases} e^x & \text{if } x \neq 2; \\ 1 & \text{if } x = 2. \end{cases}$$

What is $h(0)$? What is $h(2)$? What is $h\big(\log_e(17)\big)$? Graph $h(x)$.

7.) Write the following numbers as integers: $|8-5|$, $|-10-5|$, and $|5-5|$. The function $|x-5|$ measures the distance between x and which number?

8.) Write the following numbers as integers: $|1-2|$, $|3-2|$, and $|2-2|$. The function $|x-2|$ measures the distance between x and which number?

9.) Write the following numbers as integers: $|3+4|$, $|-1+4|$, $|-4+4|$. The function $|x+4|$ measures the distance between x and which number?

10.) The function $|x-y|$ measures the distance between x and which number?

11.) Solve for x if $|5x-2| < 7$.

12.) Solve for x if $|3x+4| < 1$.

13.) Solve for x if $|-2x+3| < 5$.

Linear Algebra

Linear Equations in Two Variables

In this chapter, we'll use the geometry of lines to help us solve equations.

Linear equations in two variables.

If a, b, and r are real numbers (and if a and b are not both equal to 0) then $ax + by = r$ is called a *linear equation in two variables.* (The "two varaibles" are the x and the y.)

The numbers a and b are called the *coefficients* of the equation $ax+by = r$. The number r is called the *constant* of the equation $ax + by = r$.

Examples. $10x - 3y = 5$ and $-2x - 4y = 7$ are linear equations in two variables.

Solutions to equations.

A *solution* to a linear equation in two variables $ax+by = r$ is a specific point in $\mathbb{R}^2$ such that when when the x-coordinate of the point is multiplied by a, and the y-coordinate of the point is multiplied by b, and those two numbers are added together, the answer equals r. (There are always infinitely many solutions to a linear equation in two variables.)

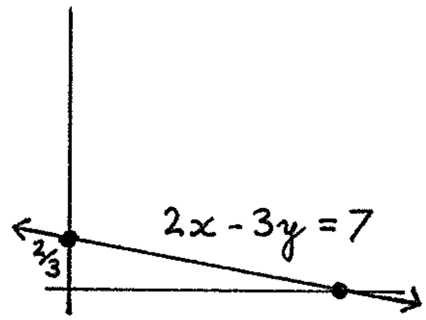
Example. Let's look at the equation $2x - 3y = 7$.

Notice that $x = 5$ and $y = 1$ is a point in $\mathbb{R}^2$ that is a solution to this equation because we can let $x = 5$ and $y = 1$ in the equation $2x - 3y = 7$ and then we'd have $2(5) - 3(1) = 10 - 3 = 7$.

The point $x = 8$ and $y = 3$ is also a solution to the equation $2x - 3y = 7$ since $2(8) - 3(3) = 16 - 9 = 7$.

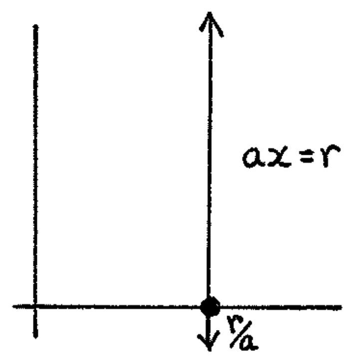
The point $x = 4$ and $y = 6$ is not a solution to the equation $2x - 3y = 7$ because $2(4) - 3(6) = 8 - 18 = -10$, and $-10 \neq 7$.

To get a geometric interpretation for what the set of solutions of $2x-3y = 7$ looks like, we can add $3y$, subtract 7, and divide by 3 to rewrite $2x - 3y = 7$ as $\frac{2}{3}x-\frac{7}{3} = y$. This is the equation of a line that has slope $\frac{2}{3}$ and a y-intercept of $-\frac{7}{3}$. In particular, the set of solutions to $2x - 3y = 7$ is a straight line. (This is why it's called a linear equation.)

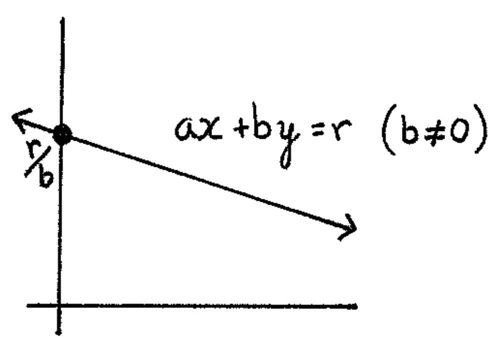


Linear equations and lines.

If $b = 0$, then the linear equation $ax + by = r$ is the same as $ax = r$. Dividing by a gives $x = \frac{r}{a}$, so the solutions to this equation consist of the points on the vertical line whose x-coordinates equal $\frac{r}{a}$.



If $b \neq 0$, then the same ideas from the $2x - 3y = 7$ example that we looked at previously shows that $ax + by = r$ is the same equation as, just written in a different form from, $-\frac{a}{b}x + \frac{r}{b} = y$. This is the equation of a straight line whose slope is $-\frac{a}{b}$ and whose y-intercept is $\frac{r}{b}$.



Systems of linear equations.

Rather than asking for the solution set of a single linear equation in two variables, we could take two different linear equations in two variables and ask for all those points that are solutions to *both* of the linear equations.

For example, the point $x = 4$ and $y = 1$ is a solution to both of the equations $x + y = 5$ and $x - y = 3$.

If you have more than one linear equation, it's called a *system* of linear equations, so that

$$x + y = 5$$
$$x - y = 3$$

is an example of a system of two linear equations in two variables. There are two equations, and each equation has the same two variables: x and y.

A *solution* to a system of equations is a point that is a solution to each of the equations in the system.

Example. The point $x = 3$ and $y = 2$ is a solution to the system of two linear equations in two variables

$$8x \ + \ 7y = 38$$
$$3x \ - \ 5y = -1$$

because $x = 3$ and $y = 2$ is a solution to $3x - 5y = -1$ *and* it is a solution to $8x + 7y = 38$.

Unique solutions.

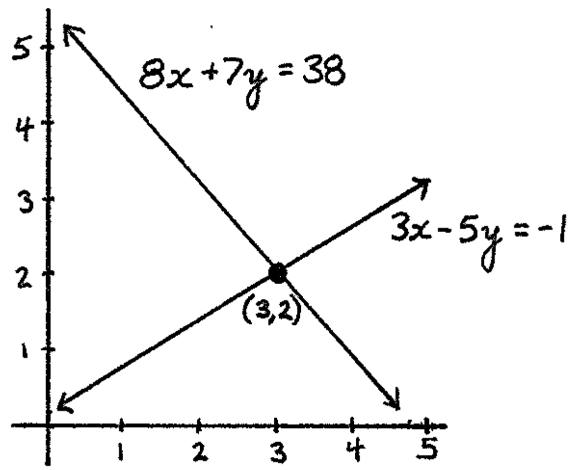
Geometrically, finding a solution to a system of two linear equations in two variables is the same problem as finding a point in $\mathbb{R}^2$ that lies on each of the straight lines corresponding to the two linear equations.

Almost all of the time, two different lines will intersect in a single point, so in these cases, there will only be one point that is a solution to both equations. Such a point is called the *unique* solution for the system of linear equations.

Example. Let's take a second look at the system of equations

$$8x \ + \ 7y = 38$$
$$3x \ - \ 5y = -1$$

The first equation in this system, $8x + 7y = 38$, corresponds to a line that has slope $-\frac{8}{7}$. The second equation in this system, $3x - 5y = 3$, is represented by a line that has slope $-\frac{3}{-5} = \frac{3}{5}$. Since the two slopes are not equal, the lines have to intersect in exactly one point. That one point will be the unique solution. As we've seen before that $x = 3$ and $y = 2$ is a solution to this system, it must be the unique solution.



Example. The system

$$\begin{aligned} 5x + 2y &= 4 \\ -2x + y &= 11 \end{aligned}$$

has a unique solution. It's $x = -2$ and $y = 7$.

It's straightforward to check that $x = -2$ and $y = 7$ is a solution to the system. That it's the only solution follows from the fact that the slope of the line $5x + 2y = 4$ is different from slope of the line $-2x + y = 11$. Those two slopes are $-\frac{5}{2}$ and $\frac{2}{11}$ respectively.

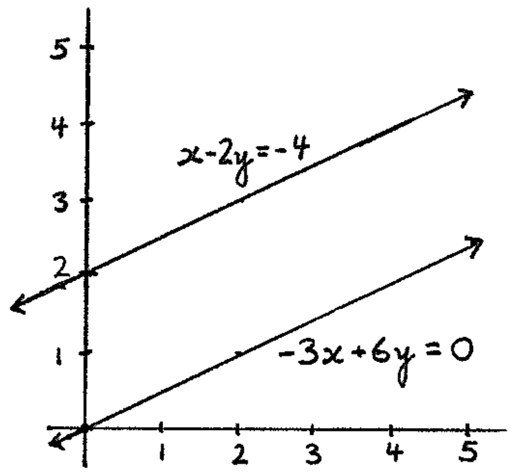
No solutions.

If you reach into a hat and pull out two different linear equations in two variables, it's highly unlikely that the two corresponding lines would have exactly the same slope. But if they did have the same slope, then there would not be a solution to the system of two linear equations since no point in $\mathbb{R}^2$ would lie on both of the parallel lines.

Example. The system

$$\begin{aligned} x \, - \, 2y &= -4 \\ -3x \, + \, 6y &= 0 \end{aligned}$$

does not have a solution. That's because each of the two lines has the same slope, $\frac{1}{2}$, so the lines don't intersect.



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Exercises

1.) What are the coefficients of the equation $2x - 5y = -23$?

2.) What is the constant of the equation $2x - 5y = -23$?

3.) Is $x = -4$ and $y = 3$ a solution to the equation $2x - 5y = -23$?

4.) What are the coefficients of the equation $-7x + 6y = 15$?

5.) What is the constant of the equation $-7x + 6y = 15$?

6.) Is $x = 3$ and $y = -10$ a solution to the equation $-7x + 6y = 15$?

7.) Is $x = 1$ and $y = 0$ a solution to the system

$$\begin{aligned} x + y &= 1 \\ 2x + 3y &= 3 \end{aligned}$$

8.) Is $x = -1$ and $y = 3$ a solution to the system

$$\begin{aligned} 7x + 2y &= -1 \\ 5x - 3y &= -14 \end{aligned}$$

9.) What's the slope of the line $30x - 6y = 3$?

10.) What's the slope of the line $-10x + 5y = 4$?

11.) Is there a unique solution to the system

$$\begin{aligned} 30x - 6y &= 3 \\ -10x + 5y &= 4 \end{aligned}$$

12.) What's the slope of the line $6x + 2y = 4$?

13.) What's the slope of the line $15x + 5y = -7$?

14.) Is there a unique solution to the system

$$\begin{aligned} 6x + 2y &= 4 \\ 15x + 5y &= -7 \end{aligned}$$

Substitution

In this chapter, we'll examine systems of two linear equations in two variables that have unique solutions. If a system has a unique solution, we can use a method called "substitution" to find the unique solution.

How to find the solution

Suppose you're given a system of two linear equations in two variables, and that the variables are named x and y. Name the equations "Equation-1" and "Equation-2" (the order doesn't matter).

Use algebra to transform Equation-1 into an equation that looks like

$$x = (\text{something with } y\text{'s and numbers})$$

Let's call this equation "New-equation-1".

Use New-equation-1 to substitute for x in Equation-2. You'll be left with a "New-equation-2" that only has y's and numbers — there won't be any x's.

Use New-equation-2 to solve for y. Once you have, substitute your solution for y into New-equation-1. That will tell you what x is.

(In the explanation above, the roles of x and y could have been switched.)

Problem 1. Find the solution to the system

$$\begin{aligned} x + 4y &= -2 \\ 2x + 7y &= -3 \end{aligned}$$

Solution. Let's name $x + 4y = -2$ Equation-1, and solve for x. Then we'll get that

$$x = -4y - 2$$

This is New-equation-1.

Equation-2 is $2x + 7y = -3$. Using New-equation-1, we can replace x in Equation-2 with $-4y - 2$ to get

$$2[-4y - 2] + 7y = -3$$

This is New-equation-2, and we can use it to solve for y:

$$-8y - 4 + 7y = -3$$

thus

$$-y = 1$$

and hence

$$y = -1$$

Now that we know y, we return to New-equation-1, replace y with -1, and we are left with

$$x = -4(-1) - 2 = 2$$

Now we know that $x = 2$ and $y = -1$ is the solution to the system of equations that we started with.

Problem 2. Find the solution to the system

$$-2x + y = -1$$
$$5x - 2y = 5$$

Solution. Use the first equation to solve for x:

$$x = \frac{y+1}{2}$$

Substitute for x in the second equation:

$$5\left[\frac{y+1}{2}\right] - 2y = 5$$

so

$$\frac{5y}{2} + \frac{5}{2} - 2y = 5$$

and then

$$\frac{y}{2} + \frac{5}{2} = 5$$

Multiplying both sides of the equation by 2 gives

$$y + 5 = 10$$

and therefore,

$$y = 5$$

Now return to the equation

$$x = \frac{y+1}{2}$$

and substitute 5 for y to get

$$x = \frac{5+1}{2}$$

which means that

$$x = 3$$

We have our solution to the system, it's $x = 3$ and $y = 5$.

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Exercises

Each of the systems below has a unique solution. Find the solution.

1.)

$$\begin{aligned} 8x + 4y &= 12 \\ x - 7y &= -21 \end{aligned}$$

2.)

$$\begin{aligned} 10x - 3y &= 52 \\ -3x + y &= -16 \end{aligned}$$

3.)

$$\begin{aligned} 3x &= 5 \\ 2x - 3y &= 12 \end{aligned}$$

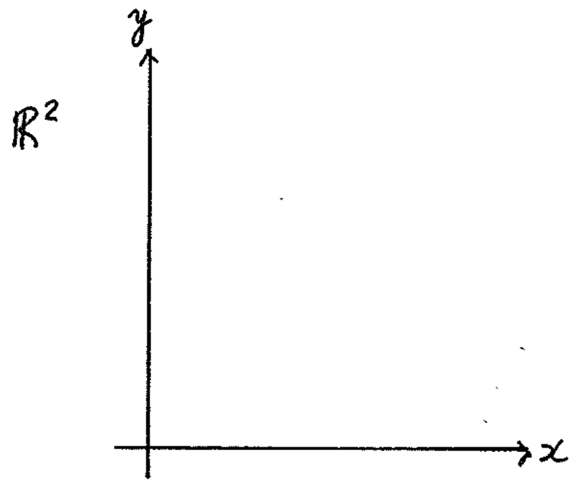
4.)

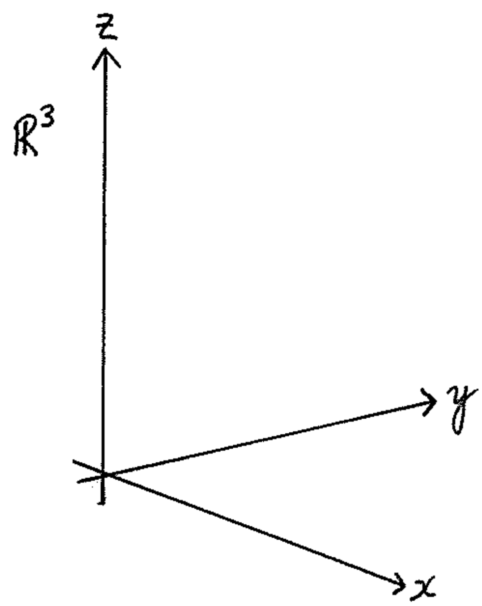
$$\begin{aligned} 2x + 8y &= -8 \\ -3x + 6y &= 12 \end{aligned}$$

Linear Equations in Three Variables

$\mathbb{R}^2$ is the space of 2 dimensions. There is an x-coordinate that can be any real number, and there is a y-coordinate that can be any real number.

$\mathbb{R}^3$ is the space of 3 dimensions. There is an x, y, and z coordinate. Each coordinate can be any real number.





Linear equations in three variables

If a, b, c and r are real numbers (and if a, b, and c are not all equal to 0) then $ax + by + cz = r$ is called a *linear equation in three variables.* (The "three variables" are the x, the y, and the z.)

The numbers a, b, and c are called the *coefficients* of the equation. The number r is called the *constant* of the equation.

Examples. $3x + 4y - 7z = 2$, $-2x + y - z = -6$, $x - 17z = 4$, $4y = 0$, and $x + y + z = 2$ are all linear equations in three variables.

Solutions to equations

A *solution* to a linear equation in three variables $ax+by+cz = r$ is a specific point in $\mathbb{R}^3$ such that when when the x-coordinate of the point is multiplied by a, the y-coordinate of the point is multiplied by b, the z-coordinate of the point is multiplied by c, and then those three products are added together, the answer equals r. (There are always infinitely many solutions to a linear equation in three variables.)

Example. The point $x = 1$, $y = 2$, and $z = -1$ is a solution to the equation

$$-2x + 5y + z = 7$$

since

$$-2(1) + 5(2) + (-1) = -2 + 10 - 1 = 7$$

The point $x = 3$, $y = -2$, and $z = 4$ is a *not* a solution to the equation $-2x + 5y + z = 7$ since

$$-2(3) + 5(-2) + (4) = -6 - 10 + 4 = -12$$

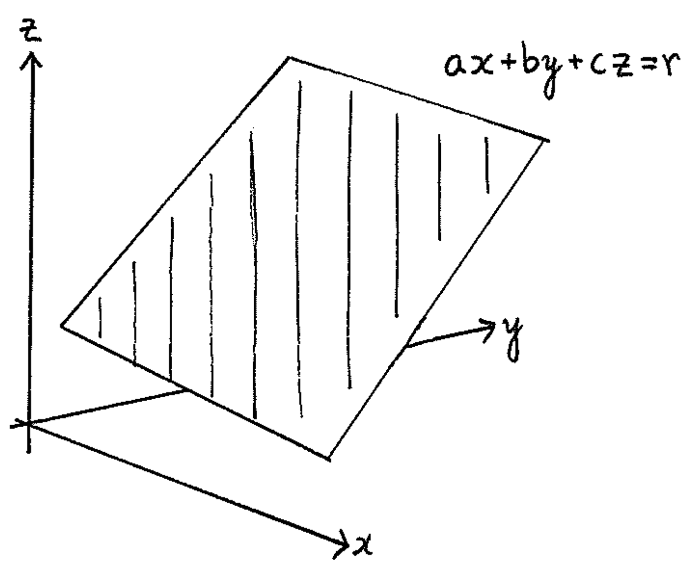
and

$$-12 \neq 7$$

Linear equations and planes

The set of solutions in $\mathbb{R}^2$ to a linear equation in two variables is a 1-dimensional line.

The set of solutions in $\mathbb{R}^3$ to a linear equation in three variables is a 2-dimensional plane.



Solutions to systems of linear equations

As in the previous chapter, we can have a system of linear equations, and we can try to find solutions that are common to each of the equations in the system.

We call a solution to a system of equations *unique* if there are no other solutions.

Example. The point $x = 3$, $y = 0$, and $z = 1$ is a solution to the following system of three linear equations in three variables

$$\begin{aligned} -3x + 2y - 5z &= -14 \\ 2x - 3y + 4z &= 10 \\ x + y + z &= 4 \end{aligned}$$

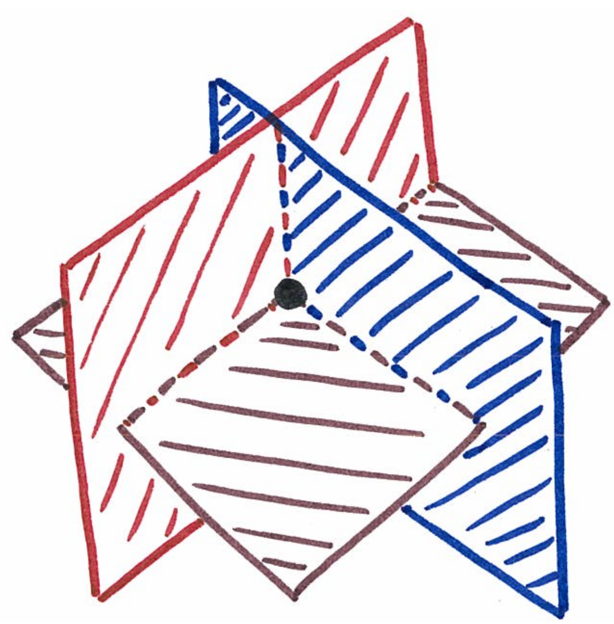
That's because we can substitute 3, 0, and 1 for x, y, and z respectively in the equations above and check that

$$\begin{aligned} -3(3) + 2(0) - 5(1) &= -9 - 5 = -14 \\ 2(3) - 3(0) + 4(1) &= 6 + 4 = 10 \\ (3) + (0) + (1) &= 3 + 1 = 4 \end{aligned}$$

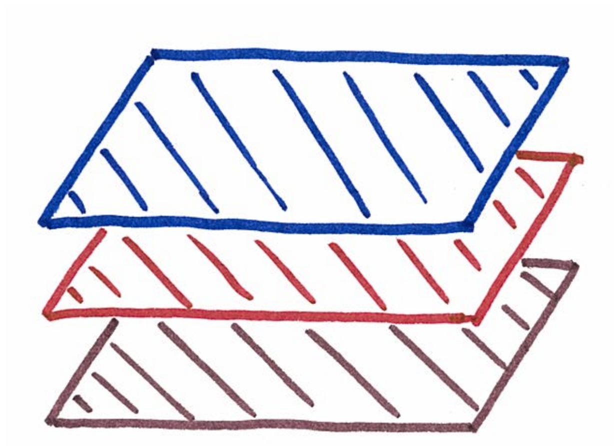
Geometry of solutions

Suppose you have a system of three linear equations in three variables. Each of the three equations has a set of solutions that's a plane in $\mathbb{R}^3$. A solution to the system of equations is a point that lies on all three of those planes. If there is only one point that lies on all three planes, then that solution is unique.

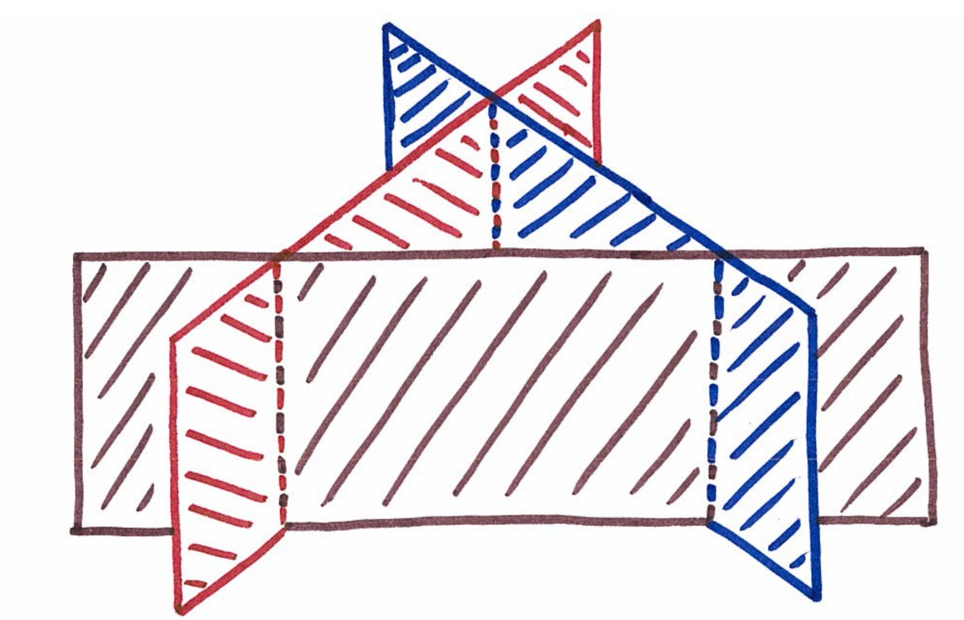
If you randomly write down three different linear equations in three variables, the odds are that the three corresponding planes will intersect in one, and only one, point. That means that for most systems of three linear equations in three variables, there will be a unique solution.



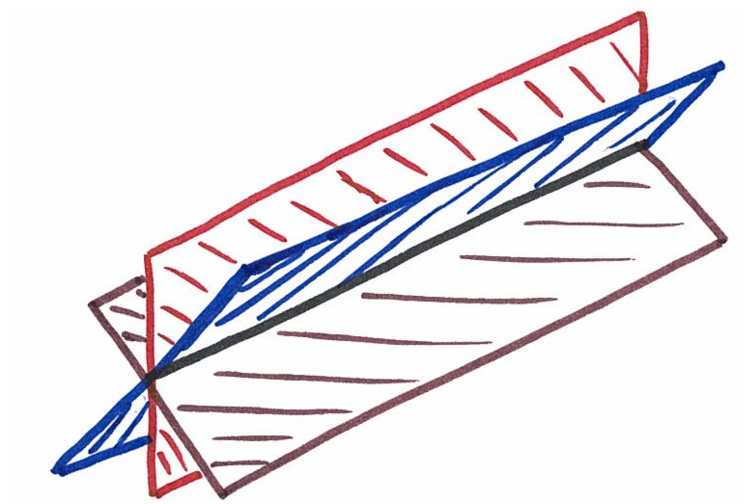
It might be that the three planes from the system of three equations will be parallel. Then the three planes wouldn't intersect. There'd be no point common to all three planes, and hence the system will not have any solutions.



There might not be a point that lies on all three planes even if the planes aren't parallel. In this case again, there'd be no solution at all.



Sometimes, the three planes will intersect in a way that allows for more than one point to be on all three planes at once. In this case, there are multiple solutions. Because there's more than one solution, there's not a unique solution.



Visualize different arrangements of three planes in $\mathbb{R}^3$ and try to convince yourself that either there is exactly one point contained in all three planes, or no points contained in all three planes, or that there are infinitely many points that are contained in all three planes. That means that a system of three linear equations in three variables will always have either a unique solution, no solution at all, or infinitely many solutions.

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Exercises

1.) What are the coefficients of the equation

$$3x - 2y + 5z = 13$$

2.) What is the constant of the equation

$$3x - 2y + 5z = 13$$

3.) Is $x = 4$, $y = -3$, and $z = -1$ a solution to the equation

$$3x - 2y + 5z = 13$$

4.) Is $x = -2$, $y = 0$, and $z = 4$ a solution to the equation

$$-x + 7y - 8z = 10$$

5.) Is $x = 5$, $y = 7$, and $z = -2$ a solution to the equation

$$3y - 5z = 31$$

Questions #6-10 refer to the following system of three linear equations in three variables.

$$\begin{aligned} 3x - 4y + 2z &= -9 \\ -4x + 4y + 10z &= 32 \\ -x + 2y - 7z &= -7 \end{aligned}$$

6.) Is $x = 1$, $y = 4$, and $z = 2$ a solution to the system?

7.) Is $x = -1$, $y = 1$, and $z = -1$ a solution to the system?

8.) Is $x = 25$, $y = 23$, and $z = 4$ a solution to the system?

9.) Does the system have a unique solution?

10.) Does the system have infinitely many solutions?

Rows & Columns

In this chapter we'll learn how to multiply a row of n numbers and a column of n numbers to obtain a single number. Actually, in this class, n will always be either 2 or 3, though it could be any natural number.

Rows

A *row* of 2 numbers is just two numbers written left-to-right. For example, $(3,4)$ and $(-2,0)$ are each a row of 2 numbers.

The *entries* of a row are the numbers that make up the row. For example, the first entry of the row $(3,4)$ is 3. The second entry of $(3,4)$ is 4.

An example of a row of three numbers is $(2,-4,6)$. The first entry of this row is 2, the second entry is -4, and the third entry is 6.

Columns

A *column* of 2 numbers is two numbers written top-to-bottom. For example,

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

is a column of two numbers. Its first entry is 4 and its second entry is 1.

(Caution: We write a column of 2 numbers exactly as we wrote binomial coefficients earlier in the semester. Although they are written in the same way, they mean different things. We just have to use the context of the problem we are working on to interpret the correct meaning. This is similar to how *there* and *they're* sound the same, but it's always clear in conversation which of these two words a person is using.)

As another example of a column of numbers, we write the column of 3 numbers with entries 5, 0, and -1 as

$$\begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$$

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Multiplying rows (on the left) and columns (on the right)

To multiply a row of numbers and a column of numbers, the row and the column must have the same number of entries. For example, we can multiply a row of 3 numbers and a column of 3 numbers, but we cannot multiply a row of 3 numbers and a column of 2 numbers.

Also, to multiply a row of numbers and a column of numbers, the row must be written on the left, and the column must be written on the right.

To perform the multiplication of the row and the column, multiply the first entry in the row and the first entry of the column. That's the first product.

Now take the product of the second entry in the row and the second entry in the column. That's the second product.

If the row and column being multiplied each have 3 entries, then there will also be a third product obtained from multiplying the third entries of the row and column.

Now sum the products: the first, and the second, and the third (if the row and column each have three entries). That's the answer.

Examples.

$$(3,4)\begin{pmatrix}1\\2\end{pmatrix} = 3\cdot 1 + 4\cdot 2 = 3+8 = 11$$

$$(-2,0)\begin{pmatrix}1\\-3\end{pmatrix} = -2\cdot 1 + 0\cdot(-3) = -2$$

More generally, the pattern for multiplying a row of two numbers and a column of two numbers is given by

$$(r_1,r_2)\begin{pmatrix}c_1\\c_2\end{pmatrix} = r_1c_1 + r_2c_2$$

Similarly, the pattern for multiplying a row of three number and a column of three numbers is given by

$$(r_1,r_2,r_3)\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = r_1c_1 + r_2c_2 + r_3c_3$$

For example,

$$(1, 2, -6)\begin{pmatrix}-2\\0\\4\end{pmatrix} = 1 \cdot (-2) + 2 \cdot 0 + (-6) \cdot 4 = -2 + 0 - 24 = -26$$

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Exercises

Find the following products of rows and columns.

1.) $(2,8)\begin{pmatrix}1\\1\end{pmatrix}$

2.) $(-1,0)\begin{pmatrix}0\\15\end{pmatrix}$

3.) $(2,3)\begin{pmatrix}8\\-2\end{pmatrix}$

4.) $(3,-2)\begin{pmatrix}-2\\-4\end{pmatrix}$

5.) $(1,0,1)\begin{pmatrix}3\\4\\1\end{pmatrix}$

6.) $(-2,2,1)\begin{pmatrix}-2\\3\\2\end{pmatrix}$

7.) $(4,6,-3)\begin{pmatrix}-2\\4\\-2\end{pmatrix}$

8.) $$(-6, -11, -13)\begin{pmatrix}5\\0\\0\end{pmatrix}$$

Vectors & Scalars

Vectors

$\mathbb{R}^2$ is the set of all pairs of real numbers. In the context of drawing graphs, the objects in $\mathbb{R}^2$ are called points, and pairs are written left-to-right, so that $(3, 2)$ is the point in $\mathbb{R}^2$ whose x-coordinate equals 3 and whose y-coordinate equals 2.

In the context of linear algebra, the objects in $\mathbb{R}^2$ are called *vectors*, and instead of being written left-to-right, they are usually written top-to-bottom. Written in this way, the vector in $\mathbb{R}^2$ whose x-coordinate is 3 and whose y-coordinate is 2 is

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$\mathbb{R}^3$ is the set of all "triples" of real numbers. An object in $\mathbb{R}^3$ – also called a vector – has an x-coordinate, a y-coordinate, and a z-coordinate. When writing vectors in $\mathbb{R}^3$, the x-coordinate is on top, the y-coordinate is directly below, and the z-coordinate is on the bottom. Thus

$$\begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$$

is the vector in $\mathbb{R}^3$ where $x = 5$, $y = 0$, and $z = -1$.

Vector addition

To add two vectors in $\mathbb{R}^2$ – or two vectors in $\mathbb{R}^3$ – add each of their coordinates.

Examples.

$$\begin{pmatrix} -5 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -5+4 \\ 1+2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ -8 \\ 0 \end{pmatrix} = \begin{pmatrix} 4+3 \\ 2-8 \\ 6+0 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \\ 6 \end{pmatrix}$$

Scalar multiplication

In linear algebra, real numbers are often called *scalars*. You cannot multiply two vectors, but you can multiply a scalar and a vector. To do so, multiply every coordinate in the vector by the scalar.

Examples.

$$2\begin{pmatrix} 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 2(7) \\ 2(-3) \end{pmatrix} = \begin{pmatrix} 14 \\ -6 \end{pmatrix}$$

and

$$5\begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 5(-1) \\ 5(0) \\ 5(4) \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 20 \end{pmatrix}$$

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Exercises

1.) Find

$$\begin{pmatrix} -5 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

2.) Find

$$\begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ -8 \\ 0 \end{pmatrix}$$

3.) Find

$$2 \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

4.) Find

$$5 \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$

2×2 Matrices

A 2×2 matrix (pronounced "2-by-2 matrix") is a square block of 4 numbers. For example,

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

is a 2×2 matrix. It's called a 2×2 matrix because it has 2 rows and 2 columns. The four numbers in a 2×2 matrix are called the *entries* of the matrix.

Two matrices are equal if the entry in any position of the one matrix equals the entry in the same position of the other matrix.

Examples.

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$$

Scalar multiplication for matrices

To take the product of a scalar and a matrix, just as with vectors, multiply every number in the matrix by the scalar. For example,

$$2\begin{pmatrix} 2 & 1 \\ 5 & 9 \end{pmatrix} = \begin{pmatrix} 2\cdot 2 & 2\cdot 1 \\ 2\cdot 5 & 2\cdot 9 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 10 & 18 \end{pmatrix}$$

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Multiplying a matrix and a vector

Suppose A is a 2×2 matrix. To be more precise, let's say that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in \mathbb{R}$.

The matrix A and a vector

$$\begin{pmatrix} u \\ w \end{pmatrix} \in \mathbb{R}^2$$

can be combined to produce a new vector in $\mathbb{R}^2$ as follows:

Imagine the matrix A as two rows of numbers

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{matrix} (a, b) \\ (c, d) \end{matrix}$$

To find the product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}$$

we simply multiply the first row, (a, b), and the column $\begin{pmatrix} u \\ w \end{pmatrix}$ to obtain the number $(a, b) \begin{pmatrix} u \\ w \end{pmatrix} = au + bw$. That will be the first entry for the new vector in $\mathbb{R}^2$.

The second entry in the new vector will be $(c, d) \begin{pmatrix} u \\ w \end{pmatrix} = cu + dw$, or in other words, the second entry in our new vector in $\mathbb{R}^2$ will be the product of the second row of the matrix A with the column $\begin{pmatrix} u \\ w \end{pmatrix}$. Our new vector in $\mathbb{R}^2$ will then be the column

$$\begin{pmatrix} au + bw \\ cu + dw \end{pmatrix}$$

To summarize, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} au + bw \\ cu + dw \end{pmatrix}$$

Example.

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \cdot 6 + 1 \cdot 4 \\ 5 \cdot 6 + 3 \cdot 4 \end{pmatrix} = \begin{pmatrix} 16 \\ 42 \end{pmatrix}$$

Matrices as functions

Notice that if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix, and if $\begin{pmatrix} u \\ w \end{pmatrix} \in \mathbb{R}^2$ is a vector, then their product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} au + bw \\ cu + dw \end{pmatrix}$$

is also a vector in $\mathbb{R}^2$.

Therefore, if we fix a particular 2×2 matrix, it defines a way to assign to any vector in $\mathbb{R}^2$ a new vector in $\mathbb{R}^2$. That is, any 2×2 matrix describes a function whose domain is $\mathbb{R}^2$ and whose target is $\mathbb{R}^2$.

Example. Let

$$B = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$

Then B can be thought of as a function $B : \mathbb{R}^2 \to \mathbb{R}^2$.

The vector

$$\begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

is in the domain of the matrix function B. If we put this vector into B, we will get out the vector

$$\begin{pmatrix} 16 \\ 42 \end{pmatrix}$$

since

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 16 \\ 42 \end{pmatrix}$$

Identity matrix

Notice that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \cdot x + 0 \cdot y \\ 0 \cdot x + 1 \cdot y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

That means that any vector we put into the matrix function

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

gets returned to us unaltered. This is the identity function whose domain is the set $\mathbb{R}^2$, so we call

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the 2×2 *identity matrix*.

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Matrix multiplication

You can "multiply" two 2×2 matrices to obtain another 2×2 matrix.

Order the columns of a matrix from left to right, so that the 1st column is on the left and the 2nd column is on the right.

To multiply two matrices, call the columns of the matrix on the right "input columns", and put each of the input columns into the matrix on the left (thinking of it as a function). The column that is assigned to the 1st input column by the matrix function will be the 1st column of the product you are trying to find.

The column that is assigned to the 2nd input column by the matrix function will be the 2nd column of the product.

Let's try an example. To find the 2×2 matrix that equals the product

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ 7 & 5 \end{pmatrix}$$

first divide the matrix on the right into columns.

$$\begin{pmatrix} 6 & 4 \\ 7 & 5 \end{pmatrix} \mapsto \begin{pmatrix} 6 \\ 7 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Enter the leftmost column into the matrix function

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

and the result is

$$\begin{pmatrix}1 & 2\\0 & 3\end{pmatrix}\begin{pmatrix}6\\7\end{pmatrix} = \begin{pmatrix}1\cdot 6+2\cdot 7\\0\cdot 6+3\cdot 7\end{pmatrix} = \begin{pmatrix}20\\21\end{pmatrix}$$

This will be the first column of the product

$$\begin{pmatrix}1 & 2\\0 & 3\end{pmatrix}\begin{pmatrix}6 & 4\\7 & 5\end{pmatrix}$$

And it's second column will equal

$$\begin{pmatrix}1 & 2\\0 & 3\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}1\cdot 4+2\cdot 5\\0\cdot 4+3\cdot 5\end{pmatrix} = \begin{pmatrix}14\\15\end{pmatrix}$$

Put the two columns together to find that

$$\begin{pmatrix}1 & 2\\0 & 3\end{pmatrix}\begin{pmatrix}6 & 4\\7 & 5\end{pmatrix} = \begin{pmatrix}20 & 14\\21 & 15\end{pmatrix}$$

Matrix multiplication is function composition

Let A and B be 2×2 matrices. We have seen that the matrix A defines a function $A:\mathbb{R}^2\to\mathbb{R}^2$, and that there is also a function $B:\mathbb{R}^2\to\mathbb{R}^2$.

Thinking of A and B as functions, and ignoring that they are matrices, we could compose them to obtain a new function $A\circ B$. This new function is also described by a matrix – the matrix AB, where AB means the matrix multiplication of A with B.

Because function composition isn't commutative, neither is matrix multiplication. Try this yourself: write down two 2×2 matrices, A and B. Probably $AB\neq BA$.

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Inverse matrices

Again let A be a 2×2 matrix. Then there is a function $A:\mathbb{R}^2\to\mathbb{R}^2$.

Thinking of A purely as a function, and not as a matrix, the inverse of A is a function $A^{-1}:\mathbb{R}^2\to\mathbb{R}^2$ such that both $A\circ A^{-1}$ and $A^{-1}\circ A$ are the identity function.

To translate the above paragraph from functions to matrices, let's now think of A as a matrix again. Remember that function composition is really matrix multiplication, and that the matrix that represents the identity function is the identity matrix. After making these translations, we are left with the definition of the *inverse* of a 2×2 matrix A as another 2×2 matrix A^{-1} such that

$$AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example. We can check that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

by observing that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Exercises

1.) The matrix

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$$

describes a function $A : \mathbb{R}^2 \to \mathbb{R}^2$.

Find the vectors

$$\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 7 \end{pmatrix}$$

2.) The matrix

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

describes a function $B : \mathbb{R}^2 \to \mathbb{R}^2$.

Find the vectors

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 6 \end{pmatrix}$$

3.) Find the product

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 5 & 6 \end{pmatrix}$$

4.) Find the product

$$\begin{pmatrix} 3 & -4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

5.) Find the product

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3×3 Matrices

Much of this chapter is similar, to the chapter on 2×2 matrices. The most substantial difference between 2×2 matrices and 3×3 matrices is that it's harder to write a 3×3 matrix than it is to write a 2×2 matrix.

3×3 matrices have 3 rows and 3 columns. They are a square block of 9 numbers, such as

$$\begin{pmatrix} 2 & 0 & 6 \\ 4 & -5 & 14 \\ -10 & 3 & 4 \end{pmatrix}$$

Two matrices are equal if the entry in any position of the one matrix equals the entry in the same position of the other matrix.

Example.

$$\begin{pmatrix} 3 & 2 & 7 \\ 5 & 0 & -1 \\ 6 & 9 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 7 \\ 5 & 0 & -1 \\ 6 & 9 & 4 \end{pmatrix}$$

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Matrices as functions

A 3×3 matrix defines a function whose domain is $\mathbb{R}^3$ and whose target is $\mathbb{R}^3$. The function is defined as follows:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} au + bv + cw \\ du + ev + fw \\ gu + hv + iw \end{pmatrix}$$

Notice that the first, second, or third entry in the vector that on the right of the above equation can be found by multiplying the first, second, or third row of the matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

and the column

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Example. The matrix

$$\begin{pmatrix} 5 & 3 & 1 \\ -2 & 2 & 4 \\ 7 & 0 & -1 \end{pmatrix}$$

has 3 rows and 3 columns, so it is a function whose domain is $\mathbb{R}^3$, and whose target is $\mathbb{R}^3$.

Because,

$$\begin{pmatrix} 2 \\ 9 \\ -3 \end{pmatrix}$$

is a vector in $\mathbb{R}^3$,

$$\begin{pmatrix} 5 & 3 & 1 \\ -2 & 2 & 4 \\ 7 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \\ -3 \end{pmatrix}$$

is also a vector in $\mathbb{R}^3$. The vector it equals is

$$\begin{aligned} \begin{pmatrix} 5 & 3 & 1 \\ -2 & 2 & 4 \\ 7 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \\ -3 \end{pmatrix} &= \begin{pmatrix} 5 \cdot 2 & + & 3 \cdot 9 & + & 1 \cdot (-3) \\ -2 \cdot 2 & + & 2 \cdot 9 & + & 4 \cdot (-3) \\ 7 \cdot 2 & + & 0 \cdot 9 & + & (-1) \cdot (-3) \end{pmatrix} \\ &= \begin{pmatrix} 10 + 27 - 3 \\ -4 + 18 - 12 \\ 14 + 0 + 3 \end{pmatrix} \\ &= \begin{pmatrix} 34 \\ 2 \\ 17 \end{pmatrix} \end{aligned}$$

Identity matrix

Notice that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \cdot x + 0 \cdot y + 0 \cdot z \\ 0 \cdot x + 1 \cdot y + 0 \cdot z \\ 0 \cdot x + 0 \cdot y + 1 \cdot z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Thus,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the identity function whose domain is $\mathbb{R}^3$. We call this matrix the 3×3 *identity matrix*.

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Matrix multiplication

You can "multiply" two 3×3 matrices to obtain another 3×3 matrix.

Order the columns of a matrix from left to right, so that the 1^{st} column is on the left, the 2^{nd} column is directly to the right of the 1^{st}, and the 3^{rd} column is to the right of the 2^{nd}.

To multiply two matrices, call the columns of the matrix on the right "input columns", and put each of the input columns into the matrix on the left (thinking of it as a function). The column that is assigned to the 1^{st} input column by the matrix function will be the 1^{st} column of the product you are trying to find.

The column that is assigned to the 2^{nd} input column by the matrix function will be the 2^{nd} column of the product, and the column that is assigned to the 3^{rd} input column by the matrix function will be the 3^{rd} column of the product.

The product

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 8 \\ 9 & 4 & 1 \end{pmatrix} \begin{pmatrix} 13 & 14 & 17 \\ 12 & 11 & 18 \\ 15 & 19 & 16 \end{pmatrix}$$

will be a 3×3 matrix whose first column (when read left-to-right) equals

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 8 \\ 9 & 4 & 1 \end{pmatrix} \begin{pmatrix} 13 \\ 12 \\ 15 \end{pmatrix}$$

whose second column equals

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 8 \\ 9 & 4 & 1 \end{pmatrix} \begin{pmatrix} 14 \\ 11 \\ 19 \end{pmatrix}$$

and whose third column is

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 8 \\ 9 & 4 & 1 \end{pmatrix} \begin{pmatrix} 17 \\ 18 \\ 16 \end{pmatrix}$$

Matrix multiplication is function composition

If A and B are 3×3 matrices, then the result of multiplying the matrices AB would determine the same function $AB : \mathbb{R}^3 \to \mathbb{R}^3$ as the function that results from composition, namely $A \circ B$.

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Inverse matrices

If B is a 3×3 matrix, then B^{-1} is the 3×3 matrix where

$$BB^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$B^{-1}B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Exercises

1.) The matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 3 & -2 & 0 \end{pmatrix}$$

describes a function $A : \mathbb{R}^3 \to \mathbb{R}^3$.

Find the vectors

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 3 & -2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 3 & -2 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

2.) The matrix

$$B = \begin{pmatrix} 3 & 2 & -2 \\ 1 & 0 & 10 \\ 4 & -5 & 7 \end{pmatrix}$$

describes a function $B : \mathbb{R}^3 \to \mathbb{R}^3$.

Find the vectors

$$\begin{pmatrix} 3 & 2 & -2 \\ 1 & 0 & 10 \\ 4 & -5 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 2 & -2 \\ 1 & 0 & 10 \\ 4 & -5 & 7 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

3.) Find the product

$$\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & -4 \end{pmatrix}$$

4.) Find the product

$$\begin{pmatrix} 2 & -1 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix}$$

5.) Find the product

$$\begin{pmatrix} 3 & -17 & 5 \\ 17 & 3 & 34 \\ 41 & 3 & 18 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Determinants & Inverse Matrices

The *determinant* of the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the number $ad - cb$.

The above sentence is abbreviated as

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

Example.

$$\det \begin{pmatrix} 4 & -2 \\ 1 & -3 \end{pmatrix} = 4(-3) - 1(-2) = -12 + 2 = -10$$

The determinant of a 3×3 matrix can be found using the formula

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Example.

$$\det \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & 0 & 1 \end{pmatrix} = 2 \det \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix} - (-1) \det \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix} + 0 \det \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$$

$$= 2[3 \cdot 1 - 0(-2)] + [0 \cdot 1 - 1(-2)] + 0$$

$$= 2 \cdot 3 + 2$$

$$= 8$$

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Determinants and inverses

A matrix has an inverse exactly when its determinant is not equal to 0.

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2×2 inverses

Suppose that the determinant of the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

does not equal 0. Then the matrix has an inverse, and it can be found using the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Notice that in the above formula we are allowed to divide by the determinant since we are assuming that it's not 0.

Example. To find

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{-1}$$

first check that

$$\det \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = 3 \cdot 2 - 1 \cdot 5 = 1$$

Then

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{1} \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$$

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3×3 inverses

There is a way to find an inverse of a 3×3 matrix – or for that matter, an $n \times n$ matrix – whose determinant is not 0, but it isn't quite as simple as finding the inverse of a 2×2 matrix. You can learn how to do it if you take a linear algebra course. You could also find websites that will invert matrices for you, and some calculators can find the inverses of matrices as long as the matrices are not too large.

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Exercises

For #1-6, compute the given determinant.

1.)

$$\det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

2.)

$$\det \begin{pmatrix} 1 & \pi \\ 0 & 1 \end{pmatrix}$$

3.)

$$\det \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$$

4.)

$$\det \begin{pmatrix} 3 & 0 & 0 \\ 107 & 1 & 0 \\ \sqrt{2} & 2 & 6 \end{pmatrix}$$

5.)

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$$

6.)

$$\det \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix}$$

7.) Which of the six matrices in the previous problems have an inverse?

8.) What is the inverse of

$$\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \quad ?$$

9.) What is the inverse of

$$\begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix} ?$$

10.) What is the inverse of

$$\begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix} ?$$

11.) Are the following pair of matrices inverses of each other?

$$\begin{pmatrix} 4 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 3 \\ -1 & 0 & 2 \end{pmatrix}$$

12.) Are the following pair of matrices inverses of each other?

$$\begin{pmatrix} -1 & 1 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix Equations

This chapter consists of 3 example problems of how to use a "matrix equation" to solve a system of three linear equations in three variables.

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Problem 1. Use that

$$\begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ -4 & 5 & 2 \end{pmatrix}$$

to find $x, y, z \in \mathbb{R}$ if

$$\begin{aligned} -x + 2y - z &= -1 \\ -2x + 2y - z &= -3 \\ 3x - \quad y + z &= 8 \end{aligned}$$

Solution. Another way to write the system of three equations above is to write a single equation of 3-dimensional vectors as follows:

$$\begin{pmatrix} -x + 2y - z \\ -2x + 2y - z \\ 3x - \quad y + z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$$

The above "vector equation" can be written without omitting the coefficients that equal 1:

$$\begin{pmatrix} -1x + 2y - 1z \\ -2x + 2y - 1z \\ 3x - 1y + 1z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$$

The vector on the left in the above equation equals

$$\begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Hence, the system of three equations we began the problem with can be rewritten as the "matrix equation"

$$\begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$$

We can "erase" the matrix on the left of the above equation by applying its inverse to the right side of the equation. That would leave us with

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$$

At the start of the problem we were told what the above inverse matrix equals, so we can substitute for it below:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ -4 & 5 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

In other words, $x = 2$, $y = 3$, and $z = 5$ is the solution to the system.

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Terminology

The matrix equation from Problem 1 was

$$\begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$$

In any such matrix equation,

$$\begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix}$$

is called the **coefficient matrix** since it is formed from the coefficients of the system of linear equations. The vector

$$\begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$$

is called the **constant vector** because it is formed from the constants of the system of linear equations. The vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is called the **variable vector**.

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Problem 2. Use that

$$\begin{pmatrix} 4 & 2 & -6 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/4 & -2 & 7/2 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

to find $x, y, z \in \mathbb{R}$ if

$$\begin{aligned} 4x + 2y - 6z &= -4 \\ 2y + z &= 4 \\ y + z &= 6 \end{aligned}$$

Solution. Write the coefficients of the system as a matrix, remembering to include the "invisible" coefficients that equal 1 or 0. To the right of that matrix, write the "variable vector", then an equal sign, and then the vector of numbers given by the constants in the system of equations. The result should look like

$$\begin{pmatrix} 4 & 2 & -6 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 6 \end{pmatrix}$$

Now use the inverse matrix from the hint, so that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/4 & -2 & 7/2 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -4 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ -2 \\ 8 \end{pmatrix}$$

The matrix equation above says that $x = 12$, $y = -2$, and $z = 8$. That's the answer.

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Problem 3. Use that

$$\begin{pmatrix} 3 & 7 & 3 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 & 1 \\ 1 & -3 & 0 \\ -1 & 5 & -1 \end{pmatrix}$$

to find $x, y, z \in \mathbb{R}$ if

$$\begin{aligned} 3x+7y+3z &= 2 \\ x+2y+\ \ z &= 4 \\ 2x+3y+\ \ z &= 1 \end{aligned}$$

Solution. The answer is

$$\begin{pmatrix} -1 & 2 & 1 \\ 1 & -3 & 0 \\ -1 & 5 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

Check that this equals the vector

$$\begin{pmatrix} 7 \\ -10 \\ 17 \end{pmatrix}$$

so the solution is $x = 7$, $y = -10$, and $z = 17$.

Exercises

1.) Write the coefficient matrix, the variable vector, and the constant vector for the system of equations below.

$$\begin{aligned} 2x + 4y \quad &= -7 \\ -x + y - 4z &= 0 \\ x \quad + 3z &= 5 \end{aligned}$$

For #2-5, find solutions for the given systems of linear equations.

2.)

$$\begin{aligned} 4x + 3y + 3z &= 2 \\ 3x + y + 2z &= 4 \\ -x - y - z &= -2 \end{aligned}$$

$$\text{Hint: } \begin{pmatrix} 4 & 3 & 3 \\ 3 & 1 & 2 \\ -1 & -1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & 1 \\ -2 & 1 & -5 \end{pmatrix}$$

3.)

$$\begin{aligned} 3x - y - z &= 3 \\ 8x + 10y + 3z &= 2 \\ 2x + 3y + z &= 1 \end{aligned}$$

$$\text{Hint: } \begin{pmatrix} 3 & -1 & -1 \\ 8 & 10 & 3 \\ 2 & 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 & 7 \\ -2 & 5 & -17 \\ 4 & -11 & 38 \end{pmatrix}$$

4.)

$$\begin{aligned} x+5y+7z &= -2 \\ y+8z &= -1 \\ z &= 5 \end{aligned}$$

$$\text{Hint:} \quad \begin{pmatrix} 1 & 5 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -5 & 33 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{pmatrix}$$

5.)

$$\begin{aligned} -x+7y+4z &= 4 \\ -3x \qquad\; - \;\; z &= 3 \\ x+ \;\; y+ \;\; z &= 1 \end{aligned}$$

$$\text{Hint:} \quad \begin{pmatrix} -1 & 7 & 4 \\ -3 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -3 & -7 \\ 2 & -5 & -13 \\ -3 & 8 & 21 \end{pmatrix}$$