More on functions

Suppose $f : \mathbb{R} \to \mathbb{R}$ is the function defined by $f(x) = x^5$. The letter x in the previous equation is just a placeholder. You are allowed to replace the x with any number, symbol, or combination of symbols that you like.

$$f(4) = 4^5 \qquad\qquad f(-1) = (-1)^5$$

$$f(\pi) = \pi^5 \qquad\qquad f\big(-\tfrac{11}{13}\big) = \big(-\tfrac{11}{13}\big)^5$$

$$f(\blacksquare) = \blacksquare^5 \qquad\qquad f(\clubsuit) = \clubsuit^5$$

$$f(y) = y^5 \qquad\qquad f(x-3) = (x-3)^5$$

$$f(g(x)) = (g(x))^5 \qquad\qquad f\left(\tfrac{1-x}{x^2+4}\right) = \left(\tfrac{1-x}{x^2+4}\right)^5$$

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Composition

Suppose that $f : A \to B$ and $g : B \to C$ are functions.

If $a \in A$, then a is in the domain of f and $f(a) \in B$. Since $f(a) \in B$, we have that $f(a)$ is in the domain of g, so $g(f(a))$ is an object in C.

This process defines a third function, named $g \circ f : A \to C$ that is defined by

$$g \circ f(a) = g(f(a))$$

The function $g \circ f$ is pronounced "g composed with f".

Examples.

- Suppose $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are functions and that $g(1) = 3$ and $h(3) = 7$. Then

$$h \circ g(1) = h(g(1)) = h(3) = 7$$

- Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions that are defined by $f(x) = x^2$ and $g(x) = x - 1$.

Then $g \circ f : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$g \circ f(x) = g(f(x)) = g(x^2) = x^2 - 1$$

And $f \circ g : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$f \circ g(x) = f(g(x)) = f(x-1) = (x-1)^2$$

Important: Notice in the previous example that $g \circ f(2) = 3$ and $f \circ g(2) = 1$. That means that $g \circ f$ is not the same function as $f \circ g$. In other words, $g \circ f \neq f \circ g$.

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Range

Recall that the target of a function $f : A \to B$ is the set B. That means that for any $a \in A$, we have that $f(a) \in B$.

But it might not be that every object in B has an object from A assigned to it by the function f. For example, you might recall that if you square a real number, the result is never a negative number ($2^2 = 4$, $(-3)^2 = 9$, etc.). Therefore, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ has $\mathbb{R}$ as its target, although none of the negative numbers in the target have a number from the domain assigned to it.

The *range* of a function f is the set of numbers that "come out of" f. For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x - 2$, then $f(3) = 3 - 2 = 1$. We put 3 in to f, and got 1 out, so 1 is an object in the range.

Another way to say what the range is, is to say that it is the smallest set that can serve as the target of the function.

Examples.

- Let $h : \{1, 2, 3, 4\} \to \{3, 4, 7, 8, 9\}$ be the function given by

$$h(1) = 9 \qquad\qquad h(2) = 4$$

$$h(3) = 4 \qquad\qquad h(4) = 8$$

If we put the numbers from the domain "in to" h, the only numbers that "come out" are 9, 4, and 8. That means that the range of h is $\{9, 4, 8\}$.

• If $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = \frac{x-1}{x^2+1}$, then

$$g(1) = \frac{1-1}{1^2+1} = \frac{0}{2} = 0$$

and

$$g(2) = \frac{2-1}{2^2+1} = \frac{1}{5}$$

Therefore, 0 and $\frac{1}{5}$ are both objects in the range of g.

There are also other numbers in the range of g. For example, $g(4)$, $g(-1)$, $g(\sqrt{2})$, etc.

• The range of $f : \mathbb{N} \to \mathbb{R}$ where $f(n) = (-1)^n$ is the set $\{-1, 1\}$.

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Implied domains

Sometimes we won't go through the trouble of writing the entire name of a function as "$f : D \to T$ where $f(x) = x^5$". This is similar to how we usually call people by their first names, omitting their middle and last names, just because it's easier.

If we are introduced to a function that is given by an equation, and its domain is not specified, we will assume that the domain for that function is the largest subset of the real numbers possible. This set will be called the *implied domain* of the function.

Examples.

• Let $h(x) = 4x - 1$. Then for any real number $r \in \mathbb{R}$, $h(r) = 4r - 1$ makes sense, because it is a real number itself. Therefore, it is safe to put any real number into h. Its implied domain is $\mathbb{R}$.

• Let $f(x) = \frac{5}{x-1}$. If r is a real number, then $r - 1$ is a real number. As long as $r - 1 \neq 0$, $\frac{5}{r-1}$ is also a real number. However, if $r - 1 = 0$, then $\frac{5}{r-1} = \frac{5}{0}$ does not make sense – it is not a real number.

To recap, $f(r)$ is a real number except when $r - 1 = 0$, or equivalently, except when $r = 1$. Therefore, the numbers that it makes sense to "put in to" f are all of the real numbers except for 1. Another way of saying the previous sentence, is that the implied domain of f is the set $\mathbb{R} - \{1\}$.

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Exercises

For #1-6, assume that $f(x) = x^2$, $g(x) = 2x - 1$, and that $h(x) = x - 5$, and match the given function with one of the following:

A. $2x - 11$	B. $x^2 - 10x + 25$	C. $2x - 6$
D. $x^2 - 5$	E. $2x^2 - 1$	F. $4x^2 - 4x + 1$

1) $f \circ g(x)$

2) $g \circ f(x)$

3) $g \circ h(x)$

4) $h \circ g(x)$

5) $f \circ h(x)$

6) $h \circ f(x)$

For #7-12, assume that $f(x) = x+3$, $g(x) = 3x-4$, and that $h(x) = x^2+1$, and match the given function with one of the following:

A. $3x + 5$	B. $9x^2 - 24x + 17$	C. $3x^2 - 1$
D. $x^2 + 6x + 10$	E. $x^2 + 4$	F. $3x - 1$

7) $f \circ g(x)$

8) $g \circ f(x)$

9) $g \circ h(x)$

10) $h \circ g(x)$

11) $f \circ h(x)$

12) $h \circ f(x)$

13) What is the implied domain of $f(x) = 3x - 4$?

14) What is the implied domain of $g(x) = x^3 - 4x^2 - 2x + 1$?

15) What is the implied domain of $h(x) = 15x - 3$?

16) What is the implied domain of

$$f(x) = \frac{1}{x-3} \ ?$$

17) What is the implied domain of

$$g(x) = \frac{2x-4}{x+3} \ ?$$

18) What is the implied domain of

$$h(x) = \frac{3x^2 - 4x + 5}{-3x + 4} \ ?$$