

Matrix Equations

This chapter consists of 3 example problems of how to use a “matrix equation” to solve a system of three linear equations in three variables.

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Problem 1. Use that

$$\begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ -4 & 5 & 2 \end{pmatrix}$$

to find $x, y, z \in \mathbb{R}$ if

$$\begin{aligned} -x + 2y - z &= -1 \\ -2x + 2y - z &= -3 \\ 3x - y + z &= 8 \end{aligned}$$

Solution. Another way to write the system of three equations above is to write a single equation of 3-dimensional vectors as follows:

$$\begin{pmatrix} -x + 2y - z \\ -2x + 2y - z \\ 3x - y + z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$$

The above “vector equation” can be written without omitting the coefficients that equal 1:

$$\begin{pmatrix} -1x + 2y - 1z \\ -2x + 2y - 1z \\ 3x - 1y + 1z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$$

The vector on the left in the above equation equals

$$\begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Hence, the system of three equations we began the problem with can be rewritten as the “matrix equation”

$$\begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$$

We can “erase” the matrix on the left of the above equation by applying its inverse to the right side of the equation. That would leave us with

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$$

At the start of the problem we were told what the above inverse matrix equals, so we can substitute for it below:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ -4 & 5 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

In other words, $x = 2$, $y = 3$, and $z = 5$ is the solution to the system.

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Terminology

The matrix equation from Problem 1 was

$$\begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$$

In any such matrix equation,

$$\begin{pmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix}$$

is called the **coefficient matrix** since it is formed from the coefficients of the system of linear equations. The vector

$$\begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$$

is called the **constant vector** because it is formed from the constants of the system of linear equations. The vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is called the **variable vector**.

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Problem 2. Use that

$$\begin{pmatrix} 4 & 2 & -6 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/4 & -2 & 7/2 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

to find $x, y, z \in \mathbb{R}$ if

$$4x + 2y - 6z = -4$$

$$2y + z = 4$$

$$y + z = 6$$

Solution. Write the coefficients of the system as a matrix, remembering to include the “invisible” coefficients that equal 1 or 0. To the right of that matrix, write the “variable vector”, then an equal sign, and then the vector of numbers given by the constants in the system of equations. The result should look like

$$\begin{pmatrix} 4 & 2 & -6 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 6 \end{pmatrix}$$

Now use the inverse matrix from the hint, so that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/4 & -2 & 7/2 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -4 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ -2 \\ 8 \end{pmatrix}$$

The matrix equation above says that $x = 12$, $y = -2$, and $z = 8$. That's the answer.

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Problem 3. Use that

$$\begin{pmatrix} 3 & 7 & 3 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 & 1 \\ 1 & -3 & 0 \\ -1 & 5 & -1 \end{pmatrix}$$

to find $x, y, z \in \mathbb{R}$ if

$$3x + 7y + 3z = 2$$

$$x + 2y + z = 4$$

$$2x + 3y + z = 1$$

Solution. The answer is

$$\begin{pmatrix} -1 & 2 & 1 \\ 1 & -3 & 0 \\ -1 & 5 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

Check that this equals the vector

$$\begin{pmatrix} 7 \\ -10 \\ 17 \end{pmatrix}$$

so the solution is $x = 7$, $y = -10$, and $z = 17$.

Exercises

1.) Write the coefficient matrix, the variable vector, and the constant vector for the system of equations below.

$$\begin{aligned}2x + 4y &= -7 \\ -x + y - 4z &= 0 \\ x + 3z &= 5\end{aligned}$$

For #2-5, find solutions for the given systems of linear equations.

2.)

$$\begin{aligned}4x + 3y + 3z &= 2 \\ 3x + y + 2z &= 4 \\ -x - y - z &= -2\end{aligned}$$

$$\text{Hint: } \begin{pmatrix} 4 & 3 & 3 \\ 3 & 1 & 2 \\ -1 & -1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & 1 \\ -2 & 1 & -5 \end{pmatrix}$$

3.)

$$\begin{aligned}3x - y - z &= 3 \\ 8x + 10y + 3z &= 2 \\ 2x + 3y + z &= 1\end{aligned}$$

$$\text{Hint: } \begin{pmatrix} 3 & -1 & -1 \\ 8 & 10 & 3 \\ 2 & 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 & 7 \\ -2 & 5 & -17 \\ 4 & -11 & 38 \end{pmatrix}$$

4.)

$$\begin{aligned}x + 5y + 7z &= -2 \\y + 8z &= -1 \\z &= 5\end{aligned}$$

$$\text{Hint: } \begin{pmatrix} 1 & 5 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -5 & 33 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{pmatrix}$$

5.)

$$\begin{aligned}-x + 7y + 4z &= 4 \\-3x \quad - \quad z &= 3 \\x + \quad y + \quad z &= 1\end{aligned}$$

$$\text{Hint: } \begin{pmatrix} -1 & 7 & 4 \\ -3 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -3 & -7 \\ 2 & -5 & -13 \\ -3 & 8 & 21 \end{pmatrix}$$