

3 × 3 Matrices

Much of this chapter is similar, to the chapter on 2 × 2 matrices. The most substantial difference between 2 × 2 matrices and 3 × 3 matrices is that it's harder to write a 3 × 3 matrix than it is to write a 2 × 2 matrix.

3 × 3 matrices have 3 rows and 3 columns. They are a square block of 9 numbers, such as

$$\begin{pmatrix} 2 & 0 & 6 \\ 4 & -5 & 14 \\ -10 & 3 & 4 \end{pmatrix}$$

Two matrices are equal if the entry in any position of the one matrix equals the entry in the same position of the other matrix.

Example.

$$\begin{pmatrix} 3 & 2 & 7 \\ 5 & 0 & -1 \\ 6 & 9 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 7 \\ 5 & 0 & -1 \\ 6 & 9 & 4 \end{pmatrix}$$

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Matrices as functions

A 3 × 3 matrix defines a function whose domain is \mathbb{R}^3 and whose target is \mathbb{R}^3 . The function is defined as follows:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} au + bv + cw \\ du + ev + fw \\ gu + hv + iw \end{pmatrix}$$

Notice that the first, second, or third entry in the vector that on the right of the above equation can be found by multiplying the first, second, or third row of the matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

and the column

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Example. The matrix

$$\begin{pmatrix} 5 & 3 & 1 \\ -2 & 2 & 4 \\ 7 & 0 & -1 \end{pmatrix}$$

has 3 rows and 3 columns, so it is a function whose domain is \mathbb{R}^3 , and whose target is \mathbb{R}^3 .

Because,

$$\begin{pmatrix} 2 \\ 9 \\ -3 \end{pmatrix}$$

is a vector in \mathbb{R}^3 ,

$$\begin{pmatrix} 5 & 3 & 1 \\ -2 & 2 & 4 \\ 7 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \\ -3 \end{pmatrix}$$

is also a vector in \mathbb{R}^3 . The vector it equals is

$$\begin{aligned} \begin{pmatrix} 5 & 3 & 1 \\ -2 & 2 & 4 \\ 7 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \\ -3 \end{pmatrix} &= \begin{pmatrix} 5 \cdot 2 + 3 \cdot 9 + 1 \cdot (-3) \\ -2 \cdot 2 + 2 \cdot 9 + 4 \cdot (-3) \\ 7 \cdot 2 + 0 \cdot 9 + (-1) \cdot (-3) \end{pmatrix} \\ &= \begin{pmatrix} 10 + 27 - 3 \\ -4 + 18 - 12 \\ 14 + 0 + 3 \end{pmatrix} \\ &= \begin{pmatrix} 34 \\ 2 \\ 17 \end{pmatrix} \end{aligned}$$

Identity matrix

Notice that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \cdot x + 0 \cdot y + 0 \cdot z \\ 0 \cdot x + 1 \cdot y + 0 \cdot z \\ 0 \cdot x + 0 \cdot y + 1 \cdot z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Thus,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the identity function whose domain is \mathbb{R}^3 . We call this matrix the 3×3 *identity matrix*.

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Matrix multiplication

You can “multiply” two 3×3 matrices to obtain another 3×3 matrix.

Order the columns of a matrix from left to right, so that the 1st column is on the left, the 2nd column is directly to the right of the 1st, and the 3rd column is to the right of the 2nd.

To multiply two matrices, call the columns of the matrix on the right “input columns”, and put each of the input columns into the matrix on the left (thinking of it as a function). The column that is assigned to the 1st input column by the matrix function will be the 1st column of the product you are trying to find.

The column that is assigned to the 2nd input column by the matrix function will be the 2nd column of the product, and the column that is assigned to the 3rd input column by the matrix function will be the 3rd column of the product.

The product

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 8 \\ 9 & 4 & 1 \end{pmatrix} \begin{pmatrix} 13 & 14 & 17 \\ 12 & 11 & 18 \\ 15 & 19 & 16 \end{pmatrix}$$

will be a 3×3 matrix whose first column (when read left-to-right) equals

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 8 \\ 9 & 4 & 1 \end{pmatrix} \begin{pmatrix} 13 \\ 12 \\ 15 \end{pmatrix}$$

whose second column equals

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 8 \\ 9 & 4 & 1 \end{pmatrix} \begin{pmatrix} 14 \\ 11 \\ 19 \end{pmatrix}$$

and whose third column is

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 8 \\ 9 & 4 & 1 \end{pmatrix} \begin{pmatrix} 17 \\ 18 \\ 16 \end{pmatrix}$$

Matrix multiplication is function composition

If A and B are 3×3 matrices, then the result of multiplying the matrices AB would determine the same function $AB : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ as the function that results from composition, namely $A \circ B$.

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Inverse matrices

If B is a 3×3 matrix, then B^{-1} is the 3×3 matrix where

$$BB^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$B^{-1}B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Exercises

1.) The matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 3 & -2 & 0 \end{pmatrix}$$

describes a function $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

Find the vectors

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 3 & -2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 3 & -2 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

2.) The matrix

$$B = \begin{pmatrix} 3 & 2 & -2 \\ 1 & 0 & 10 \\ 4 & -5 & 7 \end{pmatrix}$$

describes a function $B : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

Find the vectors

$$\begin{pmatrix} 3 & 2 & -2 \\ 1 & 0 & 10 \\ 4 & -5 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 2 & -2 \\ 1 & 0 & 10 \\ 4 & -5 & 7 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

3.) Find the product

$$\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & -4 \end{pmatrix}$$

4.) Find the product

$$\begin{pmatrix} 2 & -1 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix}$$

5.) Find the product

$$\begin{pmatrix} 3 & -17 & 5 \\ 17 & 3 & 34 \\ 41 & 3 & 18 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$