2×2 Matrices

A 2×2 matrix (pronounced "2-by-2 matrix") is a square block of 4 numbers. For example,

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

is a 2×2 matrix. It's called a 2×2 matrix because it has 2 rows and 2 columns. The four numbers in a 2×2 matrix are called the *entries* of the matrix.

Two matrices are equal if the entry in any position of the one matrix equals the entry in the same position of the other matrix.

Examples.

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$$

Scalar multiplication for matrices

To take the product of a scalar and a matrix, just as with vectors, multiply every number in the matrix by the scalar. For example,

$$2 \begin{pmatrix} 2 & 1 \\ 5 & 9 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 & 2 \cdot 1 \\ 2 \cdot 5 & 2 \cdot 9 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 10 & 18 \end{pmatrix}$$

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Multiplying a matrix and a vector

Suppose A is a 2×2 matrix. To be more precise, let's say that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in \mathbb{R}$.

The matrix A and a vector

$$\begin{pmatrix} u \\ w \end{pmatrix} \in \mathbb{R}^2$$

can be combined to produce a new vector in $\mathbb{R}^2$ as follows:

Imagine the matrix A as two rows of numbers

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{matrix} (a, b) \\ (c, d) \end{matrix}$$

To find the product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}$$

we simply multiply the first row, (a, b), and the column $\begin{pmatrix} u \\ w \end{pmatrix}$ to obtain the number $(a, b) \begin{pmatrix} u \\ w \end{pmatrix} = au + bw$. That will be the first entry for the new vector in $\mathbb{R}^2$.

The second entry in the new vector will be $(c, d) \begin{pmatrix} u \\ w \end{pmatrix} = cu + dw$, or in other words, the second entry in our new vector in $\mathbb{R}^2$ will be the product of the second row of the matrix A with the column $\begin{pmatrix} u \\ w \end{pmatrix}$. Our new vector in $\mathbb{R}^2$ will then be the column

$$\begin{pmatrix} au + bw \\ cu + dw \end{pmatrix}$$

To summarize, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} au + bw \\ cu + dw \end{pmatrix}$$

Example.

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \cdot 6 + 1 \cdot 4 \\ 5 \cdot 6 + 3 \cdot 4 \end{pmatrix} = \begin{pmatrix} 16 \\ 42 \end{pmatrix}$$

Matrices as functions

Notice that if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix, and if $\begin{pmatrix} u \\ w \end{pmatrix} \in \mathbb{R}^2$ is a vector, then their product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} au + bw \\ cu + dw \end{pmatrix}$$

is also a vector in $\mathbb{R}^2$.

Therefore, if we fix a particular 2×2 matrix, it defines a way to assign to any vector in $\mathbb{R}^2$ a new vector in $\mathbb{R}^2$. That is, any 2×2 matrix describes a function whose domain is $\mathbb{R}^2$ and whose target is $\mathbb{R}^2$.

Example. Let

$$B = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$

Then B can be thought of as a function $B : \mathbb{R}^2 \to \mathbb{R}^2$.

The vector

$$\begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

is in the domain of the matrix function B. If we put this vector into B, we will get out the vector

$$\begin{pmatrix} 16 \\ 42 \end{pmatrix}$$

since

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 16 \\ 42 \end{pmatrix}$$

Identity matrix

Notice that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \cdot x + 0 \cdot y \\ 0 \cdot x + 1 \cdot y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

That means that any vector we put into the matrix function

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

gets returned to us unaltered. This is the identity function whose domain is the set $\mathbb{R}^2$, so we call

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the 2×2 *identity matrix*.

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Matrix multiplication

You can "multiply" two 2×2 matrices to obtain another 2×2 matrix.

Order the columns of a matrix from left to right, so that the 1st column is on the left and the 2nd column is on the right.

To multiply two matrices, call the columns of the matrix on the right "input columns", and put each of the input columns into the matrix on the left (thinking of it as a function). The column that is assigned to the 1st input column by the matrix function will be the 1st column of the product you are trying to find.

The column that is assigned to the 2nd input column by the matrix function will be the 2nd column of the product.

Let's try an example. To find the 2×2 matrix that equals the product

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ 7 & 5 \end{pmatrix}$$

first divide the matrix on the right into columns.

$$\begin{pmatrix} 6 & 4 \\ 7 & 5 \end{pmatrix} \mapsto \begin{pmatrix} 6 \\ 7 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Enter the leftmost column into the matrix function

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

and the result is
$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \cdot 6 + 2 \cdot 7 \\ 0 \cdot 6 + 3 \cdot 7 \end{pmatrix} = \begin{pmatrix} 20 \\ 21 \end{pmatrix}$$
This will be the first column of the product
$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ 7 & 5 \end{pmatrix}$$
And it's second column will equal
$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \cdot 4 + 2 \cdot 5 \\ 0 \cdot 4 + 3 \cdot 5 \end{pmatrix} = \begin{pmatrix} 14 \\ 15 \end{pmatrix}$$
Put the two columns together to find that
$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} 20 & 14 \\ 21 & 15 \end{pmatrix}$$

Matrix multiplication is function composition

Let A and B be 2×2 matrices. We have seen that the matrix A defines a function $A : \mathbb{R}^2 \to \mathbb{R}^2$, and that there is also a function $B : \mathbb{R}^2 \to \mathbb{R}^2$.

Thinking of A and B as functions, and ignoring that they are matrices, we could compose them to obtain a new function $A \circ B$. This new function is also described by a matrix – the matrix AB, where AB means the matrix multiplication of A with B.

Because function composition isn't commutative, neither is matrix multiplication. Try this yourself: write down two 2×2 matrices, A and B. Probably $AB \neq BA$.

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Inverse matrices

Again let A be a 2×2 matrix. Then there is a function $A : \mathbb{R}^2 \to \mathbb{R}^2$.

Thinking of A purely as a function, and not as a matrix, the inverse of A is a function $A^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ such that both $A \circ A^{-1}$ and $A^{-1} \circ A$ are the identity function.

To translate the above paragraph from functions to matrices, let's now think of A as a matrix again. Remember that function composition is really matrix multiplication, and that the matrix that represents the identity function is the identity matrix. After making these translations, we are left with the definition of the *inverse* of a 2×2 matrix A as another 2×2 matrix A^{-1} such that

$$AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example. We can check that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

by observing that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Exercises

1.) The matrix
$$A = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$$
describes a function $A : \mathbb{R}^2 \to \mathbb{R}^2$.
  Find the vectors
$$\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 7 \end{pmatrix}$$

2.) The matrix
$$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$
describes a function $B : \mathbb{R}^2 \to \mathbb{R}^2$.
  Find the vectors
$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 6 \end{pmatrix}$$

3.) Find the product
$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 5 & 6 \end{pmatrix}$$

4.) Find the product
$$\begin{pmatrix} 3 & -4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

5.) Find the product
$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$