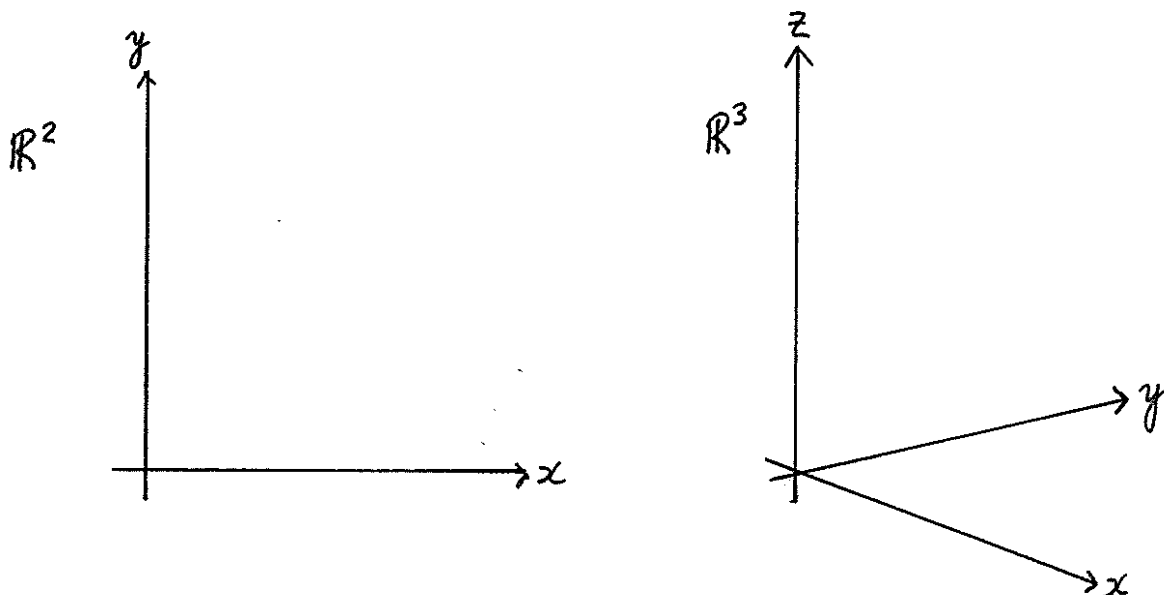


Linear Equations in Three Variables

\mathbb{R}^2 is the space of 2 dimensions. There is an x -coordinate that can be any real number, and there is a y -coordinate that can be any real number.

\mathbb{R}^3 is the space of 3 dimensions. There is an x , y , and z coordinate. Each coordinate can be any real number.



Linear equations in three variables

If a , b , c and r are real numbers (and if a , b , and c are not all equal to 0) then $ax + by + cz = r$ is called a *linear equation in three variables*. (The “three variables” are the x , the y , and the z .)

The numbers a , b , and c are called the *coefficients* of the equation. The number r is called the *constant* of the equation.

Examples. $3x + 4y - 7z = 2$, $-2x + y - z = -6$, $x - 17z = 4$, $4y = 0$, and $x + y + z = 2$ are all linear equations in three variables.

Solutions to equations

A *solution* to a linear equation in three variables $ax + by + cz = r$ is a specific point in \mathbb{R}^3 such that when the x -coordinate of the point is multiplied by a , the y -coordinate of the point is multiplied by b , the z -coordinate of the point is multiplied by c , and then those three products are added together, the answer equals r . (There are always infinitely many solutions to a linear equation in three variables.)

Example. The point $x = 1$, $y = 2$, and $z = -1$ is a solution to the equation

$$-2x + 5y + z = 7$$

since

$$-2(1) + 5(2) + (-1) = -2 + 10 - 1 = 7$$

The point $x = 3$, $y = -2$, and $z = 4$ is a *not* a solution to the equation $-2x + 5y + z = 7$ since

$$-2(3) + 5(-2) + (4) = -6 - 10 + 4 = -12$$

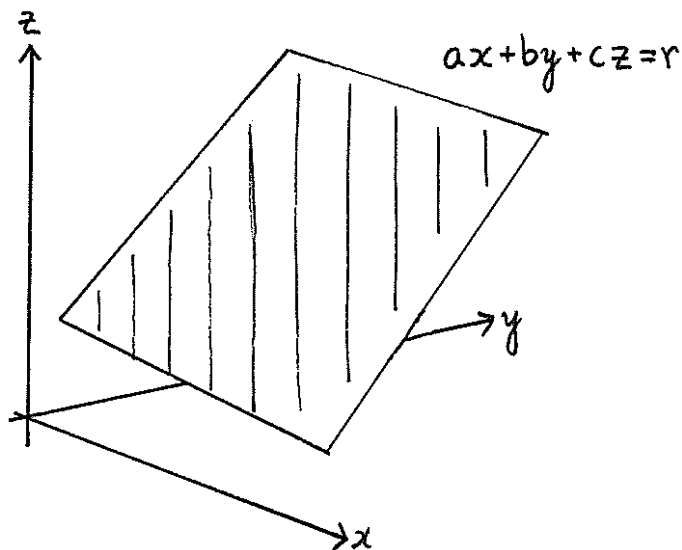
and

$$-12 \neq 7$$

Linear equations and planes

The set of solutions in \mathbb{R}^2 to a linear equation in two variables is a 1-dimensional line.

The set of solutions in \mathbb{R}^3 to a linear equation in three variables is a 2-dimensional plane.



Solutions to systems of linear equations

As in the previous chapter, we can have a system of linear equations, and we can try to find solutions that are common to each of the equations in the system.

We call a solution to a system of equations *unique* if there are no other solutions.

Example. The point $x = 3$, $y = 0$, and $z = 1$ is a solution to the following system of three linear equations in three variables

$$-3x + 2y - 5z = -14$$

$$2x - 3y + 4z = 10$$

$$x + y + z = 4$$

That's because we can substitute 3, 0, and 1 for x , y , and z respectively in the equations above and check that

$$-3(3) + 2(0) - 5(1) = -9 - 5 = -14$$

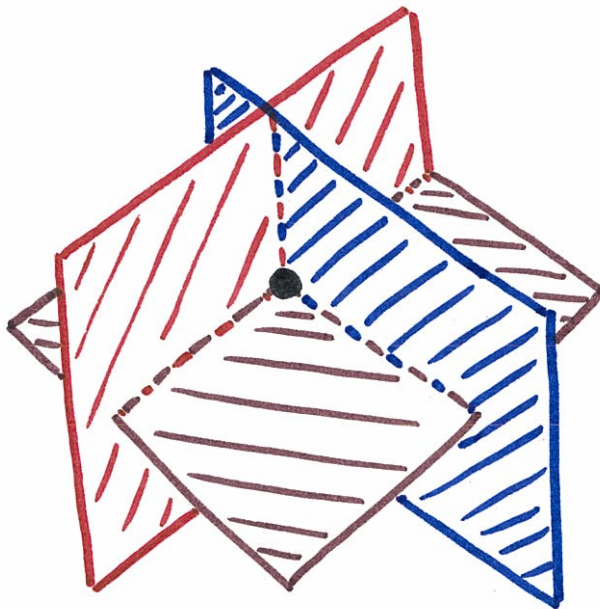
$$2(3) - 3(0) + 4(1) = 6 + 4 = 10$$

$$(3) + (0) + (1) = 3 + 1 = 4$$

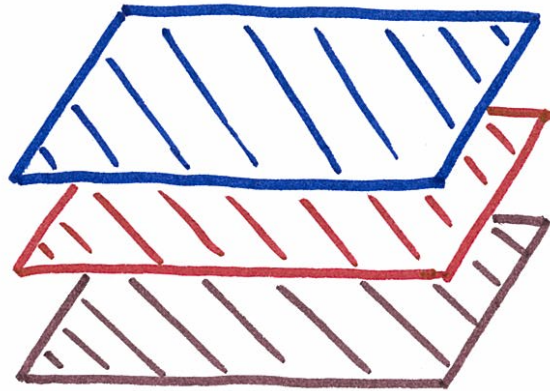
Geometry of solutions

Suppose you have a system of three linear equations in three variables. Each of the three equations has a set of solutions that's a plane in \mathbb{R}^3 . A solution to the system of equations is a point that lies on all three of those planes. If there is only one point that lies on all three planes, then that solution is unique.

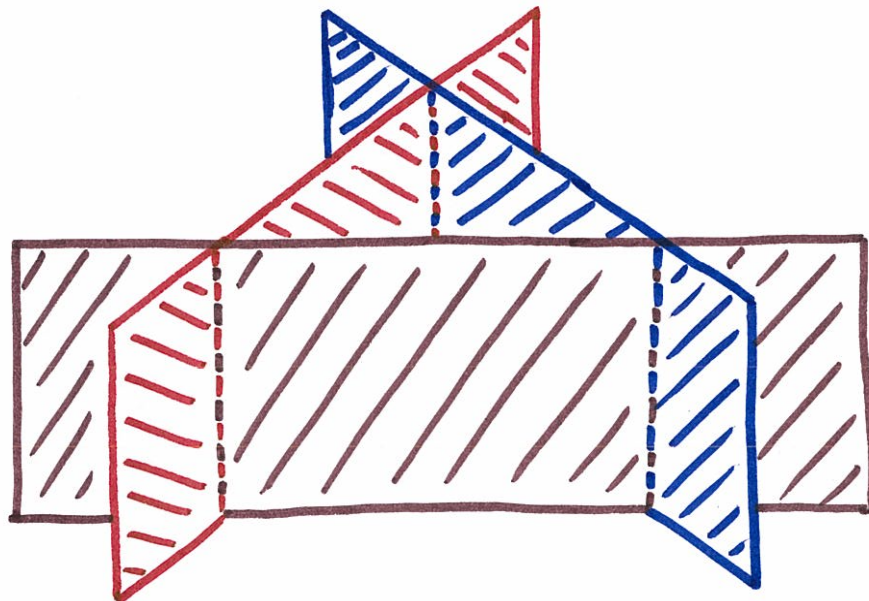
If you randomly write down three different linear equations in three variables, the odds are that the three corresponding planes will intersect in one, and only one, point. That means that for most systems of three linear equations in three variables, there will be a unique solution.



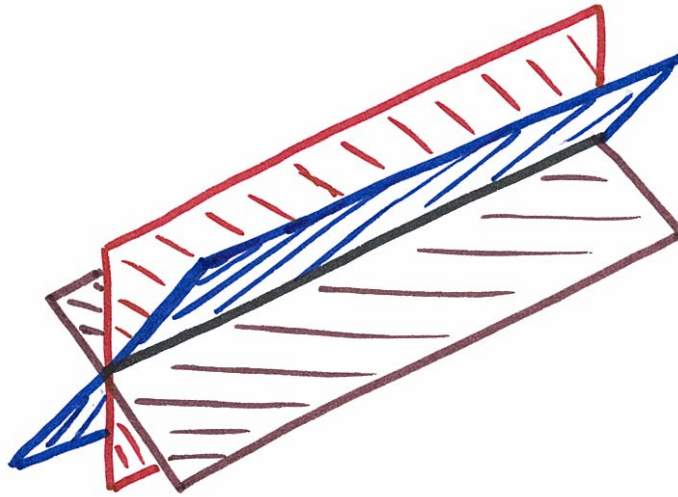
It might be that the three planes from the system of three equations will be parallel. Then the three planes wouldn't intersect. There'd be no point common to all three planes, and hence the system will not have any solutions.



There might not be a point that lies on all three planes even if the planes aren't parallel. In this case again, there'd be no solution at all.



Sometimes, the three planes will intersect in a way that allows for more than one point to be on all three planes at once. In this case, there are multiple solutions. Because there's more than one solution, there's not a unique solution.



Visualize different arrangements of three planes in \mathbb{R}^3 and try to convince yourself that either there is exactly one point contained in all three planes, or no points contained in all three planes, or that there are infinitely many points that are contained in all three planes. That means that a system of three linear equations in three variables will always have either a unique solution, no solution at all, or infinitely many solutions.

* * * * *

Exercises

1.) What are the coefficients of the equation

$$3x - 2y + 5z = 13$$

2.) What is the constant of the equation

$$3x - 2y + 5z = 13$$

3.) Is $x = 4$, $y = -3$, and $z = -1$ a solution to the equation

$$3x - 2y + 5z = 13$$

4.) Is $x = -2$, $y = 0$, and $z = 4$ a solution to the equation

$$-x + 7y - 8z = 10$$

5.) Is $x = 5$, $y = 7$, and $z = -2$ a solution to the equation

$$3y - 5z = 31$$

Questions #6-10 refer to the following system of three linear equations in three variables.

$$\begin{aligned}3x - 4y + 2z &= -9 \\ -4x + 4y + 10z &= 32 \\ -x + 2y - 7z &= -7\end{aligned}$$

6.) Is $x = 1$, $y = 4$, and $z = 2$ a solution to the system?

7.) Is $x = -1$, $y = 1$, and $z = -1$ a solution to the system?

8.) Is $x = 25$, $y = 23$, and $z = 4$ a solution to the system?

9.) Does the system have a unique solution?

10.) Does the system have infinitely many solutions?