

Division

We saw in the last chapter that if you add two polynomials, the result is a polynomial. If you subtract two polynomials, you get a polynomial. And the product of two polynomials is a polynomial.

Division doesn't work as well. Sometimes when we divide polynomials the result is a polynomial as is the case for

$$\frac{x^2 + x}{x} = \frac{x(x + 1)}{x} = x + 1$$

but more often than not, when we divide two polynomials the result is not a polynomial. For example, $\frac{1}{x}$ is not a polynomial even though 1 and x are polynomials.

Dividing by constant polynomials

Dividing a polynomial by a constant – or degree 0 – polynomial turns out to be the same as multiplying a polynomial by a constant:

$$\begin{aligned}\frac{4x^2 + x - 8}{2} &= \frac{1}{2}(4x^2 + x - 8) \\ &= \frac{1}{2}4x^2 + \frac{1}{2}x - \frac{1}{2}8 \\ &= 2x^2 + \frac{1}{2}x - 4\end{aligned}$$

Division becomes more complicated when we divide by polynomials whose degree is greater than 0.

Dividing by non-constant polynomials

The best way to learn polynomial division is to work through a few examples. We'll take a look at a couple of examples here, and we'll work out some examples in class as well.

Before we begin, if $p(x)$ and $q(x)$ are polynomials, then $p(x)$ is called the *numerator* of the fraction $\frac{p(x)}{q(x)}$, and $q(x)$ is the *denominator*.

Example 1: $\frac{6x^2 + 5x + 1}{3x + 1}$

Step 1. Write a division sign. Write the denominator to the left of the division sign. Write the numerator inside of the division sign.

$$3x+1 \overline{) 6x^2 + 5x + 1}$$

Step 2. Divide the leading term of the numerator by the leading term of the denominator and write the answer on top of the division sign.

$$\frac{6x^2}{3x} = 2x \quad \xrightarrow{\hspace{2cm}} \quad 2x$$

$$3x+1 \overline{) 6x^2 + 5x + 1}$$

Step 3. Multiply the denominator by what you just wrote on top of the division sign. Write this product below the numerator.

$$2x(3x+1) = 6x^2 + 2x \quad \xrightarrow{\hspace{2cm}} \quad 6x^2 + 2x$$

$$3x+1 \overline{) 6x^2 + 5x + 1}$$

Step 4. Subtract what you just wrote below the numerator from the numerator. Write the answer below.

$$\begin{array}{r} 2x \\ 3x+1 \overline{) 6x^2+5x+1} \\ \underline{-(6x^2+2x)} \\ 3x+1 \end{array}$$

Repeat Steps 2, 3, and 4 with the same denominator, but this time use the difference you found in Step 4 as your *new numerator*.

Step 2. Divide the leading term of the new numerator by the leading term of the denominator and write the answer on top of the division sign.

$$\frac{3x}{3x} = 1 \quad \begin{array}{r} 2x \quad 1 \\ 3x+1 \overline{) 6x^2+5x+1} \\ \underline{-(6x^2+2x)} \\ 3x+1 \end{array}$$

Step 3. Multiply the denominator by what you just wrote on top of the division sign. Write this product below the new numerator.

$$1(3x+1) = 3x+1 \quad \begin{array}{r} 2x \quad 1 \\ 3x+1 \overline{) 6x^2+5x+1} \\ \underline{-(6x^2+2x)} \\ 3x+1 \\ \rightarrow 3x+1 \end{array}$$

Step 4. Subtract what you just wrote below the new numerator from the new numerator. Write the answer below.

$$\begin{array}{r}
 2x \quad 1 \\
 \hline
 3x+1 \overline{) 6x^2+5x+1} \\
 \underline{-(6x^2+2x \quad)} \\
 3x+1 \\
 \underline{-(3x+1)} \\
 0
 \end{array}$$

The division process has ended because we ended Step 4 by writing 0. That means that $\frac{6x^2+5x+1}{3x+1}$ has no remainder. The solution is found by adding together all of the terms that were written on top of the division sign. That is,

$$\frac{6x^2 + 5x + 1}{3x + 1} = 2x + 1$$

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Example 2: $\frac{10x^4 - 4x^3 + 5x - 4}{2x^3 - 3x}$

There are two added wrinkles in this example that did not appear in the first example: In this example our division will have a remainder, and we will have to leave spaces where terms of a polynomial are missing – that last part should make sense soon.

Step 1. Write a division sign. Write the denominator to the left of the division sign. Write the numerator inside of the division sign, but in writing the numerator, leave a space wherever a term is “missing”.

At this step, our numerator is $10x^4 - 4x^3 + 5x - 4$. There is no x^2 term in $10x^4 - 4x^3 + 5x - 4$, so we'll leave a space where it would have been, between the x^3 and x terms.

$$2x^3 - 3x \overline{) 10x^4 - 4x^3 \quad + 5x - 4}$$

Step 2. Divide the leading term of the numerator by the leading term of the denominator and write the answer on top of the division sign.

$$\frac{10x^4}{2x^3} = 5x \quad \xrightarrow{\hspace{2cm}} \quad 5x$$

$$2x^3 - 3x \overline{) 10x^4 - 4x^3 \quad + 5x - 4}$$

Step 3. Multiply the denominator by what you just wrote on top of the division sign. Write this product below the numerator, again, leaving a space wherever a term is missing.

$$5x(2x^3 - 3x) = 10x^4 - 15x^2 \quad \xrightarrow{\hspace{1cm}} \quad 10x^4 \quad - 15x^2$$

$$2x^3 - 3x \overline{) 10x^4 - 4x^3 \quad + 5x - 4}$$

Step 4. Subtract what you just wrote below the numerator from the numerator. Write the answer below.

$$\begin{array}{r}
 2x^3 - 3x \overline{) 10x^4 - 4x^3 \quad + 5x - 4} \\
 \underline{-(10x^4 \quad - 15x^2 \quad)} \\
 -4x^3 + 15x^2 + 5x - 4
 \end{array}$$

Repeat Steps 2, 3, and 4 with the same denominator, but this time use the difference you found in Step 4 as your *new numerator*.

Step 2. Divide the leading term of the new numerator by the leading term of the denominator and write the answer on top of the division sign.

$$\begin{array}{r}
 \frac{-4x^3}{2x^3} = -2 \quad \xrightarrow{\hspace{10em}} \\
 2x^3 - 3x \overline{) 10x^4 - 4x^3 \quad + 5x - 4} \\
 \underline{-(10x^4 \quad - 15x^2 \quad)} \\
 -4x^3 + 15x^2 + 5x - 4
 \end{array}$$

Step 3. Multiply the denominator by what you just wrote on top of the division sign. Write this product below the new numerator, leaving a space where any missing terms would have been.

$$\begin{array}{r}
 2x^3 - 3x \overline{) 10x^4 - 4x^3 \quad + 5x - 4} \\
 \underline{-(10x^4 \quad - 15x^2 \quad)} \\
 -4x^3 + 15x^2 + 5x - 4
 \end{array}$$

$-2(2x^3 - 3x) = -4x^3 + 6x$

$$\begin{array}{r}
 \xrightarrow{\hspace{10em}} \\
 -4x^3 \quad + 6x
 \end{array}$$

Step 4. Subtract what you just wrote below the new numerator from the new numerator. Write the answer below.

$$\begin{array}{r}
 2x^3 - 3x \overline{) 10x^4 - 4x^3 } \\
 \underline{-(10x^4 - 15x^2)} \\
 -4x^3 + 15x^2 + 5x - 4 \\
 \underline{-(-4x^3 + 6x)} \\
 15x^2 - x - 4
 \end{array}$$

Important: Once you finish Step 4 with a polynomial whose degree is smaller than the degree of the denominator, you are done. It's the remainder. Otherwise it's the new numerator and you have to repeat Steps 2, 3, and 4 again.

In the first example in this chapter we ended with the polynomial 0 (the polynomial 0 always has smaller degree than the denominator) and that means that the remainder equaled 0, which is sometimes expressed by saying there is no remainder.

In this second example, the remainder is $15x^2 - x - 4$. We know it's the remainder because its degree is smaller than the degree of $2x^3 - 3x$.

We write the solution to our division problem by summing the terms on top of the division sign, and adding the remainder divided by the denominator. In other words,

$$\frac{10x^4 - 4x^3 + 5x - 4}{2x^3 - 3x} = 5x - 2 + \frac{15x^2 - x - 4}{2x^3 - 3x}$$

The way the answer above is written is the way the answer would have to be written on exams.

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Remainders from linear denominators: If you divide a polynomial by a linear polynomial, the remainder is always a constant! That's because the remainder in polynomial division has to have a smaller degree than the degree of the denominator, and the only polynomials with smaller degree than a linear polynomial are the constant polynomials.

Synthetic division

If $\alpha \in \mathbb{R}$, here's how to do synthetic division to find $\frac{p(x)}{x-\alpha}$:

Write α , and to the left of that write the coefficients of $p(x)$ in order, even the coefficients that equal 0.

$$\frac{3x^2 + 2x - 6}{x - 2} \qquad 2 \quad 3 \quad 2 \quad -6$$

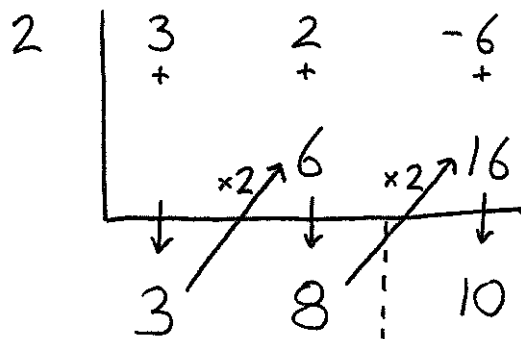
Under the coefficients of $p(x)$, write +-signs, and then leaving space for another row of numbers below that, draw an upside-down long division symbol. (I like to put a dashed horizontal line below all of that to separate the last column from the column preceding it.)

$$\begin{array}{r|rrr} 2 & 3 & 2 & -6 \\ \hline & & & \end{array}$$

There are two types of moves: going down is addition, moving up and to the right one spot is multiplication by α .

$$\begin{array}{r|rrr} 2 & 3 & 2 & -6 \\ & + & + & + \\ \hline & & & \end{array}$$

Start with the first coefficient of $p(x)$. Go down, then up and to the right, then down, then up and to the right, then down until you fill in the last space to the right of the dashed horizontal line. (This last spot will be the remainder.)



The numbers to the left of the dashed horizontal line are the coefficients, in order, of the polynomial that is your answer, except you also have to add the remainder (which is the number to the right of the dashed horizontal line) divided by $x - \alpha$.

$$\frac{3x^2 + 2x - 6}{x - 2} = 3x + 8 + \frac{10}{x - 2}$$

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Exercises

Divide. You can use synthetic division if the denominator (the polynomial in the bottom of the fraction) is degree 1, and has a leading coefficient of 1.

$$1.) \quad \frac{12x^3 - 13x^2 + 9x - 2}{3x - 1}$$

$$2.) \quad \frac{x^2 + x - 12}{x - 3}$$

$$3.) \quad \frac{15x^2 - 27x + 13}{x + 1}$$

$$4.) \quad \frac{2x^3 + 2x^2 - 48x + 72}{x + 6}$$

$$5.) \quad \frac{12x^4 - 8x^3 - 22x^2 + 4x + 8}{4x^2 - 2}$$

$$6.) \quad \frac{-x^2 + 15x - 1}{x - 1}$$

$$7.) \quad \frac{x^3 + 4x^2 + x - 6}{x - 1}$$

$$8.) \quad \frac{x^4 - 3x^3 + 5x^2 - 2x + 9}{x^2 + 1}$$

$$9.) \quad \frac{4x^3 - x^2 - x - 1}{x + 5}$$

$$10.) \quad \frac{6x^5 + 5x^4 - 2x^2 + 50x - 13}{x - 3}$$