Graph Transformations

There are many times when you'll know very well what the graph of a particular function looks like, and you'll want to know what the graph of a very similar function looks like. In this chapter, we'll discuss some ways to draw graphs in these circumstances.

Transformations "after" the original function

Suppose you know what the graph of a function $f(x)$ looks like. Suppose $d \in \mathbb{R}$ is some number that is greater than 0, and you are asked to graph the function $f(x) + d$. The graph of the new function is easy to describe: just take every point in the graph of $f(x)$, and move it up a distance of d. That is, if (a, b) is a point in the graph of $f(x)$, then $(a, b + d)$ is a point in the graph of $f(x) + d$. Let's see why:

If (a, b) is a point in the graph of $f(x)$, then that means $f(a) = b$. Hence, $f(a) + d = b + d$, which is to say that $(a, b + d)$ is a point in the graph of $f(x) + d$.

The chart on the next page describes how to use the graph of $f(x)$ to create the graph of some similar functions. Throughout the chart, $d > 0$, $c > 1$, and (a, b) is a point in the graph of $f(x)$.

Notice that all of the "new functions" in the chart differ from $f(x)$ by some algebraic manipulation that happens after f plays its part as a function. For example, first you put x into the function, then $f(x)$ is what comes out. The function has done its job. Only after f has done its job do you add d to get the new function $f(x) + d$.

Because all of the algebraic transformations occur after the function does its job, all of the changes to points in the second column of the chart occur in the second coordinate. Thus, all the changes in the graphs occur in the vertical measurements of the graph.

New function	How points in graph of $f(x)$ become points of new graph	visual effect
$f(x)+d$	$(a,b)\mapsto(a,b+d)$	shift up by d
$f(x)-d$	$(a,b)\mapsto(a,b-d)$	shift down by d
$cf(x)$	$(a,b)\mapsto(a,cb)$	stretch vertically by c
$\frac{1}{c}f(x)$	$(a,b)\mapsto(a,\frac{1}{c}b)$	shrink vertically by $\frac{1}{c}$
$-f(x)$	$(a,b)\mapsto(a,-b)$	flip over the x-axis

Transformations "before" the original function

We could also make simple algebraic adjustments to $f(x)$ *before* the function f gets a chance to do its job. For example, $f(x+d)$ is the function where you first add d to a number x, and only after that do you feed a number into the function f.

On the next page is a chart that is similar to the chart above. The difference in the next chart is that the algebraic manipulations occur before you feed a number into f, and thus all of the changes occur in the first coordinates of points in the graph. All of the visual changes affect the horizontal measurements of the graph.

One important point of caution to keep in mind is that most of the visual horizontal changes described in the next chart are the exact opposite of the effect that most people anticipate after having seen the chart above. To get an idea for why that's true let's work through one example.

Suppose that $d>0$. If (a,b) is a point that is contained in the graph of $f(x)$, then $f(a)=b$. Hence, $f((a-d)+d)=f(a)=b$, which is to say that

$(a-d, b)$ is a point in the graph of $f(x+d)$. The visual change between the point (a, b) and the point $(a-d, b)$ is a shift to the left a distance of d.

In the chart below, just as in the previous chart, $d > 0$, $c > 1$, and (a, b) is a point in the graph of $f(x)$.

New function	How points in graph of $f(x)$ become points of new graph	visual effect
$f(x+d)$	$(a, b) \mapsto (a-d, b)$	shift left by d
$f(x-d)$	$(a, b) \mapsto (a+d, b)$	shift right by d
$f(cx)$	$(a, b) \mapsto (\frac{1}{c}a, b)$	shrink horizontally by $\frac{1}{c}$
$f(\frac{1}{c}x)$	$(a, b) \mapsto (ca, b)$	stretch horizontally by c
$f(-x)$	$(a, b) \mapsto (-a, b)$	flip over the y-axis

Transformations before and after the original function

As long as there is only one type of operation involved "inside the function" – either multiplication or addition – and only one type of operation involved "outside of the function" – either multiplication or addition – you can apply the rules from the two charts above to transform the graph of a function.

Examples.

• Let's look at the function $-2f(x+3)$. There is only one kind of operation inside of the parentheses, and that operation is addition – you are adding 3.

There is only one kind of operation outside of the parentheses, and that operation is multiplication – you are multiplying by 2, and you are multiplying by -1.

So to find the graph of $-2f(x+3)$, take the graph of $f(x)$, shift it to the left by a distance of 3, stretch vertically by a factor of 2, and then flip over the x-axis.

• The graph of $2f(3x)$ is obtained by shrinking the horizontal coordinate by 3, and stretching the vertical coordinate by 2.

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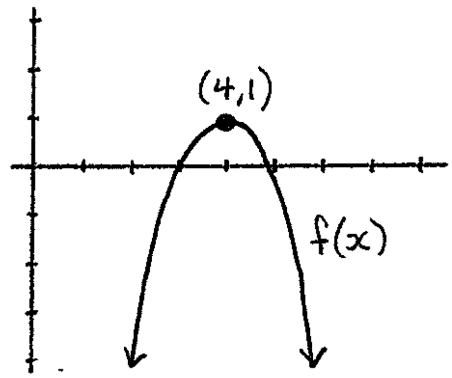
Exercises

For #1-10, suppose $f(x) = x^8$. Match each of the numbered functions on the left with the lettered function on the right that it equals.

1.) $f(x) + 2$	A.) $(-x)^8$
2.) $3f(x)$	B.) $\frac{1}{3}x^8$
3.) $f(-x)$	C.) $x^8 - 2$
4.) $f(x-2)$	D.) $x^8 + 2$
5.) $\frac{1}{3}f(x)$	E.) $(\frac{x}{3})^8$
6.) $f(3x)$	F.) $-x^8$
7.) $f(x) - 2$	G.) $(x-2)^8$
8.) $-f(x)$	H.) $(3x)^8$
9.) $f(x+2)$	I.) $3x^8$
10.) $f(\frac{x}{3})$	J.) $(x+2)^8$

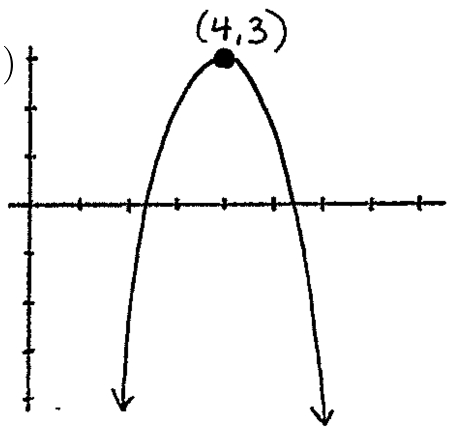
For #11 and #12, suppose $g(x) = \frac{1}{x}$. Match each of the numbered functions on the left with the lettered function on the right that it equals.

11.) $-4g(3x-7) + 2$	A.) $\frac{6}{-2x+5} - 3$
12.) $6g(-2x+5) - 3$	B.) $\frac{-4}{3x-7} + 2$



Given the graph of $f(x)$ above, match the following four functions with their graphs.

13.) $f(x)+2$ 14.) $f(x)-2$ 15.) $f(x+2)$ 16.) $f(x-2)$

A.)


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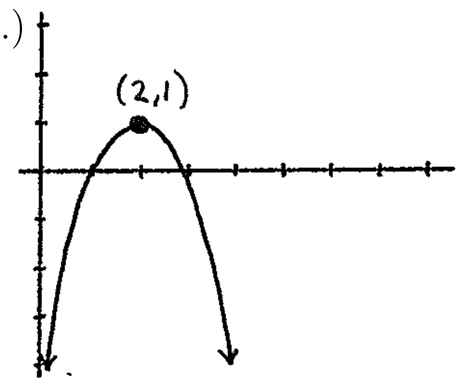

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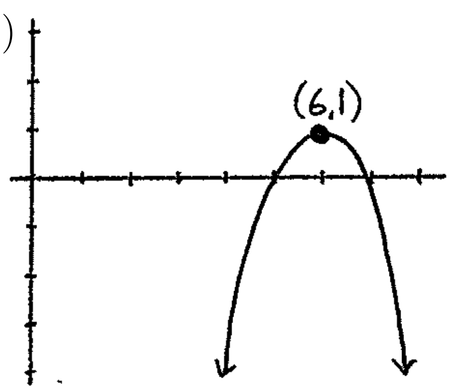

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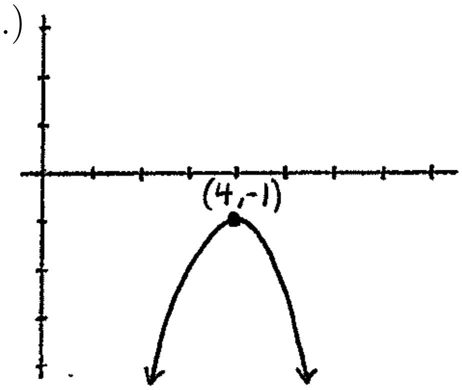
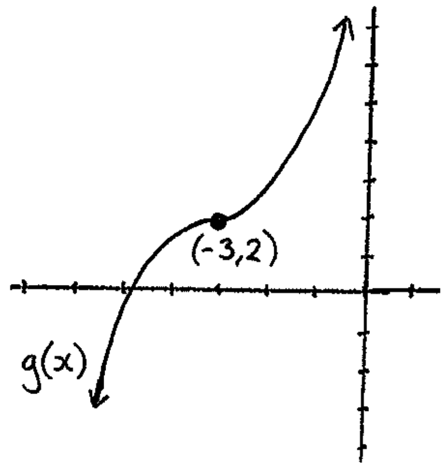
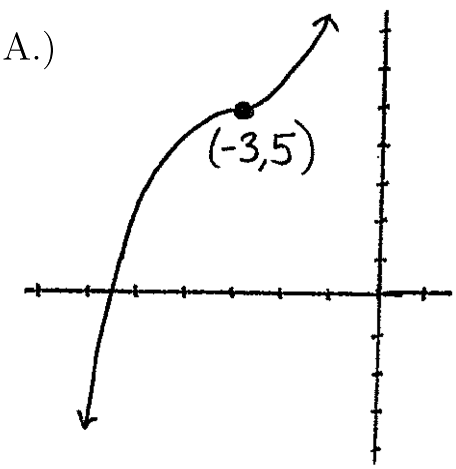




Given the graph of $g(x)$ above, match the following four functions with their graphs.

17.) $g(x)+3$ 18.) $g(x)-3$ 19.) $g(x+3)$ 20.) $g(x-3)$

A.)



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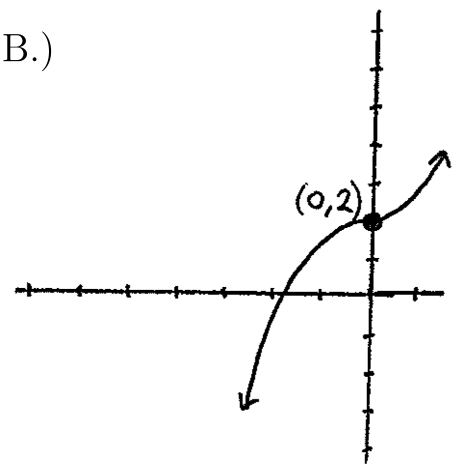

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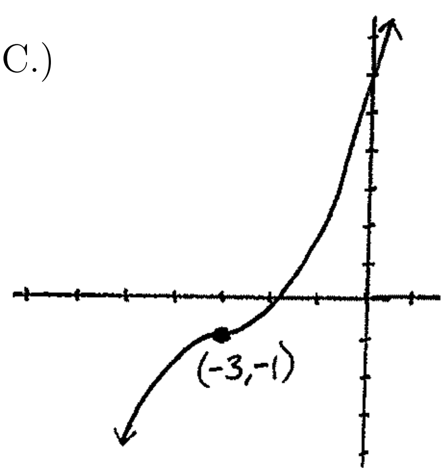

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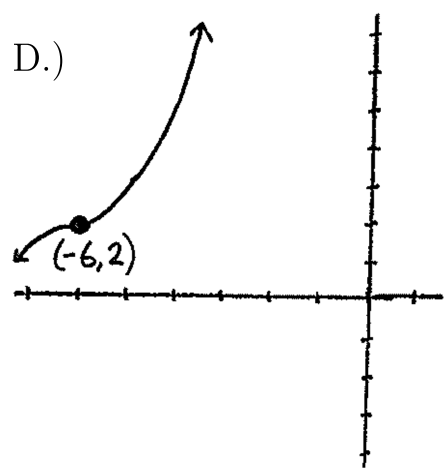

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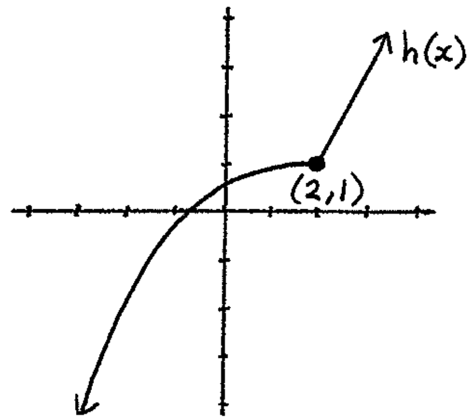
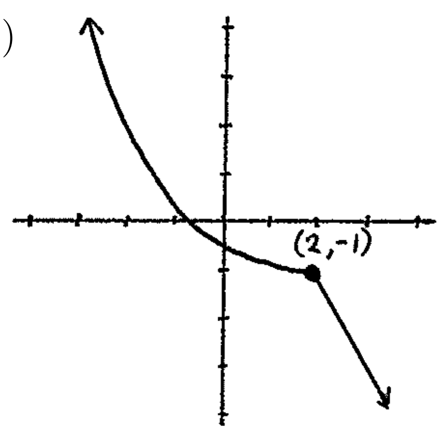


Given the graph of $h(x)$ above, match the following two functions with their graphs.

21.) $-h(x)$ 22.) $h(-x)$

A.)



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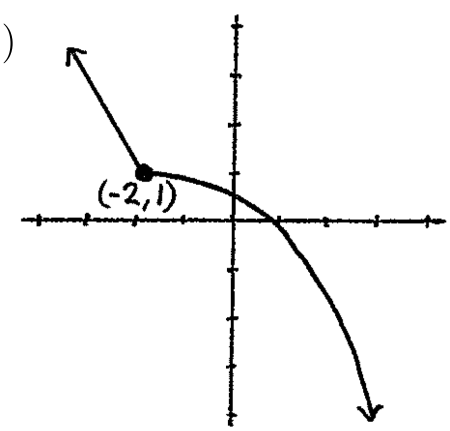

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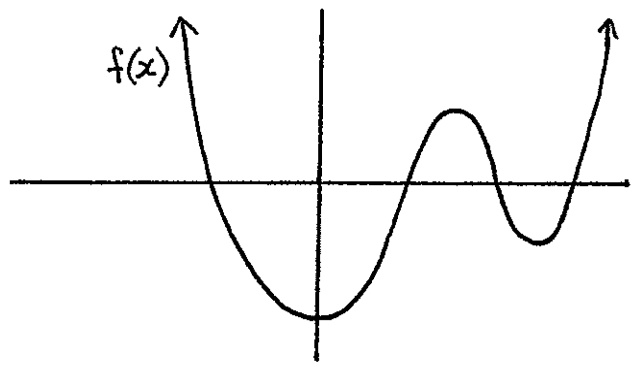
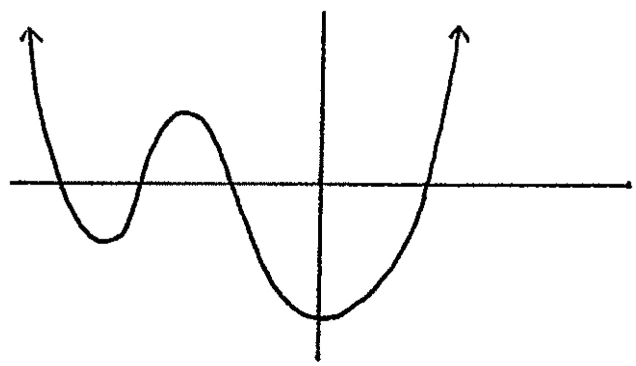


Given the graph of $f(x)$ above, match the following two functions with their graphs.

23.) $-f(x)$ 24.) $f(-x)$

A.)



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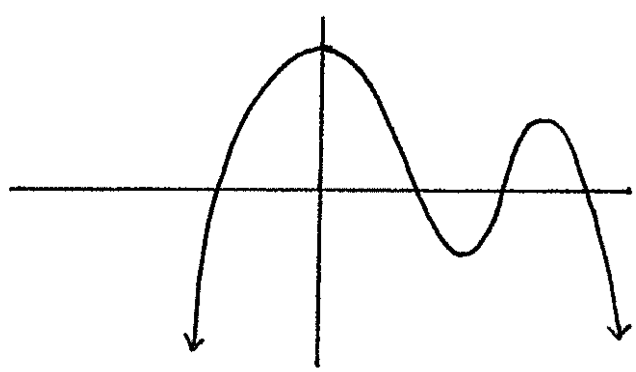

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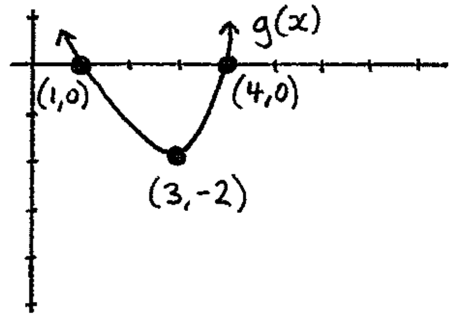
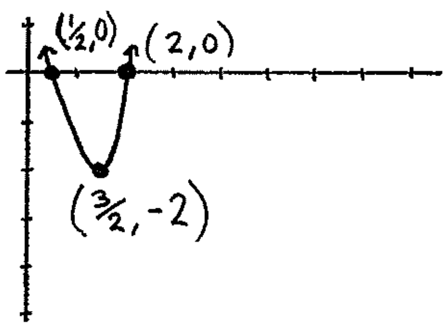


Given the graph of $g(x)$ above, match the following four functions with their graphs.

25.) $2g(x)$ 26.) $\frac{1}{2}g(x)$ 27.) $g(2x)$ 28.) $g(\frac{x}{2})$

A.)



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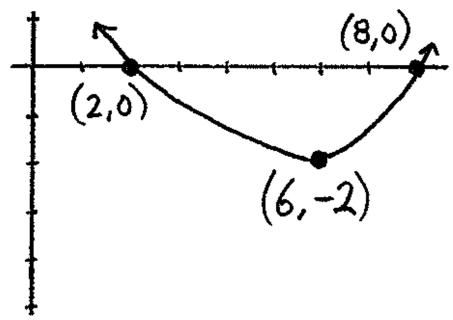

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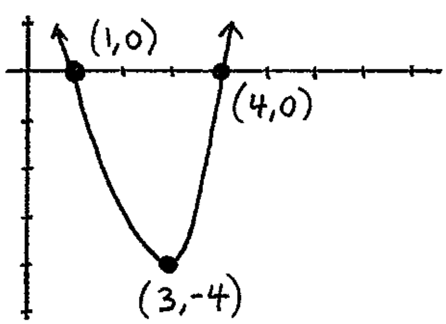

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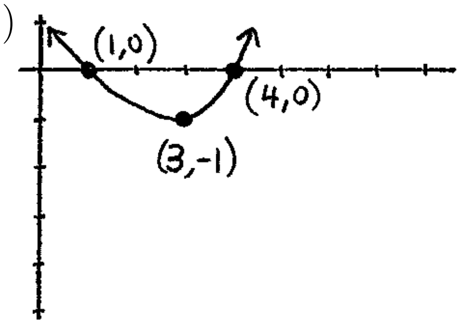
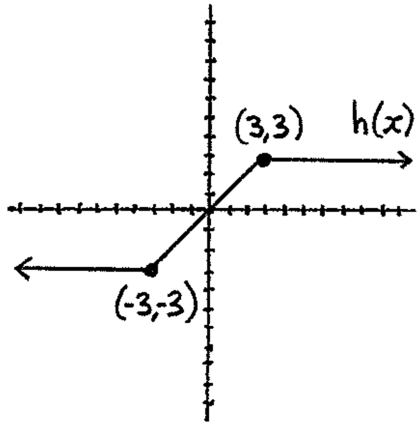
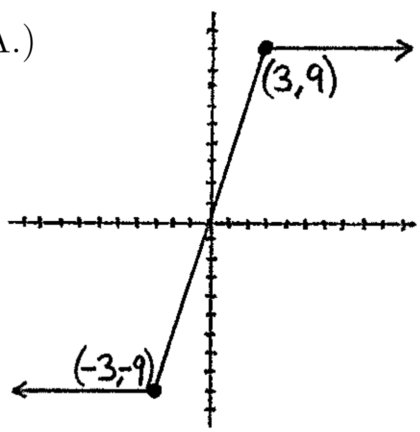




Given the graph of $h(x)$ above, match the following four functions with their graphs.

29.) $3h(x)$ 30.) $\frac{1}{3}h(x)$ 31.) $h(3x)$ 32.) $h(\frac{x}{3})$

A.)



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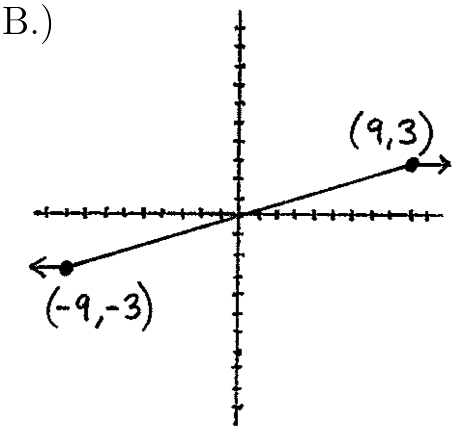

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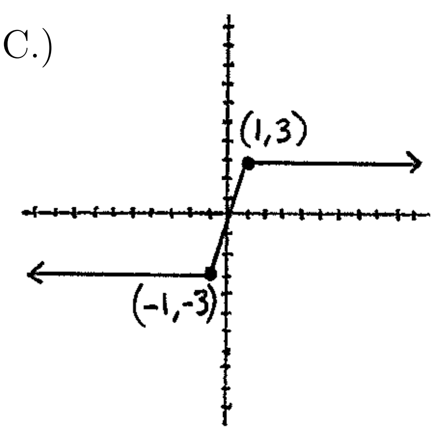

D.)