

Math 2270
Spring 2007
Dr. David M. Goulet

EXAM III SOLUTIONS

Wednesday, April 11, 2007

Problem	Points	Score
1	18	
2	32	
3	25	
4	25	
bonus	10	
TOTAL	100+10	

1. Find the determinants of the following matrices. It may be helpful to consider some of the properties of the determinant in your calculations. Be sure to explain your work.

$$(a) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{pmatrix}$$

Use Gaussian elimination to simplify the determinant. Subtract the first row from the second and the third row from the fourth, then exchange rows 2 and 3. This produces a block matrix

$$\begin{aligned} \det A &= \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 1 & -1 & 2 & -2 \\ 0 & 0 & 6 & -6 \end{pmatrix} = -\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & -6 \end{pmatrix} \\ &= -\det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \det \begin{pmatrix} 3 & 3 \\ 6 & -6 \end{pmatrix} = -(-2)(-36) = -72. \end{aligned}$$

$$(b) \begin{pmatrix} 0 & 1 & 0 & 2 & 15 & 10 \\ 1 & 0 & 0 & 7 & -1 & 7 \\ 0 & 0 & 1 & 41 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Exchange the row 1 and 2 and rows 4 and 6 to get an upper triangular matrix

$$\det A = (-1)^2 \det \begin{pmatrix} 1 & 0 & 0 & 7 & -1 & 7 \\ 0 & 1 & 0 & 2 & 15 & 10 \\ 0 & 0 & 1 & 41 & 2 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1.$$

$$(c) \begin{pmatrix} 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 \\ 6 & 8 & 10 & 12 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Since the third row is a multiple of the first row, the determinant is 0.

2. Let V be the subspace of \mathbb{R}^4 spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}.$$

(a) Find an orthonormal basis for V .

First, normalize \vec{v}_1 . $\|\vec{v}_1\| = \sqrt{2^2 + 2^2 + 2^2 + 2^2} = 4$, so take

$$\vec{u}_1 = \frac{1}{4}\vec{v}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}. \text{ Now find the orthogonal part of } \vec{v}_2:$$

$$\vec{v}_2^\perp = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right) \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}.$$

This vector already has norm 1, so we relabel it \vec{u}_2 . Finally, find the orthogonal part of \vec{v}_3 .

$$\begin{aligned} \vec{v}_3^\perp &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2 \\ &= \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right) \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right) \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} - 1 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + 2 \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} \end{aligned}$$

This is already normalized, and the third vector in the orthonormal basis.

$$\begin{aligned}
\text{(b) If } \vec{x} &= \begin{bmatrix} 8 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \text{ find } \text{proj}_V(\vec{x}). \\
&= (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + (\vec{x} \cdot \vec{u}_2)\vec{u}_2 + (\vec{x} \cdot \vec{u}_3)\vec{u}_3 \\
&= \left(\begin{bmatrix} 8 \\ 1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right) \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + \left(\begin{bmatrix} 8 \\ 1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right) \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} + \left(\begin{bmatrix} 8 \\ 1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} \right) \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} \\
&= 7 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + 2 \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} + 2 \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}
\end{aligned}$$

3. Consider the inner product in \mathbb{R}^2 defined by

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \vec{w}.$$

(a) What is the norm of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ according to this inner product?

$$\|\vec{v}\|^2 = \vec{v}^T A \vec{v} = [1 \quad 1] \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [6 \quad 3] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 9.$$

So the norm of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is 3.

(b) Find a vector that is orthogonal to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ according to this inner product.

$$0 = [1 \quad 1] \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = [6 \quad 3] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 6v_1 + 3v_2.$$

We can take, for example $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

4. For each statement below, decide whether it is True or False. If the statement is true, explain why it is true. If the statement is false, explain why it is false or give a counter example which shows why it is false.

(a) $\det(A + B) = \det(A) + \det(B)$ for all 3×3 matrices A and B .
False. Let $A = B = I_3$. Then $\det(A) + \det(B) = 2$, but

$$\det(A + B) = \det(2I_3) = 8.$$

(b) If A is symmetric, then so is $7A$.
True. $(7A)^T = 7A^T = 7A$.

(c) There exists a 3×3 matrix A such that $A^2 = -I_3$.
False. Take the determinant of each side. $\det(A)^2 = \det(-I_3) = -1$. This has no real solution.

(d) $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ is orthogonal.
False. $\det \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} = 25 \neq \pm 1$.

(e) The matrix $\begin{bmatrix} k^2 & 1 & 4 \\ k & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$ is invertible for all values of the constant k .

A matrix is invertible if and only if its determinant is nonzero.

Take the determinant of the given matrix.

$$\begin{aligned} \det A &= k^2 \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} -k & -2 \\ 1 & 1 \end{vmatrix} + 4 \begin{vmatrix} k & -1 \\ 1 & 1 \end{vmatrix} \\ &= k^2 + 3k + 2 = (k + 1)(k + 2) \end{aligned}$$

This expression is 0 when k is -1 or -2 . Thus the matrix is not invertible for all k .

(Bonus) If $\|\vec{v}\|$ denotes the standard norm in \mathbb{R}^n , does

$$\langle \vec{v}, \vec{w} \rangle = \|\vec{v} + \vec{x}\| - \|\vec{v}\| - \|\vec{x}\|$$

define an inner product in \mathbb{R} ?

This does not define an inner product. For any nonzero vector, the

square of the norm would be

$$\|\vec{v}\|^2 = \langle \vec{v}, \vec{v} \rangle = \|\vec{v} + \vec{v}\| - \|\vec{v}\| - \|\vec{v}\| = \|2\vec{v}\| - 2\|\vec{v}\| = 0.$$

So any nonzero vector has norm zero, and this is not an inner product.

(Bonus) Find the derivative, with respect to x of the function

$$f(x) = \det \begin{vmatrix} 1 & 1 & 2 & 3 & 4 \\ 9 & 0 & 2 & 3 & 4 \\ 9 & 0 & 0 & 3 & 4 \\ x & 1 & 2 & 9 & 1 \\ 7 & 0 & 0 & 0 & 4 \end{vmatrix}.$$

Compute the determinant by minors, going down the first column.

All of these terms will be constant with respect to x except for the fourth one, which has the form

$$-x \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{vmatrix} = -24x.$$

Thus, the derivative of f is -24 .