On the Prediction of Failure in Pressurized Vessels

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ABSTRACT

This report discusses a fracture criterion for the prediction of failure in flawed pressurized vessels of arbitrary shape. A comparison with some of the existing experimental data in the literature substantiates its potential use.

NOMENCLATURE

- \( c \) half crack length
- \( D \) flexural rigidity \( \left( \frac{Eh^3}{12(1-\nu^2)} \right) \)
- \( E \) Young’s modulus
- \( G \) shear modulus
- \( h \) thickness
- \( k \) fracture toughness (for flat sheets)
- \( J, J_1 \) coefficients as defined in text
- \( P_0 \) uniform internal pressure
- \( R \) radius of the shell
- \( y^* \) surface energy per unit area
- \( A^4 \) \( \frac{Eh^3}{R^2D} \left( \frac{12(1-\nu^2)c^4}{R^2h^2} \right) \)
- \( \lambda_i \) \( \frac{Eh^3}{R^2D} (j = 1, 2, 3, x, y) \)
- \( \nu \) Poisson’s ratio
- \( n \) 3.14
- \( \sigma_y \) uniaxial yield stress
- \( \sigma_{yB} \) biaxial yield stress
- \( \sigma_u \) ultimate tensile stress (for a flat sheet)
- \( \sigma^* \) \( \frac{\sigma_y + \sigma_u}{2} \)
- \( \sigma_h \) hoop stress
- \( \bar{\sigma}^{(a)}_j, \bar{\sigma}^{(b)}_j, \bar{\sigma}_j \) applied to the crack stress components

INTRODUCTION

It is well known [1-5] that initially curved sheets containing through cracks present a reduced resistance to fracture initiation. Consequently, the presence of a crack in the walls of a pressure vessel can severely reduce the strength of the structure and can cause sudden failure at nominal tensile stresses less than the material yield strength. Therefore, to ensure the integrity of a structure, the designer must have cognizance of the relation between fracture load, flaw shape and size, and structural geometry.

Such a relation has been derived by the author; and, for cylindrical and spherical pressure vessels, the results are reported in Ref. [6]. In this paper the author, in view of some recent developments [5],
**Table 1.**

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long cylinder axial crack</td>
<td>$I = 1 + \frac{5\pi}{64} \lambda^2$</td>
</tr>
<tr>
<td></td>
<td>$J \approx 0$</td>
</tr>
<tr>
<td></td>
<td>$\sigma(e) = \sigma_h = q_0 R/h$</td>
</tr>
<tr>
<td>Long cylinder peripheral crack</td>
<td>$I = 1 + \frac{\pi \lambda^2}{64}$</td>
</tr>
<tr>
<td></td>
<td>$J \approx 0$</td>
</tr>
<tr>
<td></td>
<td>$\sigma(e) = \frac{\sigma_h}{2} = q_0 R/2h$</td>
</tr>
<tr>
<td>Spherical cap</td>
<td>$I = 1 + \frac{3\pi \lambda^2}{32}$</td>
</tr>
<tr>
<td></td>
<td>$J \approx 0$</td>
</tr>
<tr>
<td></td>
<td>$\sigma(e) = \sigma_h = q_0 R/2h$</td>
</tr>
<tr>
<td>Conical circular</td>
<td>$I = 1 + \frac{5\pi}{64} \lambda^2$</td>
</tr>
<tr>
<td></td>
<td>$J \approx 0$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_h = R - c \tan \epsilon$</td>
</tr>
<tr>
<td></td>
<td>$R_1 = R + c \tan \epsilon$</td>
</tr>
<tr>
<td>Toroidal</td>
<td>$I = 1 + \frac{5\pi \lambda^2}{64} + \frac{\pi \lambda^2}{64}$</td>
</tr>
<tr>
<td></td>
<td>$J \approx 0$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_h = R - c \tan \epsilon$</td>
</tr>
<tr>
<td></td>
<td>$R_1 = R + c \tan \epsilon$</td>
</tr>
<tr>
<td>Arbitrary Geometry</td>
<td>$I = 1 + \frac{5\pi \lambda^2}{64} + \frac{5\pi \lambda^2}{64}$</td>
</tr>
<tr>
<td></td>
<td>$J \approx 0$</td>
</tr>
<tr>
<td></td>
<td>(where we assumed $\frac{\sigma}{h} = \frac{\tau}{h} = 0$.)</td>
</tr>
</tbody>
</table>

*The expressions in this table are valid for $\lambda \leq 1$ only. For larger values of $\lambda$, see Refs. [8] and [9].*
generalizes his fracture criterion to include other shell geometries and compares his theoretical results with the experimental data existing in the literature.

**FRACTURE CRITERION**

In deriving a fracture criterion two ingredients are necessary: (1) the stress distribution due to the presence of the crack and (2) an energy balance for crack initiation. Such a criterion incorporating correction factors for geometry and plasticity effects has been derived by the author. The details are given in Ref. [6], and an outline of the derivation is given in the Appendix. The criterion reads:

\[
\sigma^{(e)} = \frac{(33 + 6\nu - 7(1 + \nu)}{3(9 - 7\nu)} \left\{ \frac{3(9 - 7\nu)}{3(9 - 7\nu)} J^2 + J^2 \right\} \frac{1}{1 + \lambda} \]

where the coefficients \( l \) and \( J \) are functions of the parameter \( \lambda \) and are given in Table 1.

By Eq. (1) one would expect, therefore, to predict failure in pressurized vessels containing through-the-thickness cracks. Inasmuch as theory in general is not useful unless there exists experimental evidence to support it, in the next few pages we will compare our results with some of the experimental data existing in the literature.

In order to utilize Eq. (1), however, one must know a priori the fracture toughness \( k \). This difficulty can be eliminated if one proceeds in the following manner: (1) use the test data and compute the \( k's \), (2) find the average \( k \), and (3) use the \( k_{av} \) to predict failing hoop stresses.

**EXPERIMENTAL DATA**

R. C. Aungst's study\(^3\) shows results of tests on 2.7-in. diameter, 0.26-in. thick \( N \)-reactor tube with \( v \)-shape cracks under the following conditions:

- Material: 30 percent cold drawn zircaloy-2.
- \( \sigma_y: 98 \text{ ksi} \)
- \( \sigma_u: 98.6 \text{ ksi} \)
- \( \sigma^*: 98.4 \text{ ksi} \)
- \( k: 240 \text{ ksi} \sqrt{\text{in}}. \)
The results are plotted in Fig. 1. Notice that the agreement is very good for large crack lengths. However, for small crack lengths the predicted values are somewhat lower. This is not contrary to our expectations for our calculations were based on through the thickness cracks.

Sopher et al. give results of tests on 9-ft diameter, 3-in. thick spheres with full-thickness cracks under the following conditions:

Material: ABS-B Steel
- \( \sigma_y^* = 30.7 \text{ ksi} \)
- \( \sigma_y = 37.9 \text{ ksi} \)
- \( \sigma_u = 59.4 \text{ ksi} \)
- \( k = 102 \text{ ksi} \sqrt{\text{in.}} \)

Anderson and Sullivan show results of tests on 6-in. diameter, 0.060-in. thick cylinders with full-thickness cracks under the following conditions:

(1) Material: 2014-T6A1
- \( \sigma_y = 90.5 \text{ ksi} \)
- \( \sigma_y^* = 91.9 \text{ ksi} \)
- \( \sigma_u = 93.9 \text{ ksi} \)
- \( k = 51.6 \text{ ksi} \sqrt{\text{in.}} \)

The results are plotted in Fig. 2. The agreement is very good.

The results are plotted in Fig. 3. The agreement is good.

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**Fig. 2 Comparison Between Theory and Experiment for ABS-B Steel Spherical Vessels**

**Fig. 3 Comparison Between Theory and Experiment for 2014-T6A1 Cylindrical Vessels at -423°F**

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(2) Material: 2014-T6Al

\[
\begin{array}{cccccc}
T^\circ & F & c \text{ (in.)} & \lambda & I & \sigma_h \text{ exp. ksi} & \sigma_h \text{ calc. ksi} \\
-320^\circ & 0.057 & 0.24 & 1.02 & 71.6 & 77.0 \\
-320^\circ & 0.075 & 0.32 & 1.03 & 70.6 & 73.0 \\
-320^\circ & 0.100 & 0.43 & 1.05 & 63.7 & 66.0 \\
-320^\circ & 0.125 & 0.53 & 1.06 & 58.5 & 61.5 \\
-320^\circ & 0.150 & 0.64 & 1.09 & 52.2 & 56.1 \\
-320^\circ & 0.200 & 0.85 & 1.15 & 47.4 & 47.7 \\
-320^\circ & 0.250 & 1.06 & 1.23 & 40.1 & 41.0 \\
-320^\circ & 0.375 & 1.60 & 1.45 & 30.2 & 29.2 \\
-320^\circ & 0.500 & 2.13 & 1.66 & 23.1 & 22.3 \\
-320^\circ & 0.625 & 2.65 & 1.93 & 18.6 & 17.6 \\
-320^\circ & 0.875 & 3.72 & 2.45 & 14.4 & 11.6 \\
-320^\circ & 1.000 & 4.26 & 2.73 & 11.3 & 9.8 \\
\end{array}
\]

The results are plotted in Fig. 4. The agreement is good.

(3) Material: 2014-T6Al

\[
\begin{array}{cccccc}
\sigma_y: & 68.0 \text{ ksi} & \sigma^*: & 70.8 \text{ ksi} \\
\sigma_u: & 79.0 \text{ ksi} & k & = 43.4 \text{ ksi} \sqrt{\text{in.}} \\
\end{array}
\]

\[
\begin{array}{cccccc}
T^\circ & F & c \text{ (in.)} & \lambda & I & \sigma_h \text{ exp. ksi} & \sigma_h \text{ calc. ksi} \\
\text{room} & 0.05 & 0.24 & 1.02 & 64.4 & 66.4 \\
\text{room} & 0.25 & 1.06 & 1.23 & 34.0 & 36.1 \\
\text{room} & 0.50 & 2.23 & 1.73 & 20.6 & 19.1 \\
\text{room} & 1.00 & 4.26 & 2.73 & 9.7 & 8.9 \\
\end{array}
\]

The results are plotted in Fig. 5. The agreement is good.

They also show results of tests on 6-in. diameter, 0.020-in. thick cylinders with full thickness cracks under these conditions:

Material: 5A1-2.5S, -Ti

\[
\begin{array}{cccccc}
\sigma_y: & 222 \text{ ksi} & \sigma^*: & 200 \text{ ksi} \\
\sigma_u: & \text{not specified} & k & = 196 \text{ ksi} \sqrt{\text{in.}} \\
\end{array}
\]

FIG. 4 COMPARISON BETWEEN THEORY AND EXPERIMENT FOR 2014-T6Al CYLINDRICAL VESSELS AT -320°F

FIG. 5 COMPARISON BETWEEN THEORY AND EXPERIMENT FOR 2014-T6Al CYLINDRICAL VESSELS AT ROOM TEMPERATURE

FIG. 6 COMPARISON BETWEEN THEORY AND EXPERIMENT FOR 5A1-2.5S-Ti CYLINDRICAL VESSELS AT -320°F
Material: B.S. 1501-161: 1958, Grade B
\( \sigma_y = 36.0 \text{ ksi at } 80^\circ \text{C} \)  
\( \sigma^* = 42.2 \text{ ksi} \)  
\( \sigma_u = 60.9 \text{ ksi at } 80^\circ \text{C} \)  
\( k = 200.0 \text{ ksi} \sqrt{\text{in.}} \)

<table>
<thead>
<tr>
<th>( T^\circ \text{C} )</th>
<th>( c \text{ (in.)} )</th>
<th>( \lambda )</th>
<th>( l )</th>
<th>( \sigma_y ) \text{ exp. ksi}</th>
<th>( \sigma_y ) \text{ calc. ksi}</th>
</tr>
</thead>
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<td>78</td>
<td>1.69</td>
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<td>1.16</td>
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<td>78</td>
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<tr>
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<td>0-10</td>
<td>3.00</td>
<td>1.42</td>
<td>1.41</td>
<td>31.90</td>
<td>30.1</td>
</tr>
</tbody>
</table>

The results are plotted in Fig. 7. The agreement is good except at one point, but this is due to a temperature change.

Irvine, Quirk, and Bevitt\(^7\) show results of tests on 5-ft diameter, 1-in. thick, cylindrical vessels with through cracks under the following conditions:

Material: 0.36 percent C Steel

\( \sigma_y = 33 \text{ ksi} \)  
\( \sigma^* = 40 \text{ ksi} \)  
\( \sigma_u = 61 \text{ ksi} \)  
\( k = 179 \text{ ksi} \sqrt{\text{in.}} \)

<table>
<thead>
<tr>
<th>( T )</th>
<th>( c \text{ (in.)} )</th>
<th>( \lambda )</th>
<th>( l )</th>
<th>( \sigma_y ) \text{ exp. ksi}</th>
<th>( \sigma_y ) \text{ calc. ksi}</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.99</td>
<td>1.20</td>
<td>27.60</td>
<td>32.2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.99</td>
<td>1.20</td>
<td>30.00</td>
<td>32.2</td>
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</tr>
<tr>
<td>3</td>
<td>0.99</td>
<td>1.20</td>
<td>32.40</td>
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</tr>
<tr>
<td>6</td>
<td>1.98</td>
<td>1.61</td>
<td>18.80</td>
<td>20.8</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1.98</td>
<td>1.61</td>
<td>21.00</td>
<td>20.8</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>3.96</td>
<td>2.58</td>
<td>9.65</td>
<td>10.2</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>3.96</td>
<td>2.58</td>
<td>11.65</td>
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<td></td>
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<tr>
<td>12</td>
<td>3.96</td>
<td>2.58</td>
<td>13.00</td>
<td>10.2</td>
<td></td>
</tr>
</tbody>
</table>

The results are plotted in Fig. 6. The agreement is fairly good. It should be pointed out that titanium alloys exhibit significant increase in biaxial yield strength relative to uniaxial yield strength. Hence, the von Mises yield criterion may or may not be applicable. However, if one uses a \( \sigma^* \) slightly less than 200 ksi, even better agreement is noticed.

Taylor and Burdekin\(^6\) give results of tests on 58-in. diameter, 0.50-in. thick spheres with through-thickness cracks.


The results are plotted in Fig. 9. The agreement is fairly good. Note that Ref. [7] does not make clear as to whether $\sigma_y = 81.9$ ksi is the biaxial or uniaxial yield stress. If, however, it represents the biaxial yield stress, then $\sigma^* = 74$ ksi and $k = 58.5$ ksi $\sqrt{\text{in.}}$, and the agreement is even better (see Fig. 10).

(2) Material: 2014-T6 aluminum alloy

$\sigma_y = 90.8$ ksi $\sigma^* = 95.2$ ksi

$\sigma_u = 108.4$ ksi $k = 51.7$ ksi $\sqrt{\text{in.}}$

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FIG. 10 COMPARISON BETWEEN THEORY AND EXPERIMENT FOR 2014-T6A1 CYLINDRICAL VESSELS AT -321°F

The results are plotted in Fig. 8. The agreement is fairly good.

Getz, Pierce and Calvert\(^6\) give results of tests on 3-in. diameter, 0.060-in. thick cylindrical vessels with through-the-thickness cracks under the following conditions.

(1) Material: 2014-T6 aluminum alloy

$\sigma_y = 81.9$ ksi $\sigma^* = 84.9$ ksi

$\sigma_u = 93.9$ ksi $k = 54.0$ ksi $\sqrt{\text{in.}}$

---

FIG. 12 COMPARISON BETWEEN THEORY AND EXPERIMENT FOR 2014-T6 Al Cylindrical Vessels at -423°F

Material: X-52 Plain Carbon (semikilled)

\[ \sigma_0^\text{p} = 55 \text{ ksi} \]
\[ \sigma_0^\text{s} = 78 \text{ ksi} \]
\[ k = 256 \text{ ksi} \sqrt{\text{in.}} \]

<table>
<thead>
<tr>
<th>T°F</th>
<th>c (in.)</th>
<th>\lambda</th>
<th>I</th>
<th>\sigma_h</th>
<th>\sigma_h</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>exp. ksi</td>
<td>calc. ksi</td>
</tr>
<tr>
<td>-423</td>
<td>0.050</td>
<td>0.30</td>
<td>1.03</td>
<td>81</td>
<td>89.0</td>
</tr>
<tr>
<td>-423</td>
<td>0.050</td>
<td>0.30</td>
<td>1.03</td>
<td>83</td>
<td>89.0</td>
</tr>
<tr>
<td>-423</td>
<td>0.050</td>
<td>0.30</td>
<td>1.03</td>
<td>86</td>
<td>89.0</td>
</tr>
<tr>
<td>-423</td>
<td>0.075</td>
<td>0.45</td>
<td>1.05</td>
<td>73</td>
<td>76.0</td>
</tr>
<tr>
<td>-423</td>
<td>0.125</td>
<td>0.75</td>
<td>1.12</td>
<td>63</td>
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<tr>
<td>-423</td>
<td>0.250</td>
<td>1.50</td>
<td>1.40</td>
<td>40</td>
<td>39.3</td>
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<tr>
<td>-423</td>
<td>0.250</td>
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<td>1.40</td>
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<tr>
<td>-423</td>
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<td>1.74</td>
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<td>26.8</td>
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<td>2.11</td>
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<td>-423</td>
<td>0.625</td>
<td>3.76</td>
<td>2.49</td>
<td>20</td>
<td>17.0</td>
</tr>
</tbody>
</table>

The results are plotted in Fig. 11. The agreement is not very good. Here again the same remark as in part a) holds. Thus, if one uses \( \sigma_0^\text{p} = 79.5 \text{ ksi} \) and \( k = 62.6 \text{ ksi} \sqrt{\text{in.}} \), the agreement is even better (see Fig. 12).

A. R. Duffy\(^9\) gives results of tests on 30-in. diameter, \( \frac{1}{4} \)-in. thick pipes with through-the-thickness cracks under the following conditions:


(2) Material: 0.36 C steel

\[ \sigma_y = 34.5 \text{ ksi} \]

\[ \sigma_u = 69.5 \text{ ksi} \]

\[ \sigma^* = 43.3 \text{ ksi} \]

\[ k = 249.0 \text{ ksi} \sqrt{\bar{\nu}} \]

\[ \sigma = 43.3 \text{ ksi} \]

The results are plotted in Fig. 15. The agreement is fairly good. There is some temperature variation that affects to some extent, the fracture toughness \( k \).

\begin{tabular}{cccccc}
\hline
T°C & c (in.) & \lambda & \beta & \sigma_{h} \text{ exp. ksi} & \sigma_{h} \text{ calc. ksi} \\
\hline
62-88 & 3 & 0.93 & 1.20 & 33.0 & 35.0 \\
62-88 & 6 & 1.97 & 1.61 & 25.7 & 25.0 \\
62-88 & 6 & 1.97 & 1.61 & 27.7 & 25.0 \\
62-88 & 12 & 3.94 & 2.57 & 12.7 & 13.3 \\
62-88 & 12 & 3.94 & 2.57 & 15.2 & 13.3 \\
\hline
\end{tabular}

CONCLUSIONS

The close agreement between the theoretically predicted fracture strengths and the experimental data suggests that Eq. (1) can be used to predict failure in pressurized vessels knowing only the structural geometry, the crack length, the ultimate and yield stresses, and the fracture toughness of the material.

An interesting point, however, should be stressed.

It seems from some of the foregoing examples that for very small crack lengths – of the order 1/100 in. – the cosine angle is greater than \( \pi/2 \) and the criterion, therefore, fails. This failure is due to the fact that the material, at least local to the crack, undergoes considerable strain-hardening; and as a result, the fracture hoop stress is higher than the \( \sigma^* \). At the present time, there are no adequate criteria to handle this problem. However, the author suggests that for such materials one can use a higher value for \( \sigma^* \), perhaps \( \sigma^* = (\sigma_u + \sigma_y) / 2 \).

This matter properly forms the subject matter for continuing study.
REFERENCES

Appendix

DERIVATION OF EQ. (1)

Following the same analysis as that of Ref. [2], one arrives at the following Griffith-type fracture criterion for initially curved sheets:

\[
\frac{(33+6\nu-7\nu^2)(1+\nu)}{3(9-7\nu)} \left[ p^{(b)} \right]^2 + \left[ p^{(e)} \right]^2
\]

where the stress coefficients \( p^{(b)} \), \( p^{(e)} \) are functions of the geometry of the shell and can be found in Refs. [1-5].

However, due to the presence of high stresses in the vicinity of the crack tip, when the appropriate yield criterion is satisfied, then localized plastic deformation occurs and a plastic zone is created. This phenomenon effectively increases the crack length and, therefore, must be accounted for. Following Dugdale [8] the plastic zone size \( \rho \) is determined by the relation

\[
\frac{c}{\rho + c} = \cos \left( \frac{\pi \sigma}{2\sigma_y} \right)
\]

This relation applies only to a perfect elastic-plastic behavior of a non-strain-hardening material. McClintock [9], however, has suggested that a strain-hardening material may be approximated by an ideally plastic one, if a stress higher than \( \sigma_y \) and lower than \( \sigma^* \), is chosen, say \( \sigma^* \). Thus, correcting the Griffith-Irwin equation so as to include yielding and geometry effect, one has

\[
\sigma_F = \frac{2\sigma^*}{n} \cos^{-1} \left[ \exp \left( -\frac{\pi k^2}{8\sigma^* c} \right) \right]
\]

which upon substituting for \( \sigma_F \) from Eq. (2) one obtains Eq. (1).