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FOREWORD

This report describes work performed at the Firestone Flight Sciences Laboratory of the Graduate Aeronautical Laboratories at the California Institute of Technology for the Aerospace Research Laboratories under Contract AF 33(616)-7806, "Research on Mechanics of Crack Initiation," Task 7063-02, "Research in the Mechanics of Structures," Project 7063, "Mechanics of Flight." Notes for this report are kept in File GALCIT SM 64-13.

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ABSTRACT

Using an integral formulation the coupled Marguerre equations for a cracked cylindrical shell of length 2c are solved for the in-plane and Kirchhoff bending stresses, and, among other things, it is found that the explicit nature of the stresses near the crack point depends upon the inverse half power of the non-dimensional distance from the point ϵ. The character of the combined extension-bending stress field near the crack tip is investigated in detail for the special case of an axial crack in a long closed cylindrical shell which is subjected to uniform internal pressure q_o. Pending a complete study of the solution, approximate results for the combined surface stresses near the crack tip normal and along the line of crack prolongation are respectively of the form:

$$\sigma_y(\epsilon, 0)\Big|_{\nu=\frac{1}{3}} \approx \frac{0.79}{\sqrt{2\epsilon}} \; \frac{q_o R}{h} + \ldots$$

and similarly

$$\sigma_x(\epsilon, 0)\Big|_{\nu=\frac{1}{3}} \approx \frac{0.97}{\sqrt{2\epsilon}} \; \frac{q_o R}{h}$$

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SYMBOLS

c	=	half crack length
D	=	$Eh^3/[12(1-\nu^2)]$ = flexural rigidity
E	=	Young's modulus of elasticity
$F(X, Y)$	=	stress function
G	=	shear modulus
h	=	thickness
i	$\equiv$	$\sqrt{-1}$
K_n	=	modified Bessel function of the third kind of order n
ℓ_s	=	$2c$ = crack length of shell
ℓ_p	=	crack length of plate
$(\ell_{cr})_s$	=	critical crack length of shell
$(\ell_{cr})_p$	=	critical crack length of plate
L_i^*	=	kernels as defined in text
L_i	$\equiv$	$\lim_{\lvert y\rvert \to 0} L_i^*$
m_o	=	constant as defined in text
M_x, M_y, M_{xy}	=	moment components
n_o	=	constant as defined in text
N_x, N_y, N_{xy}	=	membrane forces
$q(X, Y)$	=	internal pressure
q_o	=	uniform internal pressure
r	=	$\sqrt{x^2 + y^2}$
R	=	radius of curvature of the shell
t_o	=	constant as defined in text
v_o	=	constant as defined in text
V_y	=	equivalent shear
$W(X, Y)$	=	displacement function

SYMBOLS

x, y, z	=	dimensionless coordinates with respect to the crack length
X, Y, Z	=	rectangular cartesian coordinates
α	$\equiv$	$(i)^{\frac{1}{2}}$
β	$\equiv$	$(-i)^{\frac{1}{2}}$
γ	=	0.5768 = Euler's constant
γ^*	=	surface energy per unit area
ϵ	$\equiv$	$\sqrt{(\frac{X-1}{c})^2 + (\frac{Y}{c})^2}$
$\varepsilon_x, \varepsilon_y, \varepsilon_z$	=	strain components
ζ	$\equiv$	$x-\xi$
θ	=	$\tan^{-1} \frac{Y}{X}$
λ^4	$\equiv$	$\frac{Ehc^4}{R^2 D} \equiv \frac{12(1-\nu^2)c^4}{R^2 h^2}$
ν	=	Poisson's ratio
ν_o	$\equiv$	$1-\nu$
ρ	$\equiv$	$\sqrt{\zeta^2 + y^2} = \sqrt{(x-\xi)^2 + y^2}$
$\sigma_{x_b}, \sigma_{y_b}, \tau_{xy_b}$	=	bending stress components
$\sigma_{x_e}, \sigma_{y_e}, \tau_{xy_e}$	=	stretching stress components
$\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy}$	=	applied stress components at the crack
σ^*	=	critical (fracture) stress

I INTRODUCTION

In the field of fracture mechanics, considerable work has been carried out on initially flat sheets subjected to either bending or extensional stresses, and for small deformations the superposition of these separate effects [1] is permissible. On the other hand, if a thin sheet is initially curved, a bending (or extensional) loading will generally produce both bending and extensional stresses. The subject of eventual concern therefore is that of the simultaneous stress fields produced in an initially curved sheet containing a crack.

Two geometries immediately come to mind: a spherical shell, and a cylindrical shell. In the former case the radius of curvature is constant in all directions. This problem was investigated by the author in a recent paper [2]. In the latter case one of the principal radii of curvature is infinite and the other constant. It appears therefore that this geometric simplicity leads to a rather straightforward analytical solution. However, the fact that the curvature varies between zero and a constant as one considers different angular positions--say around the point of a crack which is aligned prallel to the cylinder axis--more than obviates the initial geometric simplification and therefore increases the mathematical complexities considerably. For this reason, Sechler and Williams [3] suggested an approximate equation, based upon the behavior of a beam on an elastic foundation, and were able to obtain a reasonable agreement with the experimental results. In this paper, we investigate this problem in a more sophisticated manner.

It is of some practical value to be able to correlate flat sheet behavior with that of initially curved specimens. In experimental work, for example, considerable time could be saved if a reliable prediction of curved sheet response behavior could be made from flat sheet tests. For this reason an exploratory study was undertaken to assess analytically how the two problems might be related. Although it is recognized that elastic analysis is not directly applicable to fracture prediction because of the plastic flow near the crack tip, considerable information can be obtained.

II GENERALITIES

In the following, we consider bending and stretching of thin shells, as described by traditional two-dimensional linear theory, with the additional assumption of shallowness. In speaking of the formulation of two-dimensional differential equations, we mean the transition from the exact three-dimensional elasticity problem to that of two-dimensional approximate formulation, which is appropriate in view of the "thinness" of the shell. In this paper, we limit ourselves to isotropic and homogeneous shallow segments of elastic cylindrical shells of constant thickness. It is also assumed that the shell is subjected to small deformations and strains so that the stress-strain relations may be established through Hooke's law.

The basic variables in the theory of shallow shells are the displacement component W(X, Y) in the direction of an axis Z, and a stress function F(X, Y) which represents the stress resultants tangent to the middle surface of the shell. Following Marguerre [4] the coupled differential equations governing W and F, with X and Y as rectangular coordinates of the base plane (see fig. 1), are given by:

$$-\frac{Eh}{R}\frac{\partial^2 W}{\partial X^2} + \nabla^4 F = 0 \qquad (2.1)$$

$$+\nabla^4 W + \frac{1}{RD}\frac{\partial^2 F}{\partial X^2} = \frac{q(X,Y)}{D} \qquad (2.2)$$

The usual bending moment components M_x, M_y, M_{xy} are defined in terms of the displacement function W as:

$$M_x = -D\left[\frac{\partial^2}{\partial X^2} + \nu \frac{\partial^2}{\partial Y^2}\right] W \qquad (2.3)$$

$$M_y = -D\left[\frac{\partial^2}{\partial Y^2} + \nu \frac{\partial^2}{\partial X^2}\right] W \qquad (2.4)$$

$$M_{xy} = -D\ (1-\nu)\frac{\partial^2 W}{\partial X \partial Y} \qquad (2.5)$$

Similarly, the membrane forces are defined in terms of the stress function F as:

$$N_x = \frac{\partial^2 F}{\partial Y^2} \qquad (2.6)$$

$$N_y = \frac{\partial^2 F}{\partial X^2} \qquad (2.7)$$

$$N_{xy} = -\frac{\partial^2 F}{\partial X \partial Y} \qquad (2.8)$$

In view of (2.3)-(2.8) the bending stress components are

$$\sigma'_{x_b} = -\frac{EZ}{(1-\nu^2)}\left[\frac{\partial^2 W}{\partial X^2} + \nu \frac{\partial^2 W}{\partial Y^2}\right] \qquad (2.9)$$

$$\sigma_{y_b} = -\frac{EZ}{(1-\nu^2)}\left[\frac{\partial^2 W}{\partial Y^2} + \nu \frac{\partial^2 W}{\partial X^2}\right] \qquad (2.10)$$

$$\tau_{xy_b} = -2GZ\frac{\partial^2 W}{\partial X \partial Y} \qquad (2.11)$$

Similarly, the extensional stress components are

$$\sigma_{x_e} = \frac{1}{h}\frac{\partial^2 F}{\partial Y^2} \qquad (2.12)$$

$$\sigma_{y_e} = \frac{1}{h}\frac{\partial^2 F}{\partial X^2} \qquad (2.13)$$

$$\tau_{xy_e} = -\frac{1}{h}\frac{\partial^2 F}{\partial X \partial Y} \qquad (2.14)$$

III CRACKED CYLINDRICAL SHELL

1. Formulation of the Problem

Consider a portion of a thin, shallow cylindrical shell of constant thickness h and subjected to internal pressure $q(X, Y)$. The material of the shell is assumed to be homogeneous and isotropic, and parallel to the axis there exists a cut of length 2c. The coupled differential equations governing the bending deflection $W(X, Y)$ and membrane stress function $F(X, Y)$ are given by (2.1) and (2.2). It is convenient at this point to introduce dimensionless coordinates, namely

$$x \equiv \frac{X}{c} , \quad y \equiv \frac{Y}{c} \tag{3.1}$$

which change the homogeneous parts of (2.1) and (2.2) to

$$-\frac{Ehc^2}{R} \frac{\partial^2 W}{\partial x^2} + \nabla^4 F = 0 \tag{3.2}$$

$$\nabla^4 W + \frac{c^2}{RD} \frac{\partial^2 F}{\partial x^2} = 0 \tag{3.3}$$

As to boundary conditions, one must require that the normal moment, equivalent vertical shear, and normal and tangential in-plane stresses vanish along the crack. However, suppose that one has already found* a particular solution satisfying (3.2) and (3.3), but that there is a residual normal moment M_y, equivalent vertical shear V_y, normal in-plane stress N_y, and in-plane tangential stress N_{xy}, along the

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* See particular solution, SECTION V.

crack $|x| < 1$, of the form:

$$M_y^{(P)} = -\frac{D}{c^2} m_o \qquad (3.4)$$

$$V_y^{(P)} = -\frac{D}{c^3} v_o \qquad (3.5)$$

$$N_y^{(P)} = -\frac{n_o}{c^2} \qquad (3.6)$$

$$N_{xy}^{(P)} = -\frac{t_o}{c^2} \qquad (3.7)$$

For simplicity, we take m_o, v_o, n_o, t_o to be constants* and furthermore we divide the problem into two parts:

Symmetrical : where $v_o = t_o = 0$

Antisymmetrical: where $m_o = n_o = 0$

2. Mathematical Statement of the Problem

Assuming therefore that a particular solution has been found, we need to find now two functions of the dimensionless coordinates (x, y), $W(x,y)$ and $F(x,y)$, such that they satisfy the partial differential equations (3.2) and (3.3) and the following boundary conditions. At $y = 0$ and $|x| < 1$:

$$M_y(x,0) = -\frac{D}{c^2}\left[\frac{\partial^2 W}{\partial y^2} + \nu\frac{\partial^2 W}{\partial x^2}\right] = \frac{Dm_o}{c^2} \qquad (3.8)$$

$$V_y(x,0) = -\frac{D}{c^3}\left[\frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial x^2 \partial y}\right] = D\frac{v_o}{c^3} \qquad (3.9)$$

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* For non-constants see discussion on page 18.

$$N_y(x,0) = \frac{1}{c^2}\frac{\partial^2 F}{\partial x^2} = \frac{n_o}{c^2} \qquad (3.10)$$

$$N_{xy}(x,0) = -\frac{1}{c^2}\frac{\partial^2 F}{\partial x \partial y} = \frac{t_o}{c^2} \qquad (3.11)$$

Next, we must satisfy the continuity requirements, i.e., for $y = 0$ and $|x| > 1$

$$\lim_{|y| \to 0} \left[\frac{\partial^n}{\partial y^n}(W^+) - \frac{\partial^n}{\partial y^n}(W^-) \right] = 0 \qquad (3.12)$$

$$\lim_{|y| \to 0} \left[\frac{\partial^n}{\partial y^n}(F^+) - \frac{\partial^n}{\partial y^n}(F^-) \right] = 0 \qquad (3.13)$$

$(n = 0, 1, 2, 3.)$

Furthermore, because we are limiting ourselves to a large radius of curvature for this shallow shell, i.e., small deviations from a flat sheet, we can apply certain boundary conditions at infinity even though we know physically that the stresses and displacements far away from the crack are finite. Therefore, to avoid infinite stresses and infinite displacements we must require that the displacement function W and the stress function F with their first derivatives be finite far away from the crack. These restrictions simplify the mathematical complexities of the problem considerably, and correspond to the usual expectations of the St. Venant Principle. It should be pointed out that the boundary conditions at infinity are not geometrically feasible. However if the crack is small compared to the dimensions of the shell, the approximation is good.

IV SOLUTION OF THE SYMMETRIC PART

1. Integral Representations of the Solution

We next construct the following representations which have the proper symmetrical behavior with respect to x,

$$W(x,y^{\pm}) = +\int_{o}^{\infty} \Big\{ P_1\, e^{-\sqrt{s(s-\lambda\alpha)}\,|y|} + P_2\, e^{-\sqrt{s(s+\lambda\alpha)}\,|y|} + P_3\, e^{-\sqrt{s(s-\lambda\beta)}\,|y|} + P_4\, e^{-\sqrt{s(s+\lambda\beta)}\,|y|} \Big\} \cos xs\, ds \qquad (4.1)$$

$$F(x,y^{\pm}) = +i\sqrt{EhD} \int_{o}^{\infty} \Big\{ P_1\, e^{-\sqrt{s(s-\lambda\alpha)}\,|y|} + P_2\, e^{-\sqrt{s(s+\alpha\lambda)}\,|y|} - P_3\, e^{-\sqrt{s(s-\lambda\beta)}\,|y|} - P_4\, e^{-\sqrt{s(s+\lambda\beta)}\,|y|} \Big\} \cos xs\, ds \qquad (4.2)$$

where the P_i, $(i = 1, 2, 3, 4)$, are arbitrary functions of s to be determined from the boundary conditions, and the $\pm$ signs refer to $y > 0$ and $y < 0$ respectively.

2. The Boundary and Continuity Conditions in Terms of the Integral Representations

Assuming that we can differentiate under the integral sign, formally substituting (4.1) and (4.2) into (3.8)-(3.11) yields respectively:

$$\lim_{|y| \to 0} \mp \int_{o}^{\infty} \Big\{ P_1 s(\nu_o s-\alpha\lambda) e^{-\sqrt{s(s-\alpha\lambda)}\,|y|} + P_2 s(\nu_o s+\alpha\lambda) e^{-\sqrt{s(s+\alpha\lambda)}\,|y|} + P_3 s(\nu_o s-\beta\lambda) e^{-\sqrt{s(s-\beta\lambda)}\,|y|} + P_4 s(\nu_o s+\lambda\beta) e^{-\sqrt{s(s+\lambda\beta)}\,|y|} \Big\} \qquad (4.3)$$

$$\cdot \cos xs\, ds = + m_o \; ; \quad |x| < 1$$

$$\pm \lim_{|y|\to 0} \int_0^\infty \Big\{ s(\nu_o s+\alpha\lambda)P_1\sqrt{s(s-\alpha\lambda)}\; e^{-\sqrt{s(s-\alpha\lambda}\,|y|} + s(\nu_o s-\alpha\lambda)\sqrt{s(s+\alpha\lambda)}$$

$$P_2\; e^{-\sqrt{s(s+\alpha\lambda)}\,|y|} + s(\nu_o s+\beta\lambda)\sqrt{s(s-\beta\lambda)}\;P_3\; e^{-\sqrt{s(s-\beta\lambda)}} \qquad (4.4)$$

$$+ s(\nu_o s-\beta\lambda)\sqrt{s(s+\beta\lambda)}\; e^{-\sqrt{s(s+\beta\lambda)}\,|y|}\Big\} \cos xs\,ds = 0; \quad |x| < 1$$

$$\lim_{|y|\to 0} - i\sqrt{EhD} \int_0^\infty \Big\{ P_1\; e^{-\sqrt{s(s-\alpha\lambda)}\,|y|} + P_2\, e^{-\sqrt{s(s+\alpha\lambda)}\,|y|}$$

$$-P_3 e^{-\sqrt{s(s-\beta\lambda)}\,|y|} - P_4\, e^{-\sqrt{s(s+\beta\lambda)}\,|y|}\Big\}\, s^2 \cos xs\,ds = n_o; \quad |x|<1 \qquad (4.5)$$

$$\pm \lim_{|y|\to 0} i\sqrt{EhD} \int_0^\infty \Big\{ \sqrt{s(s-\alpha\lambda)}\; P_1\, e^{-\sqrt{s(s-\alpha\lambda}\,|y|} + \sqrt{s(s+\alpha\lambda)}$$

$$P_2\; e^{-\sqrt{s(s+\alpha\lambda)}\,|y|} - \sqrt{s(s-\lambda\beta)}\; P_3\, e^{-\sqrt{s(s-\lambda\beta}\,|y|} \qquad (4.6)$$

$$-\sqrt{s(s+\beta\lambda)}\; P_4\; e^{-\sqrt{s(s+\beta\lambda)}\,|y|}\Big\}\, s\, \sin xs\,ds = 0; \quad |x| < 1$$

where again the $\pm$ signs refer to $y > 0$ and $y < 0$ respectively. A sufficient condition for eqs. (4.4) and (4.6) to be satisfied is to set the integrands equal to zero. This leads to

$$\sqrt{s(s-\lambda\beta)}\; P_3 = -\left(\frac{\nu_o s}{\beta\lambda} - \frac{1}{2}\right)\left[\sqrt{s(s-\alpha\lambda)}\; P_1 + \sqrt{s(s+\alpha\lambda)}\; P_2\right]$$

$$- \frac{\alpha}{2\beta}\left[\sqrt{s(s-\alpha\lambda)}\; P_1 - \sqrt{s(s+\alpha\lambda)}\; P_2\right] \qquad (4.7)$$

$$\sqrt{s(s+\lambda\beta)}\,P_4 = \left(\frac{\nu_o s}{\beta\lambda} + \frac{1}{2}\right)\left[\sqrt{s(s-\alpha\lambda)}\; P_1 + \sqrt{s(s+\alpha\lambda)}\; P_2\right]$$

$$+ \frac{\alpha}{2\beta}\left[\sqrt{s(s-\alpha\lambda)}\; P_1 - \sqrt{s(s+\alpha\lambda)}\; P_2\right] \qquad (4.8)$$

Further it may easily be shown that the continuity conditions lead to:

$$\int_0^\infty \left\{\sqrt{s(s-\alpha\lambda)}\ P_1 + \sqrt{s(s+\alpha\lambda)}\ P_2\right\} \cos xs\, ds = 0\ ; \quad |x| > 1 \quad (4.9a)$$

$$\int_0^\infty \left\{\nu_o s^2 \left(\sqrt{s(s-\alpha\lambda)}\ P_1 + \sqrt{s(s+\alpha\lambda)}\ P_2\right)\right.$$

$$\left. + \alpha\lambda s\left(\sqrt{s(s-\alpha\lambda)}\ P_1 - \sqrt{s(s+\alpha\lambda)}\ P_2\right)\right\} \cos xs\, ds = 0; \quad |x| > 1 \quad (4.10a)$$

which are satisfied if we consider the following combinations to vanish

$$\int_0^\infty \left\{\sqrt{s(s-\alpha\lambda)}\ P_1 + \sqrt{s(s+\alpha\lambda)}\ P_2 + \frac{\alpha\lambda}{s}\right.$$

$$\left. \cdot \left(\sqrt{s(s-\alpha\lambda)}\ P_1 - \sqrt{s(s+\alpha\lambda)}\ P_2\right)\right\} \cos xs\, ds = 0\ ; \quad |x| > 1 \quad (4.9b)$$

$$\int_0^\infty \left\{\sqrt{s(s-\alpha\lambda)}\ P_1 + \sqrt{s(s+\alpha\lambda)}\ P_2 - \frac{\alpha\lambda}{s}\right.$$

$$\left. \cdot \left(\sqrt{s(s-\alpha\lambda)}\ P_1 - \sqrt{s(s+\alpha\lambda)}\right)\right\} \cos xs\, ds = 0\ ; \quad |x| > 1 \quad (4.10b)$$

Therefore we have reduced our problem to solving the dual integral equations (4.3), (4.5), (4.9b), (4.10b) for the unknown functions $P_1(s)$ and $P_2(s)$.

3. Reduction to Single Integral Equations

Because we are unable to solve dual integral equations of the type discussed in the previous section, therefore we will reduce the problem to singular integral equations. Let for $|x| < 1$ and $y = 0$

$$u_1(x) = \int_0^\infty \left\{\sqrt{s(s-\alpha\lambda)} P_1 + \sqrt{s(s+\alpha\lambda)}\ P_2 + \frac{\alpha\lambda}{s}\left(\sqrt{s(s-\alpha\lambda)} P_1 - \sqrt{s(s+\alpha\lambda)}\right) P_2\right\} \cos xs\, ds \quad (4.11)$$

$$u_2(x) = \int_0^\infty \{ \sqrt{s(s-\alpha\lambda)}\, P_1 + \sqrt{s(s+\alpha\lambda)}\, P_2 - \frac{\alpha\lambda}{s} (\sqrt{s(s-\alpha\lambda)}P_1 - \sqrt{s(s+\alpha\lambda)}P_2)\} \cos xs\, ds \tag{4.12}$$

which by Fourier Inversion gives:

$$\sqrt{s(s-\alpha\lambda)}P_1 + \sqrt{s(s+\alpha\lambda)}P_2 + \frac{\alpha\lambda}{s} (\sqrt{s(s-\alpha\lambda)}P_1 - \sqrt{s(s+\alpha\lambda)}P_2) = \frac{2}{\pi} \int_0^1 u_1(\xi) \cos\xi s\, d\xi \tag{4.13}$$

$$\sqrt{s(s-\alpha\lambda)}P_1 + \sqrt{s(s+\alpha\lambda)}\, P_2 - \frac{\alpha\lambda}{s} (\sqrt{s(s-\alpha\lambda)}\, P_1 - \sqrt{s(s+\alpha\lambda)}P_2) = \frac{2}{\pi} \int_0^1 u_2(\xi) \cos\xi s\, d\xi \tag{4.14}$$

And hence

$$P_1 = \frac{1}{2\pi\sqrt{s(s-\alpha\lambda)}} \int_0^1 \left\{ u_1 + \frac{su_1}{\alpha\lambda} + u_2 - \frac{s}{\alpha\lambda}\, u_2 \right\} \cos\xi s\, d\xi \tag{4.15}$$

$$P_2 = \frac{1}{2\pi} \frac{1}{\sqrt{s(s+\alpha\lambda)}} \int_0^1 \left\{ u_1 + u_2 - \frac{su_1}{\alpha\lambda} + \frac{s}{\alpha\lambda}\, u_2 \right\} \cos\xi s\, d\xi \tag{4.16}$$

$$P_3 = -\left(\frac{\nu_o s}{\beta\lambda} - \frac{1}{2}\right) \frac{1}{\pi\sqrt{s(s-\beta\lambda)}} \int_0^1 (u_1 + u_2) \cos\xi d\xi - \frac{s}{2\beta\lambda\pi} \frac{1}{\sqrt{s(s-\beta\lambda)}} \int_0^1 (u_1 - u_2) \cos\xi s\, d\xi \tag{4.17}$$

$$P_4 = \left(\frac{\nu_o s}{\beta\lambda} + \frac{1}{2}\right) \frac{1}{\pi} \frac{1}{\sqrt{s(s+\beta\lambda)}} \int_0^1 (u_1 + u_2) \cos\xi s\, d\xi + \frac{s}{2\beta\lambda\pi} \frac{1}{\sqrt{s(s+\beta\lambda)}} \int_0^1 (u_1 - u_2) \cos\xi s\, d\xi \tag{4.18}$$

where the functions $u_1(\xi)$ and $u_2(\xi)$ due to the symmetry of the problem, are even. Formally substituting (4.15)-(4.18) into (4.3) and (4.5) we find after changing the order of integration and rearranging

$$N_y = -\frac{i\sqrt{EhD}}{\pi}\int_{-1}^{1}\{L_1^* u_1 + L_2^* u_2\}\,d\xi \tag{4.19}$$

$$M_y = -\frac{D}{\pi}\int_{-1}^{1}\{u_1(\xi)\,L_3^* + L_4^* u_2\}\,d\xi \tag{4.20}$$

where the kernels are

$$\begin{aligned} L_1^* \equiv \int_0^\infty \Big\{ & (1+\frac{s}{\alpha\lambda})\frac{s^2}{4}\frac{e^{-\sqrt{s(s-\alpha\lambda)}\,|y|}}{\sqrt{s(s-\alpha\lambda)}} + (1-\frac{s}{\alpha\lambda})\frac{s^2}{4}\frac{e^{-\sqrt{s(s+\alpha\lambda)}\,|y|}}{\sqrt{s(s+\alpha\lambda)}} \\ & + (\frac{\nu_o s}{\beta\lambda} - \frac{1}{2})\frac{s^2}{2}\frac{e^{-\sqrt{s(s-\lambda\beta)}\,|y|}}{\sqrt{s(s-\lambda\beta)}} + \frac{s^3}{4\beta\lambda}\frac{e^{-\sqrt{s(s-\beta\lambda)}\,|y|}}{\sqrt{s(s-\beta\lambda)}} \\ & - (\frac{\nu_o s}{\beta\lambda} + \frac{1}{2})\frac{s^2}{2}\frac{e^{-\sqrt{s(s+\lambda\beta)}\,|y|}}{\sqrt{s(s+\lambda\beta)}} - \frac{s^3}{4\beta\lambda}\frac{e^{-\sqrt{s(s+\beta\lambda)}\,|y|}}{\sqrt{s(s+\beta\lambda)}}\Big\}\cos\zeta s\,ds \end{aligned} \tag{4.21}$$

$$\begin{aligned} L_2^* \equiv \int_0^\infty \Big\{ & (1-\frac{s}{\alpha\lambda})\frac{s^2}{4}\frac{e^{-\sqrt{s(s-\alpha\lambda)}\,|y|}}{\sqrt{s(s-\alpha\lambda)}} + (1+\frac{s}{\alpha\lambda})\frac{s^2}{4}\frac{e^{-\sqrt{s(s+\alpha\lambda)}\,|y|}}{\sqrt{s(s+\alpha\lambda)}} + \\ & + (\frac{\nu_o s}{\beta\lambda} - \frac{1}{2})\frac{s^2}{2}\frac{e^{-\sqrt{s(s-\beta\lambda)}\,|y|}}{\sqrt{s(s-\beta\lambda)}} - \frac{s^3}{4\beta\lambda}\frac{e^{-\sqrt{s(s-\beta\lambda)}\,|y|}}{\sqrt{s(s-\beta\lambda)}} \\ & - (\frac{\nu_o s}{\beta\lambda} + \frac{1}{2}(+)\frac{s}{2\beta\lambda})\frac{s^2}{2}\frac{e^{-\sqrt{s(s+\lambda\beta)}\,|y|}}{\sqrt{s(s+\lambda\beta}}\Big\}\cos\zeta s\,ds \end{aligned} \tag{4.22}$$

?

$$\begin{aligned} L_3^* \equiv \int_0^\infty \Big\{ & \frac{s(s\nu_o-\alpha\lambda)}{4}(1+\frac{s}{\alpha\lambda})\frac{e^{-\sqrt{s(s-\alpha\lambda)}\,|y|}}{\sqrt{s(s-\alpha\lambda)}} + \frac{s(s\nu_o+\alpha\lambda)}{4}(1-\frac{s}{\alpha\lambda})\frac{e^{-\sqrt{s(s+\alpha\lambda)}\,|y|}}{\sqrt{s(s+\alpha\lambda)}} \\ & - \frac{s(s\nu_o-\lambda\beta)}{2}(\frac{\nu_o s}{\beta\lambda} - \frac{1}{2} + \frac{s}{2\beta\lambda})\frac{e^{-\sqrt{s(s-\beta\lambda)}\,|y|}}{\sqrt{s(s-\beta\lambda)}} \\ & + \frac{s(s\nu_o+\lambda\beta)}{2}(\frac{\nu_o s}{\beta\lambda} + \frac{1}{2} + \frac{s}{2\beta\lambda})\frac{e^{-\sqrt{s(s+\beta\lambda)}\,|y|}}{\sqrt{s(s+\beta\lambda)}}\Big\}\cos\zeta s\,ds \end{aligned} \tag{4.23}$$

$$L_4^* \equiv \int_0^\infty \Bigg\{ \frac{s(s\nu_o - a\lambda)}{4} \left(1 - \frac{s}{a\lambda}\right) \frac{e^{-\sqrt{s(s-a\lambda)}\,|y|}}{\sqrt{s(s-a\lambda)}} + \frac{s(\nu_o s + a\lambda)}{4} \left(1 + \frac{s}{a\lambda}\right) \frac{e^{-\sqrt{s(s+a\lambda)}\,|y|}}{\sqrt{s(s+a\lambda)}}$$

$$- \frac{s(\nu_o s - \lambda\beta)}{2} \left(\frac{\nu_o s}{\beta\lambda} - \frac{1}{2} - \frac{s}{2\beta\lambda}\right) \frac{e^{-\sqrt{s(s-\beta\lambda)}\,|y|}}{\sqrt{s(s-\beta\lambda)}}$$

$$+ \frac{s(\nu_o s + \beta\lambda)}{2} \left(\frac{\nu_o s}{\beta\lambda} + \frac{1}{2} - \frac{s}{2\beta\lambda}\right) \frac{e^{-\sqrt{s(s+\beta\lambda)}\,|y|}}{\sqrt{s(s+\beta\lambda)}} \Bigg\} \cos\zeta s \, ds \qquad (4.24)$$

The integrations in (4.21)-(4.24) may be carried out explicitly by making use of the Fourier cosine transforms*

$$I_1(\alpha) = \frac{1}{2}\int_0^\infty \left\{ \frac{e^{-\sqrt{s(s-a\lambda)}\,|y|}}{\sqrt{s(s-a\lambda)}} + \frac{e^{-\sqrt{s(s+a\lambda)}\,|y|}}{\sqrt{s(s+a\lambda)}} \right\} \cos\zeta s\, ds = \cos\frac{a\lambda\zeta}{2} \cdot K_o\left(\frac{\beta\lambda\rho}{2}\right) \qquad (4.25)$$

$$I_2(\alpha) = \frac{1}{2}\int_0^\infty \left\{ \frac{e^{-\sqrt{s(s-a\lambda)}\,|y|}}{\sqrt{s(s-a\lambda)}} - \frac{e^{-\sqrt{s(s+a\lambda)}\,|y|}}{\sqrt{s(s+a\lambda)}} \right\} \sin\zeta s\, ds = \sin\frac{a\lambda\zeta}{2} \cdot K_o\left(\frac{\beta\lambda\rho}{2}\right) \qquad (4.26)$$

and similar results obtained by differentiating them with respect to x and $|y|$ (see Appendix). In these formulas $\rho^2 = \zeta^2 + |y|^2$, and K_n denotes the modified Bessel function of the third kind of order n.

The expressions (4.21)-(4.24) can be written in the form

----

* The equations (4.25) and (4.26) are proven in the Appendix.

$\frac{\partial}{\partial x}\{L_i^*(\zeta, |y|, \lambda)\}$ and in the limit as $|y| \to 0$, N_y and M_y become

$$\lim_{|y|\to 0} N_y \equiv \frac{i\sqrt{EhD}}{\pi} \frac{d}{dx} \int_{-1}^{1} \{u_1 L_1 + u_2 L_2\} d\xi \tag{4.27}$$

$$\lim_{|y|\to 0} M_y = -\frac{D}{\pi} \frac{d}{dx} \int_{-1}^{1} \{u_1 L_3 + u_2 L_4\} d\xi \tag{4.28}$$

where the integrals are understood to be of Cauchy principal value and $L_i = \lim_{|y|\to 0} L_i^*$. If in the limit $|y| \to 0$ we replace N_y, M_y by n_o and m_o respectively, integrate with respect to x, then we find that they must satisfy the integral equations

$$\int_{-1}^{1} \{u_1(\xi)L_1 + u_2(\xi)L_2\} d\xi = \frac{-\pi n_o}{i\sqrt{EhD}} x ; \quad |x| < 1 \tag{4.29}$$

$$\int_{-1}^{1} \{u_1(\xi)L_3 + u_2(\xi)L_4\} d\xi = -\pi m_o x ; \quad |x| < 1 \tag{4.30}$$

The kernels L_1, L_2, L_3, L_4 have singularities of the order $\frac{1}{\zeta} \equiv \frac{1}{x-\xi}$ and their behavior for small arguments is:

$$L_1 = \frac{1+\nu_o}{2(x-\xi)} + \frac{a^2\lambda^2(x-\xi)}{16}\left[\frac{7\nu_o - 6}{6} + \frac{5\pi i}{4} + 5\nu_o\left(\gamma + \ln\frac{\lambda a|x-\xi|}{4}\right) - 6\left(\gamma + \ln\frac{\lambda|x-\xi|}{4}\right)\right] \tag{4.31}$$

$$+ O(\lambda^4(x-\xi)^3 \ln \lambda|x-\xi|) - \frac{\lambda^4(x-\xi)^3}{30720}\Big\{-683(1+\nu_o) + 1260(1+\nu_o)\gamma + (1260\nu_o - 70)\ln\frac{\lambda\alpha(x-\xi)}{4} + (1330)\ln\frac{\lambda\beta(x-\xi)}{4}\Big\}$$

$$L_2 = -\frac{1-\nu_o}{2(x-\xi)} + \frac{a^2\lambda^2(x-\xi)}{16}\left[\frac{7\nu_o - 6}{6} - \frac{5\pi i}{4} + 5\nu_o\left(\gamma + \ln\frac{\lambda a|x-\xi|}{4}\right) - 6\left(\gamma + \ln\frac{\lambda|x-\xi|}{4}\right)\right] \tag{4.32}$$

$$+ O(\lambda^4(x-\xi)^3 \ln \lambda|x-\xi|)$$

$$- \frac{\lambda^4(x-\xi)^3}{30720}\Big\{683(1-\nu_o) - 1260(1-\nu_o)\gamma + (-1330 + 1260\nu_o)\ln\frac{\lambda\alpha(x-\xi)}{4} + 70\ln\frac{\lambda\beta(x-\xi)}{4}$$

$$L_3 = \frac{(4-\nu_o)\nu_o}{2(x-\xi)} + \frac{\beta^2\lambda^2(x-\xi)}{16}\left[\frac{7\nu_o-6}{6}(1+\nu_o)+2\pi i+\left(\frac{10\nu_o^2-13\nu_o-6}{2}\right)(\gamma+\ell n\frac{\lambda a|x-\xi|}{4}) + (\frac{11\nu_o-6}{2})(\gamma+\ell n\frac{\lambda\beta|x-\xi|}{4})\right] + O(\lambda^4(x-\xi)^3\ell n\,\lambda|x-\xi|) \quad \checkmark \tag{4.33}$$

$$-\frac{\lambda^4(x-\xi)^3}{30720}\Big\{(683\nu_o^2-1580\nu_o+960)+(-1260\nu_o^2+2,800\nu_o-1,600)\gamma+(-1260\nu_o^2+1470\nu_o-100)\ln\frac{\alpha\lambda(x-\xi)}{4} +(1330\nu_o-1500)\ln\frac{\beta\lambda(x-\xi)}{4}\Big\}$$

$$L_4 = \frac{(4-\nu_o)\nu_o}{2(x-\xi)} + \frac{\beta^2\lambda^2(x-\xi)}{16}\left[-(\frac{7\nu_o-6}{6})(1-\nu_o)+2\pi i+(\frac{6+\nu_o}{2})(\gamma+\ell n\frac{\lambda\beta|x-\xi|}{4}) + (\frac{10\nu_o^2-23\nu_o+6}{2})(\gamma+\ell n\frac{\lambda a|x-\xi|}{4})\right] + O(\lambda^4(x-\xi)^3\ell n\,\lambda|x-\xi|) \tag{4.34}$$

$$-\frac{\lambda^4(x-\xi)^3}{30720}\Big\{(1580\nu_o-960-683\nu_o^2)+(1600+1260\nu_o^2-2800\nu_o)\gamma+(1500+1260\nu_o^2-2730\nu_o)\ln\frac{\alpha\lambda(\;)}{4}+(100-70\nu_o)\ln\frac{\lambda\beta(\;)}{4}\Big\}+\cdots$$

We require that the solutions $u_1(x)$, $u_2(x)$ be Hölder continuous for some positive Hölder indices μ_1 and μ_2 for all x in the closed interval $[-1,1]$. Thus in particular $u_1(x)$, $u_2(x)$ are to be bounded near the ends of the crack.

The problem of obtaining a solution to the coupled integral equations (4.29) and (4.30) can be reduced to the problem of solving two coupled Fredholm integral equations with bounded kernels. These details are discussed in reference [2].

4. Solution of Integral Equations for Small λ.

Because of the complicated nature of the kernels L_i $(i=1,2,3,4)$, a closed form solution of the unknown functions $u_1(x)$ and $u_2(x)$ is extremely difficult. On the other hand, for most practical applications the parameter λ assumes small values as follows from the definition of λ, namely

$$\lambda \equiv \frac{\sqrt[4]{12(1-\nu^2)}}{\sqrt{R/h}}(c/h) = \sqrt[4]{12(1-\nu^2)}\,(c/R)\,(R/h)^{1/2}$$

It is clear that λ is small for large ratios of R/h and small crack lengths. As a practical matter, if we consider crack length less than one tenth of the periphery, i.e. $2c < \frac{2\pi R}{10}$, and for $\frac{R}{h} < 10^3$ a corresponding upper bound for λ can be obtained, namely $\lambda < 20$. Thus the range of λ becomes $0 < \lambda < 20$ and for most practical cases is between 0 and 2, depending upon the size of the crack.

Following reference [2] we assume solutions of the form

$$u_1(\xi) = \sqrt{1-\xi^2}\left[A_1 + \lambda^2 A_2(1-\xi^2) + \ldots\right]; \quad |\xi| < 1 \tag{4.35}$$

$$u_2(\xi) = \sqrt{1-\xi^2}\left[B_1 + \lambda^2 B_2(1-\xi^2) + \ldots\right]; \quad |\xi| < 1 \tag{4.36}$$

where the coefficients $A_1, A_2, \ldots, B_1, B_2, \ldots$ can be functions of λ but not of ξ.

Substituting (4.35) and (4.36) into (4.29) and (4.30) and making use of the integrals given in the Appendix we obtain two algebraic equations with four unknowns — good up to $O(\lambda^2)$. Next we equate coefficients, in particular we first require the coefficients of the x^3 terms to vanish which gives:

$$\left(\frac{1+\nu_o}{2}\right) A_2 - \left(\frac{1-\nu_o}{2}\right) B_2 = \frac{a^2}{2.48}(5\nu_o - 6)(A_1 + B_1) \tag{4.37}$$

$$\left(\frac{4-\nu_o}{2}\right)\nu_o (A_2 + B_2) = \frac{\beta^2}{2.48}\left\{\frac{10\nu_o^2 - 2\nu_o - 12}{2} A_1 + \frac{10\nu_o^2 - 22\nu_o + 12}{2} B_1\right\} \tag{4.38}$$

Hence we are left with two equations and two unknowns, namely:

$$A_1\left\{\frac{1+\nu_o}{2} + \frac{a^2\lambda^2}{32}\left[\frac{37\nu_o - 42}{6} + \frac{5\pi i}{4} + 5\nu_o\left(\gamma + \ell n\frac{\lambda a}{8}\right) - 6\left(\gamma + \ell n\frac{\lambda}{8}\right)\right]\right\}$$
$$+ B_1\left\{-\frac{1-\nu_o}{2} + \frac{a^2\lambda^2}{32}\left[\frac{37\nu_o - 42}{6} - \frac{5\pi i}{4} + 5\nu_o\left(\gamma + \ell n\frac{\lambda a}{8}\right) - 6\left(\gamma + \ell n\frac{\lambda}{8}\right)\right]\right\} \tag{4.39}$$
$$+ O(\lambda^4 \ell n\lambda) = \frac{-n_o}{i\sqrt{EhD}}$$

Similarly

$$A_1 \left\{ \left(\frac{4-\nu_o}{2}\right)\nu_o + \frac{\beta^2\lambda^2}{32}\left[\frac{7\nu_o(1+\nu_o)}{6} + \frac{9\nu_o-8}{2} + \frac{10\nu_o^2-13\nu_o-6}{2} + 2\pi i \right.\right.$$

$$\left.\left. + \frac{11\nu_o-6}{2}\left(\gamma+\ell n\,\frac{\lambda\beta}{8}\right) + \frac{10\nu_o^2-13\nu_o-6}{2}\left(\gamma+\ell n\,\frac{\lambda a}{8}\right)\right]\right\}$$

$$+B_1 \left\{ \left(\frac{4-\nu_o}{2}\right)\nu_o + \frac{\beta^2\lambda^2}{32}\left[-\frac{7\nu_o(1-\nu_o)}{6} + \frac{8-\nu_o}{2} + \frac{10\nu_o^2-23\nu_o+6}{2} + 2\pi i \right.\right. \tag{4.40}$$

$$\left.\left. + \left(\frac{6+\nu_o}{2}\right)\left(\gamma+\ell n\,\frac{\lambda\beta}{8}\right) + \frac{10\nu_o^2-13\nu_o+6}{2}\left(\gamma+\ell n\,\frac{\lambda a}{8}\right)\right]\right\} + O(\lambda^4 \ell n\lambda) = \mp m_o$$

And solving for A_1 and B_1 we find:

$$A_1 = \frac{-n_o}{i\sqrt{EhD}}\left\{1 + \frac{\pi\lambda^2}{16}\left[\frac{5}{4} + \frac{12\nu_o-5\nu_o^2-8}{4(4-\nu_o)\nu_o}\right] + \frac{\beta^2\lambda^2}{16(4-\nu_o)\nu_o}\left[-\frac{7\nu_o(1-\nu_o)}{6} + \frac{8-\nu_o}{2}\right.\right.$$

$$\left.\left. + \frac{10\nu_o^2-23\nu_o+6}{2} + 2\pi i + \frac{6+\nu_o}{2}\left(\gamma+\ell n\,\frac{\lambda\beta}{8}\right) + \frac{10\nu_o^2-23\nu_o+6}{2}\left(\gamma+\ell n\,\frac{\lambda a}{8}\right)\right]\right\}$$

$$- \frac{m_o(1-\nu_o)}{(4-\nu_o)\nu_o}\left\{1 + \frac{\pi\lambda^2}{16}\left[\frac{5}{4} + \frac{12\nu_o-5\nu_o^2-8}{4(4-\nu_o)\nu_o}\right] - \frac{a^2\lambda^2}{16(1-\nu_o)}\left[\frac{37\nu_o-42}{6} - \frac{5\pi i}{4} + 5\nu_o\left(\gamma+\ell n\,\frac{\lambda a}{8}\right)\right.\right. \tag{4.41}$$

$$\left.\left. - 6\left(\gamma+\ell n\,\frac{\lambda}{8}\right)\right]\right\} + O(\lambda^4 \ell n\lambda)$$

$$B_1 = +\frac{n_o}{i\sqrt{EhD}}\left\{1 + \frac{\pi\lambda^2}{16}\left[\frac{5}{4} + \frac{12\nu_o-5\nu_o^2-8}{4(4-\nu_o)\nu_o}\right] + \frac{\beta^2\lambda^2}{16(4-\nu_o)\nu_o}\left[\frac{7\nu_o(1+\nu_o)}{6} + \frac{9\nu_o-8}{2}\right.\right.$$

$$\left.\left. + \frac{10\nu_o^2-13\nu_o-6}{2} + 2\pi i + \frac{11\nu_o-6}{2}\left(\gamma+\ell n\,\frac{\lambda\beta}{8}\right) + \frac{10\nu_o^2-13\nu_o-6}{2}\left(\gamma+\ell n\,\frac{\lambda a}{8}\right)\right]\right\}$$

$$- \frac{m_o(1+\nu_o)}{(4-\nu_o)\nu_o}\left\{1 + \frac{\pi\lambda^2}{16}\left[\frac{5}{4} + \frac{12\nu_o-5\nu_o^2-8}{4(4-\nu_o)\nu_o}\right] + \frac{a^2\lambda^2}{16(1+\nu_o)}\left[\frac{37\nu_o-42}{6} + \frac{5\pi i}{4} + 5\nu_o\left(\gamma+\ell n\,\frac{\lambda a}{8}\right)\right.\right. \tag{4.42}$$

$$\left.\left. -6\left(\gamma+\ell n\,\frac{\lambda}{8}\right)\right]\right\} + O(\lambda^4 \ell n\lambda)$$

It should be pointed out that if coefficients A_1, B_1 of higher accuracy are desired, say up to order λ^{2n}, then it is necessary to solve an n x n algebraic system. In effect, this is a method of successive approximations. Reference [2] shows that for $\lambda < \lambda^*$ the power series solutions of the form

$$u_1^{(N)}(\xi) = \sqrt{1-\xi^2} \sum_{n=0}^{N} A_{n+1} \lambda^{2n} (1-\xi^2)^n \qquad (4.43)$$

$$u_2^{(N)}(\xi) = \sqrt{1-\xi^2} \sum_{n=0}^{N} B_{n+1} \lambda^{2n} (1-\xi^2)^n \qquad (4.44)$$

in the limit as $N \to \infty$, will converge to the exact solutions $u_1(\xi)$ and $u_2(\xi)$ of the integral equations (4.29) and (4.30). However since most particular solutions will give us a non-uniform residual moment and normal membrane stress along the crack, it is only natural to ask how the solution changes. Suppose, for $|x| < 1$, we expand m_o and n_o in the form $\sum_n a_n x^{2n}$ (even powers because of the symmetry of the problem), then our previous method of solution will still be applicable. And as can easily be seen from equations (4.29) and (4.30) although the coefficients A_n, B_n in this case may change, the character of the solution will still remain the same. Finally, because we desire to focus our attention upon the singular stresses around the neighborhood of the crack point, we need only to compute the coefficients A_1 and B_1.

5. Determination of W and F

In view of equations (4.11), (4.12), (4.35), (4.36) and the relation

$$\int_0^\infty s^{-\mu} J_\mu(as)\cos xs\,ds = \begin{cases} \sqrt{\pi}\,(2a)^{-\mu}\left[\Gamma(\mu+\frac{1}{2})\right]^{-1}(a^2-x^2)^{\mu-\frac{1}{2}} & ;\ 0<x<a \\ & ;\ \mathrm{Re}\mu > -\frac{1}{2} \\ 0 & ;\ a<x<\infty \end{cases} \tag{4.45}$$

which can be found on page 44 of [5] we have:

$$\sqrt{s(s-\alpha\lambda)}P_1 + \sqrt{s(s+\alpha\lambda)}\,P_2 + \frac{\alpha\lambda}{s}\left(\sqrt{s(s-\alpha\lambda)}\,P_1 - \sqrt{s(s+\alpha\lambda)}\,P_2\right) = A_1\frac{J_1(s)}{s} + \frac{\lambda^2 A_2}{3}\frac{J_2(s)}{s^2} + \ldots \tag{4.46}$$

$$\sqrt{s(s-\alpha\lambda)}\,P_1 + \sqrt{s(s+\alpha\lambda)}\,P_2 - \frac{\alpha\lambda}{s}\left(\sqrt{s(s-\alpha\lambda)}\,P_1 - \sqrt{s(s+\alpha\lambda)}\,P_2\right) = B_1\frac{J_1(s)}{s} + \frac{\lambda^2 B_2}{3}\frac{J_2(s)}{s^2} + \ldots \tag{4.47}$$

where A_1 and B_1 are given by (4.41) and (4.42) respectively. From this we find that

$$4P_1(s) = \frac{(A_1+B_1)}{\sqrt{s(s-\alpha\lambda)}}\frac{J_1(s)}{s} + \frac{(A_1-B_1)}{\sqrt{s(s-\alpha\lambda)}}\frac{J_1(s)}{\alpha\lambda} + \ldots \tag{4.48}$$

$$4P_2(s) = \frac{(A_1+B_1)}{\sqrt{s(s+\alpha\lambda)}}\frac{J_1(s)}{s} - \frac{(A_1-B_1)}{\sqrt{s(s+\alpha\lambda)}}\frac{J_1(s)}{\alpha\lambda} + \ldots \tag{4.49}$$

and

$$\sqrt{s(s-\lambda\beta)}\,P_3(s) = -\left(\frac{\nu_o s}{\beta\lambda} - \frac{1}{2}\right)\left(\frac{A_1+B_1}{2}\right)\frac{J_1(s)}{s} - \frac{\alpha}{2\beta}\left(\frac{A_1-B_1}{2}\right)\frac{J_1(s)}{\alpha\lambda} + \ldots \tag{4.50}$$

$$\sqrt{s(s+\lambda\beta)}\,P_4(s) = \left(\frac{\nu_o s}{\beta\lambda} + \frac{1}{2}\right)\left(\frac{A_1+B_1}{2}\right)\frac{J_1(s)}{s} + \frac{\alpha}{2\beta}\left(\frac{A_1-B_1}{2}\right)\frac{J_1(s)}{\alpha\lambda} + \ldots \tag{4.51}$$

Therefore, a substitution of the above relations into (4.1) and (4.2) will determine the bending deflection W and membrane stress function F.

6. Determination of the Singular Stresses

In view of equations (4.1), (4.2), (4.48)-(4.51), the bending and extensional stresses defined by (2.9)-(2.14) can be expressed in integral forms which may then be evaluated using the relations (11-14) of the Appendix. Without going into the details we list below the results.

Bending Stresses: On the surface $Z = +\frac{h}{2}$

$$\sigma_{x_b} = -\frac{Eh}{2(1+\nu)c^2}\frac{\tilde{P}_{10}}{\sqrt{2\epsilon}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{5\theta}{2}\right) + O(\epsilon^o) \qquad (4.52)$$

$$\sigma_{y_b} = +\frac{Eh}{2(1-\nu^2)c^2}\frac{\tilde{P}_{10}}{\sqrt{2\epsilon}}\left(\frac{11+5\nu}{4}\cos\frac{\theta}{2} + \frac{1-\nu}{4}\cos\frac{5\theta}{2}\right) + O(\epsilon^o) \qquad (4.53)$$

$$\tau_{xy_b} = -\frac{Gh}{(1-\nu)c^2}\frac{\tilde{P}_{10}}{\sqrt{2\epsilon}}\left(\frac{7+\nu}{4}\sin\frac{\theta}{2} + \frac{1-\nu}{4}\sin\frac{5\theta}{2}\right) + O(\epsilon^o) \qquad (4.54)$$

where

$$\tilde{P}_{10} = \mp\frac{\nu_o}{2}(A_1+B_1) = \frac{-n_o}{\sqrt{EhD}}\frac{\lambda^2}{32(4-\nu_o)}\left\{\frac{42-37\nu_o}{3} + (12-10\nu_o)\left(\gamma + \ln\frac{\lambda}{8}\right)\right\} + \frac{m_o}{(4-\nu_o)}\left\{1 + \frac{12\nu_o - 5\nu_o^2 - 8}{4(4-\nu_o)\nu_o}\frac{\pi\lambda^2}{16}\right\} + O(\lambda^4 \ln\lambda) \qquad (4.55)$$

Note because of the Kirchhoff boundary conditions, the bending shear stress does not vanish in the free edge. For the flat sheet this problem was discussed by Knowles and Wang [6].

Similarly we find through the thickness

Extensional Stresses:

$$\sigma_{x_e} = \frac{\widetilde{P}_{20}}{hc^2\sqrt{2\epsilon}} \left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{5\theta}{2}\right) + O(\epsilon^o) \tag{4.56}$$

$$\sigma_{y_e} = \frac{\widetilde{P}_{20}}{hc^2\sqrt{2\epsilon}} \left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{5\theta}{2}\right) + O(\epsilon^o) \tag{4.57}$$

$$\tau_{xy_e} = -\frac{\widetilde{P}_{20}}{hc^2\sqrt{2\epsilon}} \left(\frac{1}{4}\sin\frac{\theta}{2} - \frac{1}{4}\sin\frac{5\theta}{2}\right) + O(\epsilon^o) \tag{4.58}$$

where

$$\widetilde{P}_{20} = i\sqrt{EhD}\left\{\left(\frac{1+\nu_o}{2}\right)A_1 - \left(\frac{1-\nu_o}{2}\right)B_1\right\}$$

$$= n_o\left\{1 + \frac{5\pi\lambda^2}{64}\right\} - \frac{m_o}{32}\frac{\lambda^2\sqrt{EhD}}{(4-\nu_o)\nu_o}\left\{\frac{37\nu_o - 42}{3} + (10\nu_o - 12)\left(\gamma + \ln\frac{\lambda}{8}\right)\right\} + O(\lambda^4\ln\lambda) \tag{4.59}$$

It is apparent from the above equations that there exists an interaction between bending and stretching, except that in the limit as $\lambda \rightarrow 0$ the stresses of a flat sheet are recovered and coincide with those obtained previously for bending [7] and extension [8] . Thus locally the stresses in a shell are expressed in terms of the stresses in a flat sheet.

7. Combined Stresses

In general, the combined stresses will depend upon the contributions of the particular solutions reflecting the magnitude and distribution of the applied normal pressure. On the other hand, the singular part of the solution, that is, the terms producing infinite elastic stresses at the crack tip, will depend only upon the local stresses existing along the locus of the crack before it is cut. These of course are precisely the stresses which must be removed or cancelled by the particular solutions described above in order to obtain the stress free edges as required physically. Hence the distribution of $q(x, y)$ does not — to the first order — affect the local character of the stresses at the crack point.

V A PARTICULAR SOLUTION

As an illustration of how the local solution may be combined in a particular case, consider a shallow cylindrical shell containing an axial crack of length 2c (see fig. 2). The shell is subjected to a uniform internal pressure q_o with an axial extension $N_x = \frac{q_o R}{2}$, $M_y = 0$, far away from the crack. For this problem, the solution of the coupled-extension-bending equations for the uncracked shell is:

$$W_P(x,y) = \frac{q_o R^2}{Eh} \tag{5.1}$$

$$F(x,y) = \frac{q_o R c^2}{2} \; y^2 \tag{5.2}$$

Hence along $\theta = 0, \pi$, the bending and extensional shear vanish by symmetry, and the circumferential bending and stretching stresses are

$$M^{(P)}_{\theta\theta} = 0 \tag{5.3}$$

$$N^{(P)}_{\theta\theta} = \frac{1}{c^2} \frac{\partial^2 F}{\partial x^2} = q_o R \tag{5.4}$$

Because the homogeneous solution must negate these values from the particular solution, therefore we choose

$$\frac{n_o}{c^2} = q_o R \tag{5.5}$$

$$\frac{D m_o}{c^2} = 0 \tag{5.6}$$

Returning now to the stresses along the crack prolongation, for example the normal stresses $\sigma_{y_{total}}$ and $\sigma_{x_{total}}$, one finds using (4.53) and (4.57) that for $\overline{\sigma}_b = 0$

$$\sigma_{y_{total}}(\epsilon, 0)\Big|_{\substack{\nu=\frac{1}{3} \\ \overline{\sigma}_b=0}} \approx \frac{\overline{\sigma}_e}{\sqrt{2\epsilon}} \left\{1 + (0.76 + 0.30 \ln \frac{\lambda}{8}) \lambda^2\right\} \tag{5.7}$$

and

$$\sigma_{x_{total}}(\epsilon, 0)\Big|_{\substack{\nu=\frac{1}{3} \\ \overline{\sigma}_b=0}} \approx \frac{\overline{\sigma}_e}{\sqrt{2\epsilon}} \left\{1 + (0.24 - 0.06 \ln \frac{\lambda}{8}) \lambda^2\right\} \tag{5.8}$$

where

$$\overline{\sigma}_b = \frac{6D}{h^2} \frac{m_o}{c^2} = \text{''applied bending''}$$

$$\overline{\sigma}_e = \frac{n_o}{c^2 h} = \text{''applied stretching''}$$

For $\lambda = 0.98$ equations (5.7) and (5.8) reduce to:

$$\sigma_{y_{total}}(\epsilon, 0)\Big|_{\substack{\nu=\frac{1}{3} \\ \overline{\sigma}_b=0 \\ \lambda=0.98}} \approx \frac{\overline{\sigma}_e}{\sqrt{2\epsilon}} \{1.12\} \tag{5.9}$$

$$\sigma_{x_{total}}(\epsilon, 0)\Big|_{\substack{\nu=\frac{1}{3} \\ \overline{\sigma}_b=0 \\ \lambda=0.98}} \approx \frac{\overline{\sigma}_e}{\sqrt{2\epsilon}} \{1.37\} \tag{5.10}$$

And for our particular example, we can associate

$$\bar{\sigma}_b = 0 \tag{5.11}$$

$$\bar{\sigma}_e = q_o \, (R/h) \tag{5.12}$$

Hence

$$\sigma_{y_{total}}(\epsilon, 0) \approx \frac{0.79}{\sqrt{\epsilon}} \, q_o \, (R/h) \tag{5.13}$$

and

$$\sigma_{x_{total}}(\epsilon, 0) \approx \frac{0.97}{\sqrt{\epsilon}} \, q_o \, (R/h) \tag{5.14}$$

where, based on the Kirchhoff theory, the stresses σ_x and σ_y have the same sign but differ in magnitude. This difference is because in a cylindrical shell the curvature varies between zero and a constant as one considers different angular positions. On the other hand, for a spherical shell [2] and for a flat plate [6], because the curvature remains constant in all directions, we obtain a "hydrostatic tension" stress field.

VI GRIFFITH'S THEORY OF FRACTURE FOR CYLINDRICAL PANELS

As is well known in fracture mechanics, the prediction of failure in the presence of sharp discontinuities is a very complicated problem. Some work has been done on flat sheets, based on the brittle fracture theory of A. A. Griffith [9]. His hypothesis is that the total energy of a cracked system subjected to loading remains constant as the crack extends an infinitesimal distance. It should of course be recognized that this is a necessary condition for failure but not sufficient.

Griffith applied his criterion to an infinite, isotropic plate containing a flat, sharp-ended crack of length 2c and under the action of external loading. A similar criterion for bending is not available for (i) there exists no exact solution and (ii) the upper fibers are under extension while the lower fibers are under compression. In view of the above the author in a previous paper [2] has developed a similar, but approximate, criterion for initially curved sheets based upon only the singular terms of the stresses and furthermore the stresses were integrated through the thickness of the shell. Without going into the details* we list below the derived local fracture criterion for an axial cracked cylindrical panel

$$\frac{33+6\nu-7\nu^2}{6(3+\nu)^2} \; \frac{1+5\nu}{9-7\nu} \left[1 - \frac{5\nu_o^2-12\nu_o+8}{(4-\nu_o)\nu_o} \; \frac{\pi\lambda^2}{32} \right] \bar{\sigma}_b^2 +$$

----

* See reference [2].

$$+\frac{1+5\nu}{2(1+\nu)}\left[1+\frac{5\pi\lambda^2}{32}\right]\bar{\sigma}_e^2+\frac{30-3\nu-7\nu^2}{(3+\nu)(9-7\nu)}\quad\frac{1+5\nu}{16\sqrt{3}\sqrt{1-\nu^2)}\;(3+\nu)}$$

$$\cdot\left[\frac{42-37\nu_o}{3}+(12-10\nu_o)(\gamma+\ln\frac{\lambda}{8})\right]\lambda^2\,\bar{\sigma}_e\,\bar{\sigma}_b \qquad (6.1)\ \text{Cont'd.}$$

$$=\frac{16G\gamma^*}{\pi c}\equiv(\sigma^*)^2$$

It should be pointed out that equation (6.1) is normalized such that for $\lambda \rightarrow 0$ and $\bar{\sigma}_b = 0$ we recover the equivalent plate fracture criterion* for $\sigma_y = \bar{\sigma}_e$ and $\sigma_x = \frac{\bar{\sigma}_e}{2}$ namely

$$\frac{1+5\nu}{2(1+\nu)}\,\bar{\sigma}_e^2=\frac{16G\gamma^*}{\pi c} \qquad (6.2)$$

In the special case where $\nu = \frac{1}{3}$ equation (6.1) reduces to

$$0.21(1-0.10\lambda^2)\left(\frac{\bar{\sigma}_b}{\sigma^*}\right)^2+(1+0.49\lambda^2)\left(\frac{\bar{\sigma}_e}{\sigma^*}\right)^2+(0.18+0.11\,\ln\frac{\lambda}{8})\,\lambda^2\left(\frac{\bar{\sigma}_b}{\sigma^*}\right)\left(\frac{\bar{\sigma}_e}{\sigma^*}\right)$$

$$+\,O(\lambda^4\,\ln\lambda)=1 \qquad (6.3)$$

This is of the same character as the one obtained for a cracked spherical shell [2] namely

$$0.21\,(1+0.12\lambda^2)\left(\frac{\bar{\sigma}_b}{\sigma^*}\right)^2+(1+0.59\lambda^2)\left(\frac{\bar{\sigma}_e}{\sigma^*}\right)^2$$

$$-(0.24+0.07\,\ln\frac{\lambda}{4})\,\lambda^2\left(\frac{\bar{\sigma}_e}{\sigma^*}\right)\left(\frac{\bar{\sigma}_b}{\sigma^*}\right)+O(\lambda^4\ln\lambda)=1 \qquad (6.4)$$

----

* See reference [10]

Notice that in (6.3) the coefficient of $\overline{\sigma}_b$ is less than one while in (6.4) greater than one. Plots of the two equations are given in figures (3) and (4) respectively. Apparently this variation is due to the different types of curvature. In the spherical shell for example, the curvature remains constant in all directions, while in the cylindrical shell it varies between zero and a constant as one considers different angular positions. For this reason, the author conjectures that the correct trend of the curves is that of figure (4), and that if stresses with higher order terms were included for the cylinder the curves will correct themselves to give the same trend. Be this as it may, for our particular example $\overline{\sigma}_b = 0$, hence equation (6.3) reduces to:

$$(1 + 0.49\lambda^2)\left(\frac{\overline{\sigma}_e}{\sigma^*}\right)^2 + O(\lambda^4 \ell n\lambda) = 1 \qquad (6.5)$$

or

$$\left(\frac{\overline{\sigma}_e}{\sigma^*}\right)^2 = \left(\frac{q_f R/h}{\sigma^*}\right)^2 \doteq 1 - 0.49\,\lambda^2 \qquad (6.6)$$

which gives an expression for the maximum internal pressure that the shell can withstand before fracture. This is plotted in figure (5). From equation (6.5) we can also derive an expression for the ratio of the critical crack length of a cylindrical shell to the critical crack length of a plate, namely

$$\left(\frac{\ell_s}{\ell_p}\right)_{cr} \approx (1 - 0.49\,\lambda^2)\,\frac{(\overline{\sigma}_e)^2_{plate}}{(\overline{\sigma}_e)^2_{cyl.\ shell}} \qquad (6.7)$$

This ratio is less than 1 for all $\lambda < 1.4$. We conjecture that for $\lambda > 1.4$ the same character will be preserved, but higher orders of λ would be required to verify this point. The same type of results were obtained experimentally by Sechler and Williams [3] for pressurized monocoque cylinders. Finally, the spherical shell gives also similar results for $\lambda < 7$, which indicates that the symmetry of the curvature speeds up the convergence of the solution.

VII CONCLUSIONS

As in the case of a spherical shell, the

(i) stresses are proportional to $1/\sqrt{\epsilon}$

(ii) stresses have the same angular distribution as that of flat plate

(iii) stress intensity factors are functions of R and in the limit as $R \rightarrow \infty$ we recover the flat plate expressions

(iv) stresses include interaction terms for bending and stretching

A typical term is

$$\frac{\sigma_{shell}}{\sigma_{plate}} \sim 1 + \left(a + b \, \ell n \frac{c}{\sqrt{Rh}}\right) \frac{c^2}{Rh} + O\left(\frac{1}{R^2}\right) \tag{7.1}$$

where the expression in parentheses is a positive quantity. From this and from the spherical shell we conjecture that the general effect of initial curvature, in reference to that of a flat sheet, is to increase the stress in the neighborhood of the crack point. Furthermore, it is of some practical value to be able to correlate flat sheet behavior with that of initially curved specimens. In experimental work on brittle fracture for example, considerable time might be saved since by (7.1) we would expect to predict the response behavior of curved sheets from flat sheet tests.

It is well known that large thin-walled pressure vessels resemble balloons and like balloons are subject to puncture and explosive loss. For any given material, under a specified stress field due to internal pressure, there will be a crack length in the material which will be self propagating. Crack lengths less than the critical value will cause leakage but not destruction. However, if the critical length is ever reached, either by penetration or by the growth of a small fatigue crack, the explosion and complete loss of the structure occurs. This critical crack length, using Griffith's criterion, was shown to depend upon the stress field, the radius and thickness of the vessel, as well as the material itself (see eq. 6.1). We were also able to obtain a relation for the ratio critical crack length of a cylindrical shell over critical crack length of a flat sheet (see eq. 6.7). In general for a spherical and cylindrical curvature this ratio is less than unity, which again indicates clearly that a cracked initially curved shell is weaker than a cracked flat sheet subjected to the same loading. Similar results were obtained experimentally by Sechler and Williams [3] for pressurized monocoque cylinders.

In conclusion it must be emphasized that the classical bending theory has been used in deducing the foregoing results. Hence it is inherent that only the Kirchhoff equivalent shear free condition is satisfied along the crack, and not the vanishing of both individual shearing stresses. While outside the local region the stress distribution should be accurate, one might expect the same type of discrepancy to exist near the crack point as that found by Knowles and Wang in comparing Kirchhoff and Reissner bending results for the flat plate case. In this case the order of the stress singularity remained unchanged but the circumferential distribution around the crack changed so as to be precisely the same as that due to solely extensional loading. Pending further investigation of this effect for initially curved plates, one is tempted to conjecture that the bending amplitude and angular distribution would be the same as that of stretching.

APPENDIX

1. Table of F. C. Transforms:

We list below the following integrals which are useful for the evaluation of the kernels L_1, L_2, L_3, L_4.

$$\int_0^{\infty} \Phi_1 \cos \zeta s ds = 2 \cos \frac{\alpha\lambda\zeta}{2} K_o \left(\frac{\beta\lambda\rho}{2}\right) \tag{1}$$

$$\int_0^{\infty} \Phi_2 \sin \zeta s ds = 2 \sin \frac{\alpha\lambda\zeta}{2} K_o \left(\frac{\beta\lambda\rho}{2}\right) \tag{2}$$

$$-\frac{1}{2}\int_0^{\infty} s\, \Phi_1 \sin \zeta s ds = -\frac{\alpha\lambda}{2} \sin \frac{\alpha\lambda\zeta}{2} K_o \left(\frac{\beta\lambda\rho}{2}\right) - \frac{\beta\lambda\zeta}{2\rho} \cos \frac{\alpha\lambda\zeta}{2} K_1\left(\frac{\lambda\beta\rho}{2}\right) \tag{3}$$

$$+\frac{1}{2}\int_0^{\infty} s^2\, \Phi_2 \sin \zeta s ds = \left[\frac{\alpha^2\lambda^2}{4} + \frac{\alpha^2\lambda^2\zeta^2}{4\rho^2}\right] \sin \frac{\alpha\lambda\zeta}{2} K_o\left(\frac{\beta\lambda\rho}{2}\right)$$

$$+\left[\frac{\alpha\beta\lambda^2}{2} \cos \frac{\alpha\lambda\zeta}{2} - \frac{\beta\lambda\zeta^2}{\rho^3} \sin \frac{\alpha\lambda\zeta}{2} + \frac{\beta\lambda}{2\rho} \sin \frac{\alpha\lambda\zeta}{2}\right] K_1\left(\frac{\beta\lambda\rho}{2}\right) \tag{4}$$

where we have defined

$$\Phi_1 \equiv \frac{e^{-\sqrt{s(s-\alpha\lambda)}\,|y|}}{\sqrt{s(s-\alpha\lambda)}} + \frac{e^{-\sqrt{s(s+\alpha\lambda)}\,|y|}}{\sqrt{s(s+\alpha\lambda)}}$$

$$\Phi_2 \equiv \frac{e^{-\sqrt{s(s-\alpha\lambda)}\,|y|}}{\sqrt{s(s-\alpha\lambda)}} - \frac{e^{-\sqrt{s(s+\alpha\lambda)}\,|y|}}{\sqrt{s(s+\alpha\lambda)}}$$

To evaluate (1) and (2) consider

$$I \equiv \int_{-\infty}^{\infty} \frac{e^{-\sqrt{s(s-\alpha\lambda)}\,|y|}}{\sqrt{s(s-\alpha\lambda)}} e^{-is\zeta} ds$$

if we let $\xi = s - \frac{\alpha\lambda}{2}$ then

$$I = e^{-\frac{i\alpha\lambda\zeta}{2}} \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\xi^2 - \frac{\alpha^2\lambda^2}{4}}\,|y|}}{\sqrt{\xi^2 - \frac{\alpha^2\lambda^2}{4}}}\, e^{-i\xi\zeta}\, d\xi$$

$$= 2e^{-\frac{i\alpha\lambda\zeta}{2}} K_o\left(\frac{\beta\lambda\rho}{2}\right)$$

2. Expansions for Small λ

$$\lim_{|y|\to 0}\left\{-\frac{1}{2}\int_0^{\infty} \Phi_2^{\alpha} \sin\zeta s\, ds\right\} = \frac{\alpha\lambda\zeta}{2}\left(\gamma + \ln\frac{\lambda\beta|\zeta|}{4}\right) + O(\lambda^3 \ln\lambda) \tag{5}$$

$$- \frac{5\alpha^3\lambda^3 z^3}{8\cdot 12}\left(\gamma + \ln\frac{\beta\lambda z}{4}\right) + \frac{\alpha^3\lambda^3 z^3}{4\cdot 8}$$

$$\lim_{|y|\to 0}\left\{-\frac{1}{2}\int_0^{\infty} s\,\Phi_1^{-\alpha} \sin\zeta s\, ds\right\} = -\frac{1}{\zeta} - \frac{3\beta^2\lambda^2\zeta}{8}\left(\gamma + \ln\frac{\lambda\beta|\zeta|}{4}\right) \tag{6}$$

$$- \frac{\beta^2\lambda^2\zeta}{16} + O(\lambda^4 \ln\lambda) \Rightarrow - \frac{35}{2^8\cdot 3}\beta^4\lambda^4 z^3\left(\gamma + \ln\frac{\beta\lambda z}{4}\right) + \frac{79}{4^5\cdot 3}\beta^4\lambda^4 z^3$$

$$\lim_{|y|\to 0}\left\{\frac{1}{2}\int_0^{\infty} s^2\,\Phi_2^{\alpha} \sin\zeta s\, ds\right\} = \frac{\alpha\lambda}{2\zeta} - \frac{7}{6}\,\frac{\alpha^3\lambda^3\zeta}{16} - \frac{5}{16}\,\alpha^3\lambda^3\zeta\left(\gamma + \ln\frac{\lambda\beta|\zeta|}{4}\right) \tag{7}$$

$$+ O(\lambda^5 \ln\lambda) \Rightarrow - \frac{683}{2^{11}\cdot 3\cdot 5}\alpha^5\lambda^5 z^3 + \frac{63}{2^9\cdot 3}\alpha^5\lambda^5 z^3\left(\gamma + \ln\frac{\beta\lambda z}{4}\right)$$

3. Table of Proper and Improper Integrals

Another set of integrals which are used in Chapter IV

$$\text{C.P.V.}\int_{-1}^{1} \frac{\sqrt{1-\xi^2}}{x-\xi}\, d\xi = \pi x \tag{8}$$

$$\text{C.P.V.}\int_{-1}^{1} \frac{(1-\xi^2)^{3/2}}{x-\xi}\, d\xi = \pi\left(\frac{3}{2}x - x^3\right) \tag{9}$$

$$\int_{-1}^{1} \sqrt{1-\xi^2}\,(x-\xi)\,\ln\frac{\lambda\alpha|x-\xi|}{4}\, d\xi = \frac{\pi}{4}\left(1 + \ln\frac{\lambda^2\alpha^2}{64}\right)x + \frac{\pi}{6}x^3 \tag{10}$$

4. Some Integrals of the Bessel Functions $J_1(s)$

$$\int_0^\infty J_1(s)\, e^{-s|y|} \cos xs\, ds = \frac{1}{\sqrt{2\epsilon}} \cos\frac{\theta}{2} + O(\epsilon^o) \tag{11}$$

$$\int_0^\infty J_1(s)\, e^{-s|y|} \sin xs\, ds = -\frac{1}{\sqrt{2\epsilon}} \sin\frac{\theta}{2} + O(\epsilon^o) \tag{12}$$

$$|y|\int_0^\infty s\, J_1(s)\, e^{-s|y|} \cos xs\, ds = \frac{1}{4\sqrt{2\epsilon}}\left[\cos\frac{\theta}{2} - \cos\frac{5\theta}{2}\right] + O(\epsilon^o) \tag{13}$$

$$|y|\int_0^\infty sJ_1(s)\, e^{-s|y|} \sin xs\, ds = -\frac{1}{4\sqrt{2\epsilon}}\left[\sin\frac{\theta}{2} - \sin\frac{5\theta}{2}\right] + O(\epsilon^o) \tag{14}$$

REFERENCES

[1] Ang, D. D. and Williams, M. L., "Combined Stresses in an Orthotropic Plate Having a Finite Crack," Journal of Applied Mechanics, vol. 28, September 1961, pp. 372-378.

[2] Folias, Efthymios S., "The Stresses in a Spherical Shell Containing a Crack," ARL 64-23, Aerospace Research Laboratories, Office of Aerospace Research, U. S. Air Force, January 1964.

[3] Sechler, E. E. and Williams, M. L., "The Critical Crack Length in Pressurized, Monocoque Cylinders," Final Report on Contract NAw-6525, California Institute of Technology, September 1959.

See also: Williams, M. L., "Some Observations Regarding the Stress Field Near the Point of a Crack," Proceedings of the Crack Propagation Symposium, vol. 1, College of Aeronautics, Cranfield (England), September 1961, pp. 130-165.

[4] Marguerre, K., "Zur Theorie der gekrümmten Platte grosser Formänderung," Proc. Fifth Congress Appl. Mech., 1938, pp. 93-101.

[5] Erdelyi, etc., *Tables of Integral Transforms*, vol. 1, Bateman Manuscript Project, McGraw-Hill, N. Y., 1954.

[6] Knowles, J. K. and Wang, N. M., "On the Bending of an Elastic Plate Containing a Crack," Journal of Mathematics and Physics, vol. 39, 1960, pp. 223-236.

[7] Williams, M. L., "The Bending Stress Distribution at the

Base of a Stationary Crack," Journal of Applied Mechanics, vol. 28, March 1961, pp. 78-82.

[8] Williams, M. L., "On the Stress Distribution at the Base of a Stationary Crack," Journal of Applied Mechanics, vol. 24, March 1957, pp. 109-114.

[9] Griffith, A. A., "The Theory of Rupture," Proceedings of the First International Congress of Applied Mechanics, Delft, 1924, pp. 55-63.

[10] Swedlow, J. L., "On Griffith's Theory of Fracture," GALCIT SM 63-8, California Institute of Technology, March 1963.

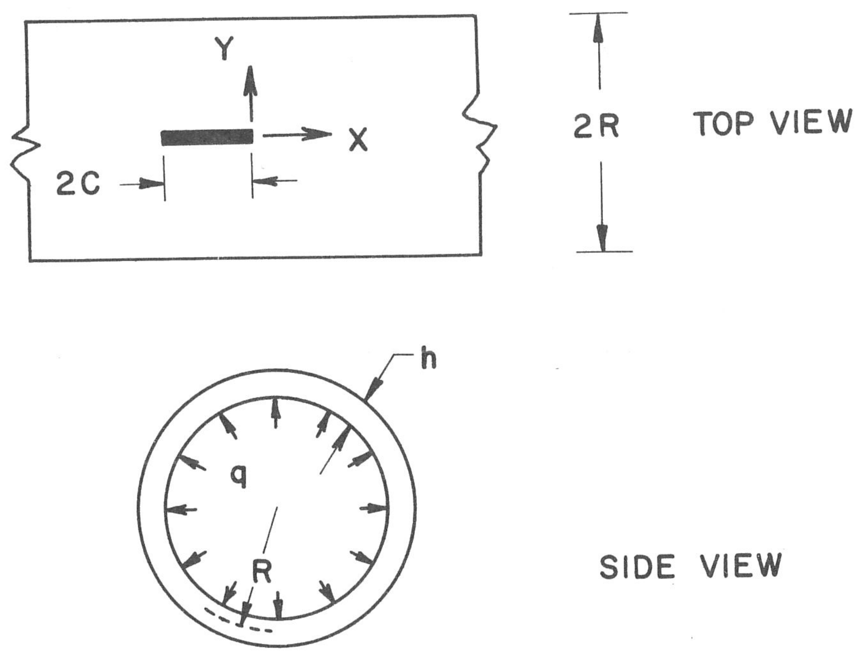

FIG. 1 - GEOMETRY AND COORDINATES

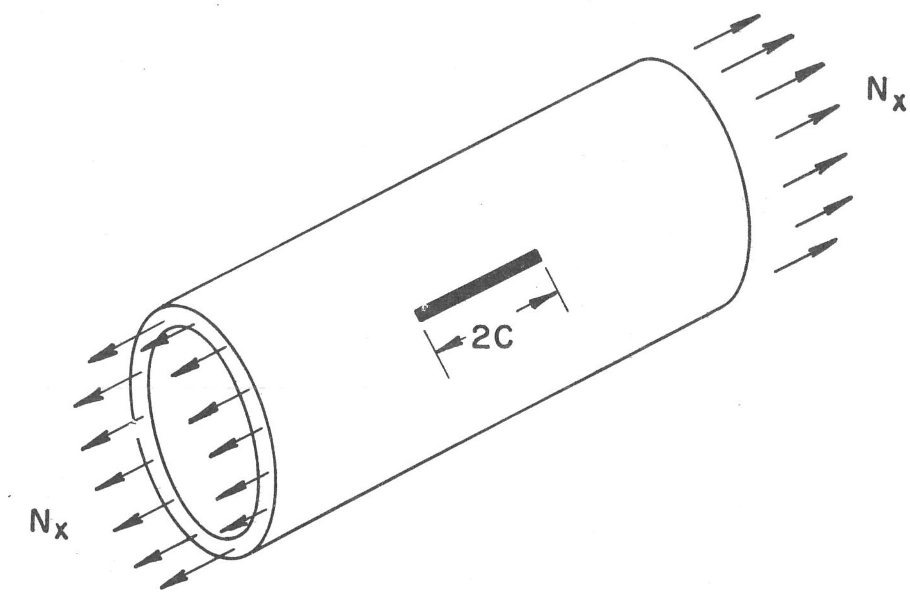

FIG. 2 - CRACKED SHELL UNDER UNIFORM AXIAL EXTENSION N_x AND INTERNAL PRESSURE q_o

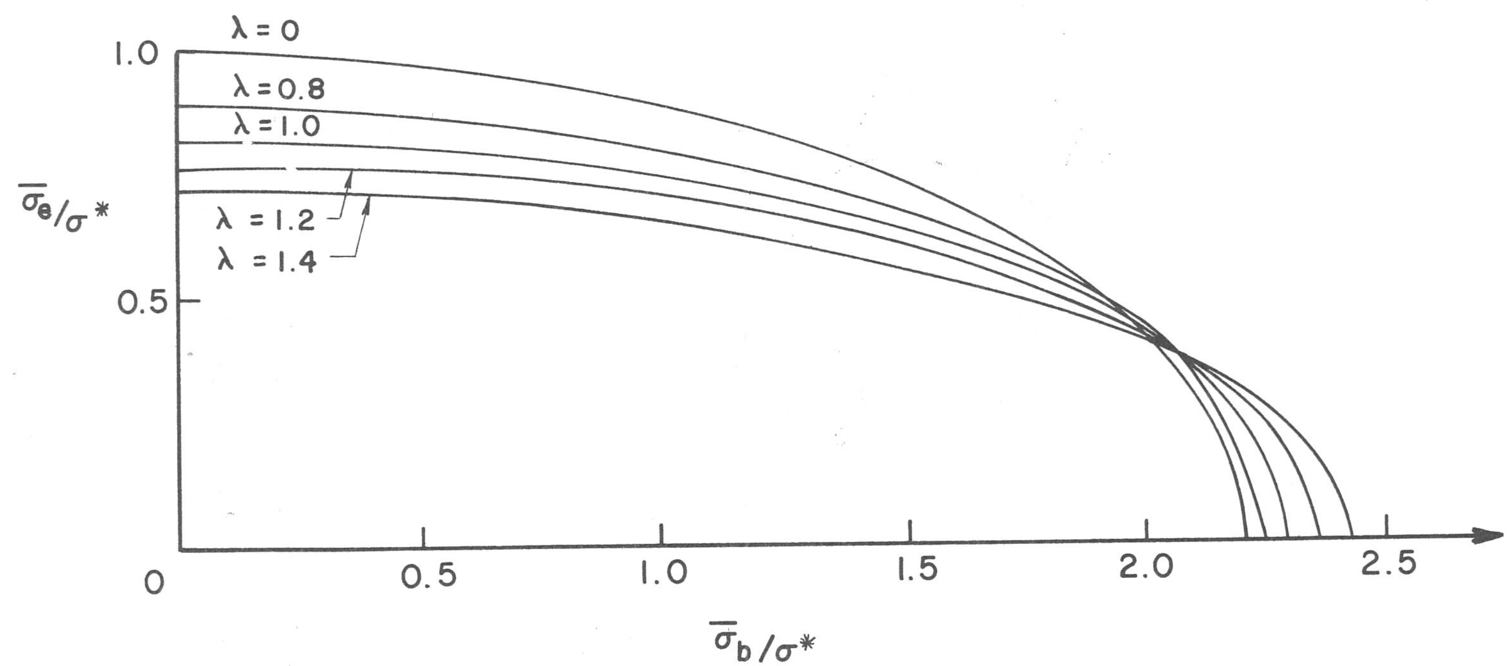

FIG. 3 - EXTENSION - BENDING INTERACTION CURVE FOR A CYLINDRICAL SHELL CONTAINING A CRACK, FOR $\nu = 1/3$; $\lambda = \sqrt[4]{12(1-\nu^2)}\ \frac{C}{\sqrt{Rh}}$

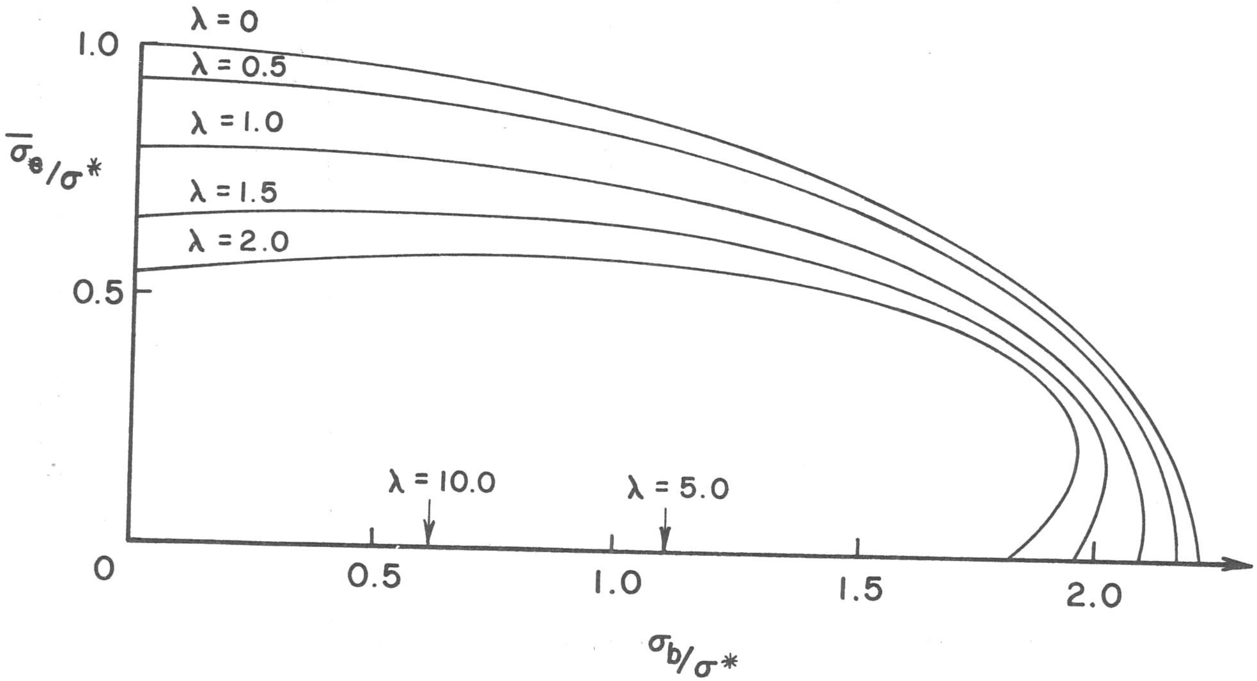

FIG. 4 - EXTENSION-BENDING INTERACTION CURVE FOR A SHALLOW SPHERICAL SHELL CONTAINING A CRACK, FOR $\nu = 1/3$;

$$\lambda = \sqrt[4]{12(1-\nu^2)} \quad \frac{C}{\sqrt{Rh}}$$

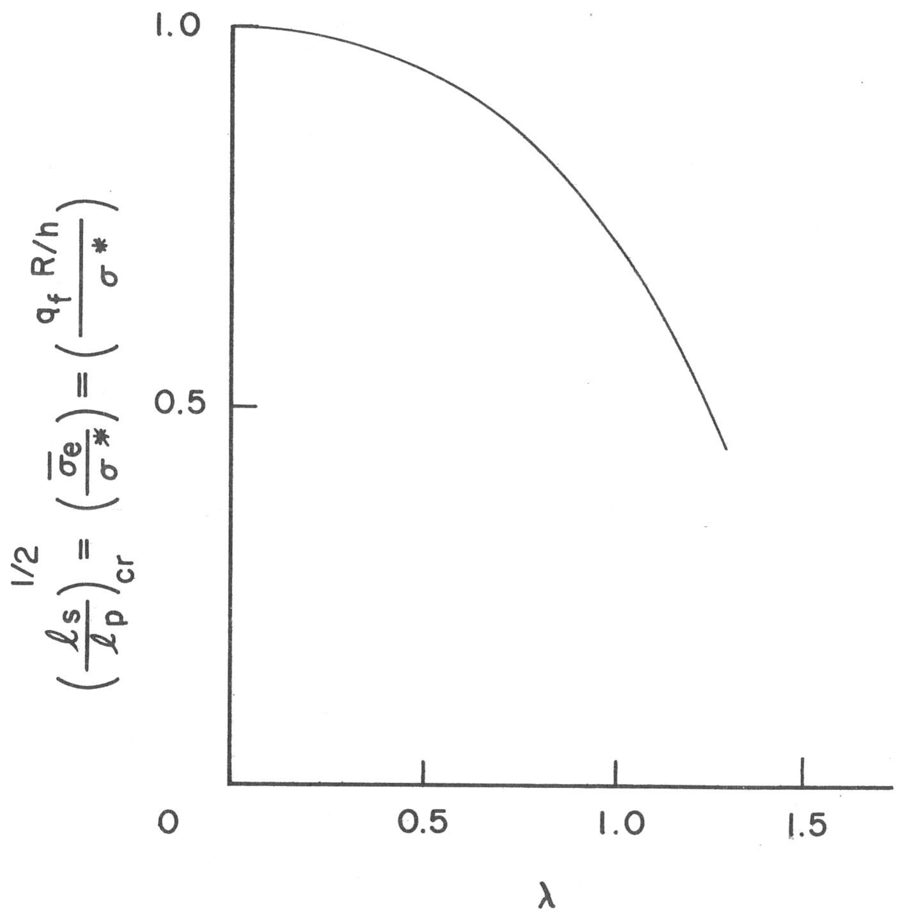

FIG. 5 - SQUARE ROOT RATIO OF CRITICAL CRACK LENGTHS IN A CYLINDRICAL SHELL AND A FLAT PLATE, FOR $\nu = 1/3$; $\lambda = \sqrt[4]{12(1-\nu^2)}\,\frac{C}{\sqrt{Rh}}$

NOTICES

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