A Gentle Introduction to the Physics and Mathematics of Incompressible Flow

Course Notes, Fall 2000

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1 Primary concepts in modeling fluids

In these notes, vectors will be denoted by bold face. They may be functions. Generally (in 3D, say) $\boldsymbol{x} = (x, y, z)$ (or (x_1, x_2, x_3)) and $\boldsymbol{a}$ are vectors denoting positions, and $\boldsymbol{u} = (u, v, w)$ (or (u_1, u_2, u_3)) is a velocity. The latter is usually a function of $\boldsymbol{x}$ and perhaps t. We will almost always work in either 2 or 3 space dimensions.

1.1 Fluid "particles"

First, a few words about classical particle mechanics. That well-studied field deals with the motion of systems of "particles". These are idealized objects which have mass but no size; at any instant of time they are located at discrete points. They move in accordance with forces which usually are given functions of all their positions; in this sense they "interact" with each other.

For a system of N particles, the masses and trajectories can be denoted by m_k, $\boldsymbol{x}_k(t)$, $k = 1, 2, \ldots, N$. The force exerted on the k-th particle is some function $f_k(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N)$. Then the motion is governed by the law $m_k \ddot{\boldsymbol{x}}(t) = f_k(\boldsymbol{x}_1(t), \ldots)$.

Continuum fluid mechanics is a sort of generalization of classical particle mechanics: instead of having discrete particles, we have a continuum of them. So rather than being indexed by an integer k, they may be indexed by a continuous variable $\boldsymbol{a}$. The trajectory of the "particle" corresponding

to some value of $\boldsymbol{a}$ is a function $\boldsymbol{x}(\boldsymbol{a}, t)$. The laws by which the particles interact in this case get a little delicate and in fact are sometimes only implicit.

It is convenient to define the parameter $\boldsymbol{a}$ to simply be the initial position of the particle under consideration, i.e.

$$\boldsymbol{a} = \boldsymbol{x}(\boldsymbol{a}, 0). \tag{1.1}$$

We shall always make this identification. Since $\boldsymbol{a}$ takes on a continuum of values, the particles corresponding to the values $\boldsymbol{a}$ and $\boldsymbol{a} + \boldsymbol{\delta a}$ are distinct if $\boldsymbol{\delta a} \neq 0$, no matter how small $\boldsymbol{\delta a}$ is. Therefore we say that the fluid is "infinitely divisible".

Conceptually, what is a "fluid particle"? On one level, you can think of it as being a bit of matter that moves around. How big a bit? Since for the purpose of simple mathematical modeling the fluid is supposed to be infinitely divisible, as indicated above, a particle should be infinitely small. But then it will have no mass (which is different from classical particle mechanics) and would not be a particle. We get into some conceptual difficulties at this point, to which we offer a solution in Section 1.6 below.

1.2 Properties of the fluid particles

However you wish to think of them, fluid particles generally have quantitative properties such as velocity

$$\boldsymbol{u} = \frac{\partial}{\partial t}\boldsymbol{x}(\boldsymbol{a}, t), \tag{1.2}$$

density, vorticity, energy, temperature, and so on.

1.3 Blobs

We follow Acheson (more or less) and use the term "blob" to denote a bounded open set of points which moves in time as though each point were a particle. (Lighthill sometimes uses "lump".) So if $V(0)$ is such a set at $t = 0$, then it evolves as a blob into $V(t)$ as time proceeds, where

$$V(t) = \{\boldsymbol{x}(\boldsymbol{a}, t), \boldsymbol{a} \in V(0)\}. \tag{1.3}$$

You will naturally think of blobs as specific sets of material points which retain their identity as they move about.

1.4 Stresses

What makes the particles move? Forces.

There are two kinds of forces to talk about: the external ones, exerted by something outside the fluid region such as gravity, and internal ones, exerted by some parts of the fluid directly on other parts. For now, we consider only the latter kind.

What is important to understand is how the material outside a given blob applies (internal) forces to that blob. It happens only at the blob's boundary (unless some of the internal forces are nonlocal, which case we shall exclude).

The boundary of a blob is a surface. Therefore the forces we consider have the property that they act on surfaces. The total resultant (internal) force acting on a blob can be found by adding together (as vectors) all the forces acting on the outer surface of the blob. In general, adding together means integrating a density of forces. To reiterate, internal forces are referred to surfaces. But to integrate them, we have to have the concept of density of forces distributed on a surface.

Another important issue is this: Given a point in space, there exist many surfaces through that point. What is the relationship among the internal forces acting, at that point, on all these surfaces? It is generally assumed that these forces depend only on the orientation of the surface, which is specified by its unit normal vector. One may ask, why don't other geometric properties of the surface, such as its curvature, affect the stress at a given point? At this stage, we simply assume it is so. Thus as far as exerting stress on the surface, the only property of the surface at a given point that the material outside the blob sees is the surface's orientation, i.e. its unit outward normal vector.

This brings us to a basic definition: **Stresses** are force densities (vectors) which act on surfaces at points. Following the above discussion, we require that the stress in a fluid be a function of only two things: the location in the fluid and a unit vector $\boldsymbol{n}$, which represents the outward normal to the surface at that location. Thus if we denote stress by $\boldsymbol{T}$, we have

$$\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{x}, \boldsymbol{n}). \tag{1.4}$$

Stress is another primary concept in modeling fluid dynamics, apart from particles.

If the stress is everywhere normal to the surface, i.e. $\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{n})$ is always parallel to $\boldsymbol{n}$, its magnitude is called a pressure (much more on this later).

It can be shown by an easy physical argument (see e.g. Batchelor, Section 1.3) that $\boldsymbol{T}$ must be a *linear* function of $\boldsymbol{n}$. That is, the i-th component of the force density is given by $\sum_j T_{ij} n_j$, where n_j are the components of $\boldsymbol{n}$ and the numbers T_{ij} (which depend on $\boldsymbol{x}$) are elements of the so-called stress tensor. Another clear physical argument based on balance of angular momentum, akin to the one I will describe below in Section 4.2, shows that the stress tensor must be symmetric:

$$T_{ij} = T_{ji} \quad \text{for all } j \text{ and } i. \tag{1.5}$$

To summarize, given a point and an orientation $\boldsymbol{n}$, we get a stress. It is a force density acting at that point on any blob enclosed by a surface which goes through the point and which has the given orientation (outward normal) at that same point. Finally, it depends linearly on $\boldsymbol{n}$.

1.5 Fields

More or less, this is a generic term for functions of position and perhaps other things. It could be scalar (such as density of the particle at that position) or vector (such as velocity and acceleration) or tensor (such as stress). The various fields are distinguished by their "tensorial properties", i.e. how they transform when the basic coordinate system is changed.

1.6 Back to the concept of a fluid

At a rather basic level, one could do away with the idea of "fluid particle" and just incorporate the other concepts above into a formal definition of a **fluid flow**. This might be a good idea, in view of the conceptual difficulties mentioned in Section 1.1 above. The fluid flow would then be a triple of fields: velocity, density, and stress. One then could define, as secondary concepts, "particle paths" and "Lagrangian coordinates" in the manner to be described below in Section 2. *However, we will still think and speak of particles as small bits of matter*, because it helps in visualizing the flow.

To differentiate among classes of fluid flows, one generally imposes restrictions on these fields. For example, in viscous fluids, the tangential components of the stress field may be required to depend on the velocity field in a certain way. Or the density may be required to be constant, or to be related to the velocity by a PDE expressing the law of conservation of mass. Or there may

be a relation between the acceleration field and the other fields, so as to satisfy conservation of momentum. Of course we will always assume these conservation laws hold. But we have to express them in mathematical terms.

What distinguishes "fluids" from "solids" (for which velocity, density, and stress are also among the primary quantities) is that the there is no "rest state" (velocity zero) with stress and density independent of time, such that the stress has a nonzero tangential component somewhere. On the other hand, elastic bodies *can* support nonzero shear (tangential) stresses in the rest state (even when external forces are present).

2 Kinematics

It would be a good idea now to review the divergence theorem—try Acheson, page 349.

2.1 Particle paths

If we know the path of a particle which we might specify as "particle $\boldsymbol{a}$", namely $\boldsymbol{x}(\boldsymbol{a}, t)$ (Section 1.1), then its velocity is $\frac{\partial}{\partial t}\boldsymbol{x}(\boldsymbol{a}, t) = \dot{\boldsymbol{x}}(\boldsymbol{a}, t)$. If we know *all* the particle paths, then in principle the velocity field, as a function of $\boldsymbol{x}$, might be reconstructed as follows: given a point $\boldsymbol{x}$ and time t, (1) find the particle $\boldsymbol{a}$ that is there at that time, (2) find its velocity by taking the time derivative, and (3) call the result $\boldsymbol{u}(\boldsymbol{x}, t)$. We can write this out in component form to obtain (in 3D, say) $\boldsymbol{u}(\boldsymbol{x}, t) = (u(\boldsymbol{x}, t), v(\boldsymbol{x}, t), w(\boldsymbol{x}, t)) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))$. We shall always assume that $\boldsymbol{u}$ is a smooth bounded function.

Conversely, if you know the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$, you can find the particle paths by solving a system of ODE's:

$$\frac{dx}{dt} = u(x, y, z, t), \quad \frac{dy}{dt} = v(x, y, z, t), \quad \frac{dz}{dt} = w(x, y, z, t). \tag{2.1}$$

Recall that the initial values of x, y, z are known by (1.1); they are simply the components of $\boldsymbol{a}$. Given such an initial condition, the solution $\boldsymbol{x}(\boldsymbol{a}, t)$ of (2.1) exists for all $t > 0$ and is unique, by the assumptions on $\boldsymbol{u}$ and a basic theorem about ODE's.

2.2 Eulerian and Lagrangian coordinates

You will have noticed that $\boldsymbol{x}$ is sometimes used as a position indicator, and $\boldsymbol{a}$ is at other times. The former is called an *Eulerian coordinate*; the latter is a *Lagrangian coordinate*.

Recall that the function $\boldsymbol{x}(\boldsymbol{a}, t)$, for each fixed t, tells how the 3D position $\boldsymbol{x}$ of the particle at the later time t depends on the 3D initial position $\boldsymbol{a}$ of the particle. It can be considered (for each t) as a transformation from the 3D coordinate system with coordinates denoted by $\boldsymbol{a}$ to the 3D system with coordinates denoted by $\boldsymbol{x}$. (This could of course also be done in 2D.)

Thus $\boldsymbol{x}(\boldsymbol{a}, t)$ provides a coordinate transformation in 3-space. In all the flow fields we will be working with, it is invertible: $\boldsymbol{a}$ can in principle be found in terms of $\boldsymbol{x}$. This means that any function of $\boldsymbol{x}$ can be also expressed as a function of $\boldsymbol{a}$, and vice versa.

This latter statement applies to any property of the fluid particles. For example the velocity $\boldsymbol{u}$. In (1.2) I skirted the issue by not writing any independent variables on the left side. They could be the Lagrangian variables $(\boldsymbol{a}, t)$ (which seems most natural in this case) or the Eulerian variables $(\boldsymbol{x}, t)$. (In Section 2.1 I used Eulerian coordinates.) But either representation can in principle be obtained in terms of the other, once all the particle paths are known.

In the case of velocity, usually the Eulerian coordinates $\boldsymbol{x}$, as in Section 2.1, are preferred. The Lagrangian representation might then be symbolized with a tilde:

$$\tilde{\boldsymbol{u}}(\boldsymbol{a},t) = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{a},t),t). \tag{2.2}$$

2.3 Rate of change "following the fluid"; convection

Consider any property P of the fluid particles, such as density, velocity, momentum, energy or temperature. As discussed in the previous section, we may express it as a function of either Eulerian or Lagrangian coordinates. To reiterate, suppose P is originally given as a function $P(\boldsymbol{x},t)$ of the Eulerian coordinates. Then the associated function $\tilde{P}(\boldsymbol{a},t)$ is defined by $P(\boldsymbol{x},t) = P(\boldsymbol{x}(\boldsymbol{a},t),t) \equiv \tilde{P}(\boldsymbol{a},t)$.

We define the rate of change of P "following the fluid", or "material time derivative" to be its time derivative with $\boldsymbol{a}$ held constant, and use the notation

$$\frac{DP}{Dt} \equiv \frac{\partial}{\partial t}\tilde{P}(\boldsymbol{a},t),$$

where $\boldsymbol{x}$ and $\boldsymbol{a}$ are related by $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a},t)$. Thus by the chain rule

$$\frac{DP}{Dt} = \frac{\partial P}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial}{\partial t}P(\boldsymbol{x},t)$$

$$= \nabla P(\boldsymbol{x},t)\cdot\frac{\partial \boldsymbol{x}}{\partial t}(\boldsymbol{a},t) + \frac{\partial}{\partial t}P(\boldsymbol{x},t) = (\boldsymbol{u}\cdot\nabla)P + \frac{\partial}{\partial t}P(\boldsymbol{x},t). \tag{2.3}$$

This is the time rate of change of property P for a given particle, irrespective of how that particle moves.

Conservation laws are often expressed as

"Time rate of change of P following the fluid = a given source F of property P". (2.4)

This source F is generally postulated or derived as a function of other flow quantities. There will be examples in the course. In mathematical terms, we have

$$\frac{DP}{Dt} = (\boldsymbol{u}\cdot\nabla)P + \frac{\partial}{\partial t}P(\boldsymbol{x},t) = F. \tag{2.5}$$

Another view of the term $(\boldsymbol{u}\cdot\nabla)P$ can be obtained by rewriting this as

$$\frac{\partial}{\partial t}P(\boldsymbol{x},t) = F - (\boldsymbol{u}\cdot\nabla)P. \tag{2.6}$$

This expresses the fact that at a fixed point in space (rather than fixed $\boldsymbol{a}$), the rate of increase of P consists of the sum of two terms: the given source F plus a *rate of change due to convection*, represented by the term $-(\boldsymbol{u}\cdot\nabla)P$. This latter term expresses the fact that the fluid particles arriving at and departing from the location $\boldsymbol{x}$ carry with them some amount of the property P which should be added to F to determine the rate at which P changes.

As a simple one-dimensional example, suppose that at some instant t_0 of time we have $P(x,t_0) = mx$ for some number $m > 0$, and $u = \text{const} > 0$. Focus attention on a specific point x_1. The positive velocity carries, to the location x_1, particles that originated to the left of x_1, hence have a smaller value of P (since P is an increasing function of x). That action, which is called *convection*,

serves to decrease the amount of P at x_1, and the rate will be proportional both to u and to m. In this case the convective rate of change is $-mu$, which is indeed $-u\nabla P$.

We shall call the term $(\boldsymbol{u}\cdot\nabla)P$ in (2.3) the "convective term".

Especially important is the case when P is velocity $\boldsymbol{u}$ or momentum density $\rho\boldsymbol{u}$, where ρ is mass density. Consider the case when $\rho =$ constant. Then

$$\frac{D(\rho\boldsymbol{u})}{Dt} = \rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right). \tag{2.7}$$

The reason this is an important case is that it figures in the law of conservation of momentum; see (4.8) and (later) the Navier-Stokes equations in Section 5. The last term in parentheses in (2.7) is a vector with components

$$(uu_x + vu_y + wu_z, uv_x + vv_y + wv_z, uw_x + vw_y + ww_z). \tag{2.8}$$

But the striking feature of the convection term in (2.7) is that $\boldsymbol{u}$ plays a dual role. It is not only the quantity being convected, but it is also the velocity of that same convection.

This dual nature is the main source of difficulty in one of the most difficult areas of applied mathematics, one about which hundreds of papers are still being written each month. It is also responsible for one of the 20 most important (according to Stephen Smale) unsolved problems in all mathematics confronting us in the 21st century.

2.4 Streamlines

These are related to particle paths, and in fact turn out to be identical to them when the flow is steady. Freeze time ($t = t_0$) and solve

$$\frac{dx}{ds} = u(x, y, z, t_0), \quad \frac{dy}{ds} = v(x, y, z, t_0), \quad \frac{dz}{ds} = w(x, y, z, t_0), \tag{2.9}$$

where now the variable s represents a parameter along a curve in space, frozen at $t = t_0$. Thus, this curve is tangent to the velocity at each point (if you forgot why, just ask me). Notice that if the flow is stationary, i.e. $\boldsymbol{u}(x, y, z)$ does not depend on t, then the two systems are identical, except that the symbol t is replaced by s. This means that *in stationary flows, the particle paths are the same as the streamlines*. Example 2.4.1a below shows that this is not always the case.

2.4.1 Examples

a. Take the following velocity field in 2D: $u \equiv 1, \quad v \equiv t$. Find the streamlines and particle paths. Show that the former are straight lines and the latter are parabolas. Find $\frac{D\boldsymbol{u}}{Dt}$.

b. Consider the steady velocity field in 2D given by $u = x, \ v = -y$. Show that the streamlines and particle paths are hyperbolas. Find $\frac{D\boldsymbol{u}}{Dt}$ (this is a vector).

c. Shear flows

These are steady (stationary) flows with velocity field of the form (in 2D)

$$u(x, y) = f(y), \quad v(x, y) = 0. \tag{2.10}$$

Then the streamlines are given by solving

$$\frac{dx}{ds} = u(x,y) = f(y), \quad \frac{dy}{ds} = v = 0. \tag{2.11}$$

So by integrating we have

$$y(s) = y(0), \quad x(s) = x(0) + sf(y) = x(0) + sf(y(0)). \tag{2.12}$$

Thus, they are straight and horizontal. If $\boldsymbol{a} = (a,b)$ is the point $(x(0), y(0))$, we have

$$\boldsymbol{x}(\boldsymbol{a}, t) = (a + sf(b), b).$$

For the same reason, the particle paths are horizontal; the particles themselves follow the streamlines. Find $\frac{D\boldsymbol{u}}{Dt}$.

d. Circular shear flows

We use polar coordinates $x = r\cos\theta, \quad y = r\sin\theta$, and set

$$u(x,y) = -\phi(r)\sin\theta, \quad v(x,y) = \phi(r)\cos\theta. \tag{2.13}$$

Thus the magnitude of the velocity depends only on r, and the direction is circular. As an exercise, solve the ODE's to find that the streamlines and particle paths are indeed circles. Hint: first show that $\dot{r} = 0$.

2.5 Changes in infinitesimal volume

2.5.1 Volume changes in an evolving blob of fluid

For integrals, we will usually use the notation $\int_V f(x)$ or $\int_a^b g(t)dt$. In the first case, the domain over which we integrate is declared to be V, which may be a manifold of any dimension, so there is no need for the differential customarily appearing along with the integrand. Sometimes I'll put in the differentials anyway for a little more clarity.

Consider the motion of a blob of fluid, occupying the initial set $V(0)$. At any later time, we set (by (1.3))

$$V(t) = \{\boldsymbol{x}(t, \boldsymbol{a}) : \boldsymbol{a} \in V(0)\}, \tag{2.14}$$

and think of it as the evolved position and shape of the given initial blob $V(0)$.

Its volume is $|V(t)| = \int_{V(t)} 1$. Let $\partial V(t)$ be the moving boundary of $V(t)$. For any continuous function $f(\boldsymbol{x})$ (independent of t), we have the general rate of change formula

$$\frac{d}{dt}\int_{V(t)} f(\boldsymbol{x}) = \int_{\partial V(t)} f(\boldsymbol{x}) v_n, \tag{2.15}$$

where v_n is the normal velocity of $\partial V(t)$, counted positive if it is directed outward. Let me explain this a little more. Think of f as being the volume density of some quantity Q like mass or energy. Then the conservation of Q means that the rate of change of Q in V (the left side) equals the flow of Q into V through the boundary ∂V. The total inward flow through ∂V is the integral over ∂V of the normal component of the "flux". This normal component of the flux is the area density of the

flow. In other words, given a small patch on ∂V, the amount of Q which passes through that patch from the outside is approximately the normal flux at that location times the area of the patch. The normal component of the flux in this case is $f(\boldsymbol{x})v_n$, the density times the normal velocity of ∂V. So we get (2.15).

Applying (2.15) in the case $f(\boldsymbol{x}) \equiv 1$, we have

$$\frac{d}{dt}|V(t)| = \int_{\partial V(t)} v_n. \tag{2.16}$$

But we know that the normal velocity in this case is simply the normal component of the velocity field: $v_n = \boldsymbol{u}\cdot\boldsymbol{n}$, where $\boldsymbol{n}$ is the unit outward normal vector. The reason is that every point of the blob, including its boundary, moves with the fluid velocity $\boldsymbol{u}$. In short, the boundary "moves with the fluid".

By this and the divergence theorem, we get

$$\frac{d}{dt}|V(t)| = \int_{\partial V(t)} \boldsymbol{u}\cdot\boldsymbol{n} = \int_{V(t)} \operatorname{div} \boldsymbol{u}. \tag{2.17}$$

2.5.2 Incompressibility

A fluid flow is said to be *incompressible* if volumes of blobs of fluid do not change. So we have the general law

A fluid flow is incompressible if and only if $\operatorname{div} \boldsymbol{u} \equiv 0$.

In fact, from (2.17), we have that incompressibility holds if and only if $\int_{V(t)} \operatorname{div} \boldsymbol{u} = 0$ for all $V(t)$. Since $\boldsymbol{u}$ is a smooth function and V can be *any* blob, this implies the assertion.

2.5.3 Evolution of the Jacobian

You will probably want to review the meaning of Jacobians J, how they express changes in infinitesimal volumes under coordinate transformations, and their use in evaluating multiple integrals when there is a change of coordinates. Refer to a calculus text on multiple integrals. In our context "Jacobian" will mean the determinant of the "Jacobian matrix".

Recall $\boldsymbol{x}(\boldsymbol{a}, 0) = \boldsymbol{a}$, so that $\boldsymbol{a}$ is simply the initial position of the particle under consideration. We fix t and consider the function $\boldsymbol{x}(\boldsymbol{a}, t)$ as a coordinate transformation (Section 2.2). It has a Jacobian

$$J = \frac{\partial \boldsymbol{x}(\boldsymbol{a}, t)}{\partial \boldsymbol{a}} \tag{2.18}$$

The Jacobian expresses the infinitesimal expansion of volume elements. So take a very small blob $\delta V(t)$, for which the indicator $\boldsymbol{a}$ is essentially unique and constant. Then J, evaluated on that small blob, is essentially a function only of t. The expansion property is expressed by

$$\frac{|\delta V(t)|}{|\delta V(0)|} \simeq J(t). \tag{2.19}$$

Moreover, an integral like $\int_{\delta V(t)} f(\boldsymbol{x})$ can be approximated by $|\delta V(t)| f(\boldsymbol{x}(\boldsymbol{a}, t))$.

These considerations may be applied to (2.17) with V replaced by δV and $f = \operatorname{div} \boldsymbol{u}$ to obtain the differential equation

$$\frac{DJ}{Dt}(\boldsymbol{x}, t) = J(\boldsymbol{x}, t) \operatorname{div} \boldsymbol{u}(\boldsymbol{x}, t). \tag{2.20}$$

Convince yourself of this. This is a fundamental relation between the divergence of the velocity and the Jacobian.

3 Conservation of mass

(See also Chorin-Marsden)

3.1 The PDE

First, let us generalize (2.15) to account for functions $f(x,t)$ which also depend on t. We have

$$\frac{d}{dt}\int_{V(t)} f(\boldsymbol{x},t) = \int_{V(t)} \frac{\partial}{\partial t} f(\boldsymbol{x},t) + \int_{\partial V(t)} f(\boldsymbol{x},t) v_n. \tag{3.1}$$

Let $\rho(\boldsymbol{x},t)$ be the mass density field of the fluid. The integral law of conservation of mass says that for any evolving blob $V(t)$,

$$\int_{V(t)} \rho(\boldsymbol{x},t) \text{ is constant in time.} \tag{3.2}$$

In fact the integral here is the mass of the blob; it shouldn't change in time.

In (3.1), choose $f = \rho$. Derive the PDE

$$\rho_t + \operatorname{div}[\rho \boldsymbol{u}] = 0. \tag{3.3}$$

This is the differential form of the law of conservation of mass. For smooth flows, this differential form turns out to be equivalent to the (non-differential) form of the law stated in (3.2).

In the special case of an incompressible fluid, of course one supposes that ρ is constant along particle paths, and it follows from (3.3) that

$$\operatorname{div} \boldsymbol{u} = 0. \tag{3.4}$$

This agrees with our previous characterization of incompressible flows in Section 2.5.1.

3.2 Another form of the law

If mass is conserved, the expansion J of a fluid particle or element should be inversely proportional to the volume density ρ, so that $J\rho =$ constant along particle paths. Thus $\frac{D}{Dt}(J\rho) = 0$. Show that this indeed follows from (3.3) and (2.20).

3.3 Examples

Show, by use of (3.4), that some of the examples in section 2.4.1 are incompressible. Before you do the calculations, decide from your own intuition which you think are incompressible.

4 Ideal fluids.

4.1 Definition; viscosity

Ideal fluids, also sometimes called "perfect fluids", are defined to be incompressible fluids with constant density which always have the property that the stress on a surface is everywhere directed normal to the surface. We shall give an argument below to show that at any given point the magnitude of that stress must be independent of orientation. Its magnitude is then called the *pressure*. It is a nonnegative scalar field, which we denote by $p(\boldsymbol{x}, t)$. Usually it is not a priori known, but rather has to be determined as part of the solution of certain PDE's.

What does it mean for the stress to always be normal to the surface? For there to exist tangential stresses, you have to have (1) friction, and (2) differential velocities. This seems clear from common experience. In the continuum model, the latter translates to a gradient in the normal direction of the tangential velocity component. We will spend more time on this later. In the fluid context, "friction" and "viscosity" are allied concepts. The simplest relation that one could conceive of is a linear relation between the normal gradient and the tangential stress. Suppose that the normal to the surface is in the direction of the y-axis. Then the tangential stress T_h is horizontal. The linearity means

$$T_h = \mu u_y, \tag{4.1}$$

with μ a proportionality constant. And the same relation holds when the coordinate system is rotated. We shall give a more general formulation later in Section 5, as well as an intuitive picture, based on random motion of molecules, as to why (4.1) seems reasonable.

But it is nearly always postulated. Another justification for this law comes from kinetic theory, which we will certainly not delve into. The coefficient μ is a material property and is called the viscosity. It depends on temperature. So *ideal fluids have zero viscosity, i.e. are* **inviscid**. Thus (if $\mu \neq 0$) the tangential stress results from a velocity gradient, whereas the normal stress (pressure) does not; it does not satisfy any a priori simple relationship with the velocity.

The theory of ideal fluids provides a good description of actual fluid flows in certain situations such as the study of water waves. But in others, such as the rapid flow past an obstacle or past a boundary, it can be very misleading, even though the fluid may have small viscosity.

4.2 The normal force is independent of orientation

Consider a 2D blob in the form of a right triangle of side lengths $ra,\ rb,\ rc$ (rc is the hypotenuse), where $a,\ b,\ c$ are fixed but r can approach 0. Let the pressure on the 3 sides be p_i, $i = 1, 2, 3$. The horizontal component of the total force is equal to $p_1 ra - p_3 rc \cos\theta \equiv r p_h$, where θ is the angle between sides ra and rc of the triangle and $p_h = p_1 a - p_3 c \cos\theta$. But the mass of this 2D particle is proportional to r^2, so the ratio force/mass approaches ∞ as $r \to 0$ if $p_h \neq 0$. In this limit it would have infinite acceleration unless $p_h = 0$. This latter must therefore be true. Similarly, the vertical component must also vanish. Therefore

$$p_1 a = p_3 c \cos\theta, \quad p_2 b = p_3 c \sin\theta.$$

But $a = c\cos\theta$ and $b = c\sin\theta$, from which we gather that $p_1 = p_2 = p_3$. So for this triangular blob, to prevent infinite acceleration, the pressure must be the same on all faces. A refinement of this argument provides a general proof, even in 3D, that the pressure does not depend on orientation—only position.

4.3 Digression into classical particle mechanics

Here we return briefly to talk about Newtonian mechanics of a finite number N of particles. Let $\boldsymbol{x}_k(t)$ be their trajectories, $k = 1, \ldots, N$, and for simplicity suppose they have the same mass m. Assume there is a force field $\boldsymbol{f}_k$ acting on the various particles; it may generally be a function of the particles' positions. Then Newton's law of motion is:

$$m\ddot{\boldsymbol{x}}_k(t) = \boldsymbol{f}_k, \quad k = 1, \ldots, N. \tag{4.2}$$

These are a set of 3N (if we are in 3D) ODE's for the determination of the trajectories. If we set $\boldsymbol{U}_k(t) = \dot{\boldsymbol{x}}_k(t)$, we get a sort of law of conservation of momentum:

$$m\dot{\boldsymbol{U}}_k(t) = \boldsymbol{f}_k. \tag{4.3}$$

There is a very important special case, namely when the force field is *conservative*. This means that there is a scalar function $V(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \boldsymbol{x}_N)$ of all $3N$ components, called the potential energy, such that for each $k = 1, \ldots, N$ and $i = 1, 2, 3$,

$$\boldsymbol{f}_{ki}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \boldsymbol{x}_N) = -\frac{\partial}{\partial x_{ki}} V(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \boldsymbol{x}_N), \tag{4.4}$$

where we have denoted $\boldsymbol{x}_k = (x_{k1}, x_{k2}, x_{k3})$.

For example, suppose $N = 2$ and $V = \frac{1}{2}K|\boldsymbol{x}_1 - \boldsymbol{x}_2|^2$ for some $K > 0$. Then check that

$$f_{1i} = -K(x_{1i} - x_{2i}). \tag{4.5}$$

This is the case of elastically interacting particles.

Take each component of (4.2), multiply it by the corresponding first time derivative, and then sum over all components, using (4.4). We get

$$\sum_{k,i} m\dot{x}_{ki}\ddot{x}_{ki} = -\sum_{k,i} \dot{x}_{ki} \frac{\partial}{\partial x_{ki}} V(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \boldsymbol{x}_N) = -\frac{d}{dt} V(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \boldsymbol{x}_N), \tag{4.6}$$

by the chain rule. Thus

$$\frac{d}{dt}\left[\frac{1}{2}m\sum_k |\dot{\boldsymbol{x}}_k|^2 + V(\ldots)\right] = 0. \tag{4.7}$$

The quantity in brackets is the total energy, kinetic plus potential. This says that if the motion is under the control of a conservative force field, then the total energy is conserved. In fact, energy conservation and Newton's equations are even more intimately entwined. By Hamilton's principle, one can start with the concepts of kinetic and potential energy and from them derive the ODE's of the trajectories.

Below, we shall find an analog of this connection in the case of a fluid, which of course might be thought of as a infinite number of particles.

4.4 Conservation of momentum

Consider again a blob of ideal fluid occupying the volume $V(t)$. Assume ρ is constant. At any one instant of time, the total force acting on it is given by the pressure on all of its boundary: $-\int_{\partial V(t)} p(\boldsymbol{x}, t)\boldsymbol{n}$, where $\boldsymbol{n}$ (as in (2.17)) is the outward normal unit vector. By a vector integral identity [see Acheson (A.15)], this can be written as $-\int_{V(t)} \nabla p$.

By Newton's law of motion, if this is the only force on the blob, it must equal the blob's mass times acceleration. But the acceleration is not uniquely defined, unless we imagine the blob to be rigid or infinitely small. Think of it as being very small, with volume δV, with initial position $\boldsymbol{a}$; its acceleration is then $\frac{\partial^2}{\partial t^2}\boldsymbol{x}(\boldsymbol{a},t) = \frac{\partial}{\partial t}\left[\frac{\partial \boldsymbol{x}}{\partial t}\right] = \frac{\partial}{\partial t}\left(\boldsymbol{u}(\boldsymbol{x}(\boldsymbol{a},t),t)\right) = \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$. Its mass times acceleration is approximately $\rho|\delta V|(\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u})$. Similarly the pressure integral, i.e. the total force, will be approximated by $-|\delta V|\nabla p(\boldsymbol{x}(\boldsymbol{a},t),t)$.

Dividing by $|\delta V|$, we obtain the law of motion

$$\rho(\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) + \nabla p = 0. \tag{4.8}$$

This is a fluid analog of (4.3), in which the force is now given by $-\nabla p$. It is important to note that it is no longer an explicit function of the particle positions. On a much deeper conceptual level, however, it is.

The above is under the condition that there is no external force such as gravity applied to the fluid. If there is, assume it is conservative, and let its volume density be denoted by $-\nabla\chi$, where χ is the external force potential; then $-\nabla\chi$ occurs as an extra term on the right of (4.8). We divide by ρ to obtain

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{\nabla p}{\rho} - \nabla\chi. \tag{4.9}$$

4.5 Energetic considerations

4.5.1 The Bernoulli relation

We will derive a sort of conservation of energy relation for ideal fluids which is analogous to (4.7). But first let us look to our intuition. The first two terms in (4.8) represent a mass density times acceleration. This is like the left side of (4.2). If we assume the forces are conservative (4.4), then the right side of (4.2) seems analogous to the term ∇p in (4.8) after it is shifted to the right side with a minus sign appended.

Then p is analogous to the potential V. But we must be careful; V was a function of the positions of all the particles. In the continuous case we have an indefinitely large number of particles. It may be appropriate, at a deeper level, to think of the fluid pressure as being a function of the positions of all the fluid particles, but we can't do that here. In (4.8), the pressure is thought of as some function of the Eulerian coordinates which in principle is obtained by solving the appropriate system of partial differential equations—as opposed to obtaining it from physical first principles.

So the analogy with finite particle mechanics breaks down in part here. But it can be salvaged in the case of steady flow.

So suppose that $\boldsymbol{u}(\boldsymbol{x})$ and $p(\boldsymbol{x})$ are a t-independent solution of (4.8). We can think of any particle as moving through space which is filled with the field p which acts like a potential energy, but which is a fixed function of $\boldsymbol{x}$. Therefore the total energy of that particle (kinetic plus potential) is conserved. But since we have a continuum of particles, the energy density is conserved. This total energy density is $\frac{1}{2}\rho|\boldsymbol{u}|^2 + p$. As the particle moves along its own streamline, this total energy does not change; it is constant on any streamline. **This statement is the Bernoulli relation:**

$$\frac{1}{2}\rho|\boldsymbol{u}|^2 + p \ \text{ is constant along streamlines of a steady flow.} \tag{4.10}$$

If there are body forces with potential χ, then we have

$$\frac{1}{2}\rho|\boldsymbol{u}|^2 + p + \rho\chi \ \text{ is constant along streamlines of a steady flow.} \tag{4.11}$$

This argument does not work for non-ideal fluids for the following reason: If there are tangential stresses as well as the pressure, then the forces on a fluid particle will simply not form a conservative force field. In fact those stresses, which come from a friction mechanism, result in the dissipation of kinetic energy into heat, and the conservation of energy in the above form does not hold.

4.5.2 A stricter derivation of the Bernoulli relation

For this, let me refer you to Acheson, page 9 and the top of page 10. Here we are considering the case $\chi = 0$, although the adding of a conservative external force field is trivial. The derivation relies on some vector identities, based on those in App. A1. Don't let the notation "$\wedge$" scare you; it is the same as "cross product", which you might be more familiar with.

You should derive for yourself 1 or 2 of these vector identities to see how it goes; then you can believe and refer to the others as well.

4.6 The complete system of Euler's equations

Up until now, we have simply talked about general properties of fluid flow. We haven't addressed the issue of how to actually find the fluid flow if we know some clues. For example, if you know the initial velocity distribution at $t = 0$ for a fluid flow, you would expect that the flow at later times would be determined. The initial data would be the clue. We shall argue now that determining the later flow will typically involve solving PDE's.

The appropriate system of PDE's to use are (4.9) and (3.4), which we rewrite here:

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{\nabla p}{\rho} - \nabla\chi, \tag{4.12}$$

$$\operatorname{div}\boldsymbol{u} = 0. \tag{4.13}$$

These are a system of PDE's which the velocity and pressure of an ideal fluid must satisfy (χ is a given function of $\boldsymbol{x}$). They constitute four equations for the three components of $\boldsymbol{u}$, together with p. So there are the same number of equations as there are unknowns. That's a good sign. In fact there is a (rather difficult) rigorous mathematical theory that proves, under natural assumptions, the existence and uniqueness of a solution $(\boldsymbol{u}, p)$ for at least some time interval $t > 0$, provided that initially $\boldsymbol{u}(\boldsymbol{x}, 0)$ is known and satisfies (4.13). This is the well-known "initial-value problem" for the Euler equations. The question of whether a global solution always exists is still open.

4.7 The question of boundary conditions

A flow is affected in important ways by the nature of the boundary of the flow region. Even in regular laminar flow, the shape of the boundary will have a direct effect on the shapes of the streamlines. But the boundary conditions which the flow is required to satisfy have consequences even more profound than that.

For inviscid flows, and ideal fluids in particular, it is usually simply assumed that the velocity at boundary points is tangential to it—thus the fluid particles cannot cross the boundary. We shall see later in Section 4.11 that if one adds the assumption of irrotationality, this boundary condition typically makes for a well-posed mathematical problem.

You might verify that the steady shear flows in Example 2.4.1c as well as 4.7.1a below in the channel domain $\{0 < y < h\}$ are solutions of the Euler equations and satisfy the boundary condition indicated above. (They are not in general irrotational, however, so the boundary condition does not

pin down the solution uniquely.) Most of these channel flows are unrealistic, I might add, because even the slightest viscosity will in the long run serve to pick out specific solutions.

If viscosity is put into the model, we shall see later that we must add higher order space derivatives to the Euler equations. This has the mathematical consequence that to get a well posed problem, extra boundary conditions must be required, beyond the one stated above. The extra condition is almost always that the fluid velocity must be zero at a stationary boundary. (Or if the boundary moves, then the velocity must be the same as the boundary's velocity.) Physically, this means the fluid particles must stick to the boundary. This is a physical statement about which there is actually a bit of controversy under some circumstances.

4.7.1 Examples

a. Shear flows Let's look for velocity fields $\boldsymbol{u} = (u, v)$ in 2D such that $v = 0$. Then since the flow is divergence-free, we get that $u = u(y, t)$. Show this. The Euler equations are then

$$u_t = -\frac{p_x}{\rho}, \qquad p_y = 0. \tag{4.14}$$

Show that $p = p(x, t)$ is linear in x and $u(y, t) = u(y, 0) - \frac{1}{\rho}\int_0^t p_x(t')dt'$. If $p_x < 0$ is constant, the fluid accelerates and becomes indefinitely fast.

[This problem was not an initial value problem (those are usually so difficult that one cannot write the solution explicitly). Instead, the clue here was to prescribe that $v = 0$. This clue does not pin down the solution; in fact there are an infinite number of them.]

b. Potential flows There will be much more about this in Section 4.11.

Let $\phi(x, y)$ be any harmonic function of (x, y) [the following can also be done for harmonic functions of 3 variables (x, y, z)], that is

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} = 0. \tag{4.15}$$

There is an extensive classical theory of harmonic functions.

Now construct a steady (t-independent) flow field in the form $\boldsymbol{u} = \nabla\phi$, i.e. $u = \phi_x$, $v = \phi_y$. Such a field is called a potential flow, and ϕ is called the velocity potential. We show that automatically this flow field satisfies the Euler equations (4.12), (4.13) with $\chi = 0$. First, note that (4.12) takes the form (since $\boldsymbol{u}_t = 0$)

$$\begin{aligned} \phi_x\phi_{xx} + \phi_y\phi_{xy} &= -\tfrac{p_x}{\rho}, \\ \phi_x\phi_{xy} + \phi_y\phi_{yy} &= -\tfrac{p_y}{\rho}. \end{aligned} \tag{4.16}$$

Check that the divergence-free condition (4.13) is automatically fulfilled by virtue of (4.15).

At this point we digress slightly to a simple fact from ODE's. There is a function $p(x, y)$ satisfying the system

$$p_x = f(x, y), \quad p_y = g(x, y) \tag{4.17}$$

if and only if $f_y = g_x$ for all (x, y). Apply this to (4.16). Differentiate the first of (4.16) with respect to y and show that it is the same as the derivative of the second with respect to x. This suffices for us to conclude that there exists a pressure function p such that (4.16) holds. Therefore every potential flow satisfies the Euler equations, for some p.

But how can one obtain the pressure $p(x, y)$? One can do it by integrating the first of (4.16) with respect to x, or equivalently the second with respect to y.

Example (i): Take $\phi(x,y) = xy$. Show that $p(x,y) = -\frac{\rho}{2}(x^2+y^2)+C$, where C is an arbitrary constant.

Example (ii): Take $\phi(x,y) = e^x \sin y$. Show that (4.15) holds. Find p in this case.

4.8 Vorticity

The vorticity $\boldsymbol{\omega}$ of a fluid flow is another property of particles, to be included in the list in Section 1.2. It is a vector function defined by

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \boldsymbol{u}. \tag{4.18}$$

Let's see what it means.

Roughly speaking, vorticity is a measure of the rotation of a particle of fluid. It turns out to play a crucial role in the theory. To describe it, consider first of all a block of material (it does not matter whether it is fluid or solid) which rotates like a rigid body with constant angular velocity α. We picture this in two dimensions for the moment. It doesn't take much thought to convince yourself that each blob and each particle embedded in this block also rotate at the same angular velocity α. For example during the time the entire block rotates 360^0, each particle will also have rotated exactly this amount. Of course unless the particle is exactly at the origin, its motion will also involve translation (in the positive angular direction) as well as rotation: it is moving around a circle at the same time that it is rotating—like the moon around the earth.

We shall make use of polar coordinates (r,θ). The path of a particle which starts at position $\boldsymbol{a} = (r\cos\theta_0, r\sin\theta_0)$ is a circle of radius r:

$$\boldsymbol{x}(\boldsymbol{a},t) = (r\cos(\theta_0+\alpha t), r\sin(\theta_0+\alpha t)). \tag{4.19}$$

Taking the time derivative, we have the velocity components $u = \dot{x} = -r\alpha\sin(\theta_0+\alpha t) = -\alpha y$ and similarly $v = \alpha x$. Therefore

$$\boldsymbol{u} = (-\alpha y, \alpha x). \tag{4.20}$$

In 2D, the "scalar vorticity" at a point is given by

$$\omega = v_x - u_y. \tag{4.21}$$

In this case, it equals 2α, so that *in a steady rigid body rotation, the scalar vorticity of each point is twice the angular velocity.* We may think of this rotating block as existing in 3D, with the axis of rotation along the third axis $\boldsymbol{k}$. Then $\boldsymbol{\omega} = \omega\boldsymbol{k}$.

Now consider rigid body rotations in 3D. The axis of rotation could be in any direction. The vorticity is a vector with magnitude twice the angular velocity, and directed along the axis, using the right hand rule (if you turn a right-handed screw in the direction of rotation, the advance will be in the direction of the vorticity vector). This is the clear connection between the vorticity of a rigidly rotating body and its angular velocity and axis of rotation.

But of course fluids usually do not act like rigid bodies. The small blobs can deform and change their positions; moreover the rotation properties generally change from place to place. In 2D, the above definition (4.21) still holds locally; it is the microscopic version of the macroscopic concept of "2 times angular velocity". It is a function of space and time.

The same conclusion holds in 3D. Definition (4.18) is simply the microscopic version of our description of a rigidly rotating body in 3D. As I mentioned, in addition to rotating, particles of fluid also deform their shapes and change their positions.

4.8.1 Vorticity form of the momentum equation

We can use the notion of vorticity to find alternative forms of the Euler momentum equation. Acheson uses a vector identity to derive his (1.14), i.e.

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \wedge \boldsymbol{u} = -\nabla H, \tag{4.22}$$

where

$$H = \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + \chi, \tag{4.23}$$

χ being the potential of any external conservative forces that may be present. As shown in Acheson, page 16, you can take the curl of (4.22) and use an assortment of vector identities to get

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}. \tag{4.24}$$

This is a conservation law for vorticity; we see that there is a source of vorticity given by $(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}$.

4.8.2 Examples

a. Steady shear flows Consider the example

$$u(x, y, z) = h(y), \quad v = w = 0.$$

Calculate the vorticity $\boldsymbol{\omega}$. If Euler's equations are to hold, what must the pressure be? Calculate each term of (4.22). Verify (4.24).

b. Find the vorticity of the velocity field $u = -\alpha y,\ v = \alpha x,\ w = \beta(x^2 + y^2)$.

c. Do the same for the 2D field whose particle paths are given by

$$\boldsymbol{x}(\boldsymbol{a}, t) = (r\cos(\theta_0 + e^{-r}t),\ r\sin(\theta_0 + e^{-r}t)).$$

Describe in words what this flow field looks like.

d. Suppose that at one instant of time, the velocity field is $\boldsymbol{u} = (z - y, z + x, 0)$. Find $\boldsymbol{\omega}$ at that instant. Find the source term given on the right of (4.24).

4.8.3 Persistence of vorticity

The source term in (4.24) turns out to vanish if we are in 2D, because we can imbed the flow in 3D; then $\boldsymbol{\omega}$ is perpendicular to the plane of the fluid flow, hence perpendicular to the 2D gradient vector ∇.

It follows that in a 2D flow satisfying the Euler equations, the vorticity is constant along each particle path, and hence in steady flow it is constant along each streamline.

In 3D Euler flow this isn't true, but nevertheless it will now be shown that if $\boldsymbol{\omega} = 0$ at some point in space-time, then $\boldsymbol{\omega} = 0$ all along the particle path emanating from that point.

Let's take the scalar product of (4.24) with with $\boldsymbol{\omega}$. The left side becomes $\frac{D}{Dt}\left(\frac{1}{2}|\boldsymbol{\omega}|^2\right)$. In fact if we set $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, then $|\boldsymbol{\omega}|^2 = \boldsymbol{\omega}\cdot\boldsymbol{\omega} = \sum_i \omega_i^2$. Therefore

$$\frac{D}{Dt}\left(\frac{1}{2}|\boldsymbol{\omega}|^2\right) = \frac{1}{2}\left(\frac{\partial}{\partial t} + (\boldsymbol{u}\cdot\nabla)\right)\boldsymbol{\omega}\cdot\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\left(\frac{\partial}{\partial t} + (\boldsymbol{u}\cdot\nabla)\right)\boldsymbol{\omega}. \tag{4.25}$$

The right side of (4.24), after the scalar product operation, becomes $\boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}$. Therefore $\frac{D}{Dt}\left(\frac{1}{2}|\boldsymbol{\omega}|^2\right) = \boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}$. If we think of $\boldsymbol{u}(x,t)$ in the last position of the expression on the right as being a known smooth bounded function, then this is a quadratic function of the components of $\boldsymbol{\omega}$, i.e. a sum of terms $\sum_{i,j} a_{ij}(x,t)\omega_i\omega_j$.

Each such term is bounded as follows, for some constant C_1:

$$|a_{ij}\omega_i\omega_j| \le C_1|\omega_i||\omega_j| \le \frac{1}{2}C_1(\omega_i^2+\omega_j^2) \le \frac{1}{2}C_1|\boldsymbol{\omega}|^2. \tag{4.26}$$

Adding up the terms, we find for another constant C_2,

$$|\boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}| \le C_2|\boldsymbol{\omega}|^2. \tag{4.27}$$

All in all, we find that

$$\frac{D}{Dt}\left(\frac{1}{2}|\boldsymbol{\omega}|^2\right) \le C_2|\boldsymbol{\omega}|^2. \tag{4.28}$$

We now concentrate on a single particle path, with fixed $\boldsymbol{a}$, and let $W(t) = |\boldsymbol{\omega}(\boldsymbol{x}(\boldsymbol{a},t),t)|^2$. The estimate (4.28) simply says that

$$\frac{d}{dt}W(t) \le 2C_2W(t). \tag{4.29}$$

Multiplying by e^{-2C_2t}, we have $\frac{d}{dt}\left[W(t)e^{-2C_2t}\right] \le 0$, so that for any $t > 0$

$$0 \le W(t)e^{-2C_2t} \le W(0). \tag{4.30}$$

If the particle starts with no vorticity, then $W(0) = 0$ and (4.30) implies that $W(t) = 0$ for all later times t. We have found that

If a particle of an ideal fluid has no vorticity at some time, then it cannot gain vorticity during its subsequent motion.

This turns out to be a very significant result. The production and spread of vorticity happens to be a main contributing factor to the complexity of some flows. This statement suggests that viscosity (which is not present in ideal fluids) is responsible for that production.

All this seems reasonable even without the mathematics. In fact it would appear that the only way to get a particle, initially without rotation, to rotate would be to apply tangential stresses on its boundary. In an ideal fluid, there are no such stresses.

The companion statement also holds: **If a particle of an ideal fluid has nonzero vorticity at some time, then it cannot have zero vorticity at any time during its subsequent motion.** The proof of this is essentially the same at the above; we just reverse the time variable, defining $\tau = -t$. The possibility of doing this is related to the time-reversibility of the Euler equations: if we replace t by $-t$ and $\boldsymbol{u}$ by $-\boldsymbol{u}$ in (4.12) and (4.13), the equations are unchanged. This is emphatically not the case for the analogous equations with nonzero viscosity.

Of course this persistence property holds for blobs of fluid as well as just particles: If the vorticity in a blob evolving in an inviscid fluid is everywhere zero at some specific time, it is at all other times (before and after) as well.

Problem. Given a blob V, if the outside force on the surface ∂V is given by some force density vector $\boldsymbol{F}(\boldsymbol{x})$, the torque is defined by

$$\boldsymbol{\tau} = \int_{\partial V} \boldsymbol{x} \wedge \boldsymbol{F}(\boldsymbol{x}). \tag{4.31}$$

Show that for an ideal fluid, the pressure of the outside fluid cannot exert a significant torque $\boldsymbol{\tau}$ on V when V is small. More exactly, show that in the limit as $|V|\to 0$, the ratio $\frac{\boldsymbol{\tau}}{|V|}\to 0$. You might find Acheson, Page 349, (A.19) useful. This, again, is compatible with the above proof that no vorticity can be generated in an ideal fluid. If it were generated, there would have to be a torque to do it.

4.9 Circulation

[See also Acheson, Chapter 5, and Chorin-Marsden, Section 1.2.]

4.9.1 Definition; connection with vorticity; Kelvin's circulation theorem

Given any velocity field $\boldsymbol{u}$ and any closed oriented curve C, the fluid's *circulation* about C is defined to be

$$\Gamma_C = \int_C \boldsymbol{u}\cdot d\boldsymbol{x}. \tag{4.32}$$

Stokes theorem says that if S is any oriented piece of surface with C as its boundary (so that S spans C), then

$$\Gamma_C = \int_S \boldsymbol{\omega}\cdot\boldsymbol{n}, \tag{4.33}$$

where at each point of S, $\boldsymbol{n}$ is the unit normal to S pointing in the positive direction (by the right hand rule) relative to the orientation of C.

Now consider such a curve $C(t)$ which moves with the flow. Then the circulation $\Gamma_{C(t)}$ is in principle a function of t, but if the flow satisfies Euler's equations, **it turns out to be constant**. This is Kelvin's circulation theorem.

To prove it, let s ($0 \le s < 1$) be a parameter for the curve $C(0)$, so that points on it are represented as $\boldsymbol{x}(s,0)$ with this point passing around the curve as s goes from 0 to 1. At later times, $C(t)$ can also be parameterized by a parameter s with the same range; namely we just require that moving particles retain their same value of s as time evolves. Then $\Gamma_{C(t)} = \int_0^1 \boldsymbol{u}(\boldsymbol{x}(s,t),t)\cdot\frac{\partial \boldsymbol{x}(s,t)}{\partial s}ds$. Differentiating, we find

$$\frac{d}{dt}\Gamma_{C(t)} = \int_0^1 \frac{D\boldsymbol{u}}{Dt}\cdot\frac{\partial \boldsymbol{x}(s,t)}{\partial s}ds + \int_0^1 \boldsymbol{u}\cdot\frac{\partial \boldsymbol{u}}{\partial s}ds. \tag{4.34}$$

Since the last integrand is a perfect derivative $\frac{1}{2}\frac{\partial}{\partial s}|\boldsymbol{u}|^2$ and $\boldsymbol{u}$ is the same at $s=0$ or 1, the last integral vanishes. We now invoke (4.12) to obtain

$$\frac{d}{dt}\Gamma_{C(t)} = -\int_0^1 \frac{\partial}{\partial s}\left(\frac{p}{\rho}+\chi\right)ds = 0. \tag{4.35}$$

This proves the claim: $\Gamma_{C(t)}$ remains constant in time.

4.9.2 Vortex tubes; Helmholtz theorems

A curve is said to be "transverse" to a given vector field F if it is nowhere tangent to F.

In a fluid flow, a surface is called a "vortex sheet" if it is tangent everywhere to the vorticity field. Vortex lines are defined the same way.

Consider a closed oriented curve C, transverse to $\boldsymbol{\omega}$. Also at some fixed time t_0, consider all the vortex lines through points of C. All together, they form a tubular surface.

Let's stop a minute and define tubes. A closed curve in 3D is a continuous mapping from the unit circle S^1 into 3-space. A "tube" is a continuous mapping from $S^1 \times I$ into 3-space, where I is any open interval on the real line. We impose an extra condition on our vortex tubes—they are supposed to be invertible maps, so the tube does not intersect itself.

Let us use the symbol s, $0 \le s < 1$, to label points on S^1, so that s is the parameter on C. Let the points on our vortex tube be parameterized (designated) by (s, σ), where $\sigma \in I$. More specifically, we construct the vortex lines as mappings $\boldsymbol{x}(s, \sigma)$ by solving, for each fixed s, $0 \le s < 1$, the following system of ODE's and initial condition:

$$\frac{d\boldsymbol{x}(s,\sigma)}{d\sigma} = \boldsymbol{\omega}(\boldsymbol{x}(s,\sigma), t_0), \quad \boldsymbol{x}(s,0) = \boldsymbol{x}_0(s), \tag{4.36}$$

where $\boldsymbol{x}_0(s)$ is the representation of C (remember, t_0 is the frozen time at which we do this construction). By the basic theory of ODE's, since $\boldsymbol{\omega}(\boldsymbol{x}(s,\sigma), t_0)$ is a smooth function, there is a unique solution for some open interval $\sigma \in I$ containing 0.

First Helmholtz theorem Let C_1 be another closed oriented curve once encircling the vortex tube, oriented the same way as C. What this really means is that there is a continuous function $\sigma(s) \in I$ and C_1 is the set $\{(s, \sigma(s)), 0 \le s < 1\}$. Then

$$\Gamma_{C_1} = \Gamma_C. \tag{4.37}$$

This is one of the theorems of Helmholtz.

To give a proof of this claim in the case $\sigma(s) > 0$, note that C, C_1, and the lateral sides of the tube between them, form a section of the tube. It is like a piece of tube open at both ends. We now close the ends with smooth surfaces S and S_1 spanning C and C_1, respectively, and oriented compatibly with C and C_1. This makes the boundary of a simply connected solid V.

We have, by the divergence theorem,

$$\int_V \operatorname{div} \boldsymbol{\omega} = \int_{\partial V} \boldsymbol{\omega} \cdot \boldsymbol{n}, \tag{4.38}$$

where $\boldsymbol{n}$ now represents the unit outward normal to V. However, the divergence of any curl (such as $\boldsymbol{\omega}$) is zero, so both sides vanish. The integral on the right (which equals zero) can be split into the integrals over S and S_1, plus the integral over the lateral boundary. The latter is zero because $\boldsymbol{\omega} \cdot \boldsymbol{n} = 0$ there. The normal vector on one of S or S_1 is the same as the normal in Stokes theorem (4.33), and on the other it is the negative of that in (4.33). Therefore, again using (4.33), we find that

$$\Gamma_C = \Gamma_{C_1}, \tag{4.39}$$

as claimed.

This circulation, common to all such curves, is called the **strength** of the vortex tube.

Second Helmholtz theorem For the other Helmholtz theorem, we will need the following result: **If a vortex line is convected with the fluid, it remains a vortex line at all future times.** For the proof of this, let me just refer you to Acheson, p. 163, "Proof of (1)".

It follows that our vortex tube, constructed at some time t_0, remains a vortex tube when it moves with the fluid.

Combine this with Kelvin's theorem and we get the other Helmholtz theorem: **the strength of any vortex tube moving with the fluid is constant in time**.

The stretching effect Consider the case of long slim vortex tubes, for which each cross section is small. Let Γ be its strength, and consider any such cross section S. In (4.33), $\boldsymbol{\omega}$ is directed approximately in the direction of $\boldsymbol{n}$, so that $\boldsymbol{\omega}\cdot\boldsymbol{n} \sim |\boldsymbol{\omega}|$. Since S is small, the integral on the right of (4.20) is approximately $|S|\overline{\omega}_S$, where $\overline{\omega}_S$ is the average value of the magnitude of the vorticity over that cross section. And by the first theorem this is independent of S. In all, we have

$$|S|\overline{\omega}_S = \Gamma. \tag{4.40}$$

Thus as we move along the thin tube, the vorticity is inversely proportional to the area of the tube's cross section.

Tornados A good application of this principle is a simple model of tornados; read Acheson, page 164. Here one envisions a circular shear flow parallel to the earth's surface. Any circular cylinder centered at the center of the flow will have the flow field tangent to its boundary, and the vortex lines are straight up. Such a cylinder is therefore a vortex tube.

Now suppose there is a vertical density gradient caused by heating of the earth's surface and a resulting updraft caused by the buoyancy effect. In this, the air moves rapidly upward in localized regions, and to satisfy conservation of mass, other air has to move radially into those regions. The density variation is actually minor although the convective winds resulting from it are not minor. So the usual assumption of incompressibility is okay.

This movement is superimposed as a perturbation of the basic circular shear flow mentioned above. Where are the updraft regions located? By considerations we won't go into, the existing vortices are the preferred regions. So we have a vortex together with an updraft at the same location. The updraft has the effect of stretching the vortex tube (see the illustration in Acheson). In (4.40), the stretching of the vortex tube means that $|S|$ contracts, so that by (4.40), this stretching results in an intensification and localization of the vorticity, hence in a strengthening of the associated swirling winds. A tornado is born.

4.9.3 Vortex filaments

Problem a. Consider a 2D circular shear flow in which the tangential velocity is given by

$$u_\theta = m(r) \tag{4.41}$$

for some function $m(r)$, $r > 0$, and of course the radial velocity is zero. Let C_r be the circle of radius r, oriented counterclockwise. Let $\Gamma_r = \Gamma_{C_r}$. Find all functions $m(r)$ such that Γ_r is independent of r. Show that for these functions, the fluid is irrotational except at $r = 0$. Given Γ, find the function m for which the circulation is Γ.

These are called point vortices. They have a singularity in the flow field at the origin. Imbed this flow in 3D with the third coordinate axis z perpendicular to the flow, and you have a **vortex filament**. This particular one is straight, namely the z-axis. In general, there exist vortex filaments in 3D which have most any shape. Some form closed loops. The flow field is singular on the filament.

Vortex patches in 2D are regions with positive vorticity, outside of which the vorticity is zero. Point vortices are often taken as approximations to vortex patches with the same strength.

Problem b. How would you define the strength of a vortex patch?

4.9.4 Variation of density

If the density ρ is not constant, but the other defining properties of an ideal fluid hold, then the momentum equation (4.12) still holds, although (4.13) may not. But (4.24) is valid if we replace $\boldsymbol{\omega}$ by $\boldsymbol{\omega}^* = \frac{\boldsymbol{\omega}}{\rho}$.

If the fluid is *barotropic*, i.e. there is a one-to-one relation between the pressure and density, so that $p = p(\rho)$, then analogs of the Kelvin and Helmhotz theorems still hold with proper modifications. In particular, the result (4.40) should be changed by replacing $\boldsymbol{\omega}$ by $\boldsymbol{\omega}^*$.

4.10 Decomposition of vector fields

By now it should be clear that the operations of gradient, divergence, and curl are important ingredients in the theory of fluid dynamics. It is time we focus on some fundamental properties of these operations, and in particular on the most important properties of vector fields which are divergence-free or curl-free. It would be a good idea to take some time now to review your multivariable calculus text for information on these concepts and facts.

4.10.1 Potentials and stream functions

The most basic facts are probably the following. They are true of any vector fields, but we shall speak in terms of the velocity $\boldsymbol{u}$.

(A) If $\nabla \wedge \boldsymbol{u} \equiv 0$, then there exists a scalar function $\phi(\boldsymbol{x}, t)$, called the **velocity potential**, such that

$$\boldsymbol{u} = \nabla\phi. \tag{4.42}$$

In this case (i.e. when the vorticity vanishes), the fluid is called **irrotational**. It is a matter of easy calculation that the curl of any gradient field is zero. If you haven't done this at some time, verify it for yourself now. The statement **(A)** is a sort of converse: it says that if the curl of something is zero, then that thing is a gradient. As I mentioned, it is proved in multivariable calculus.

The next assertion is most meaningful to us in the case of two space dimensions, and so we restrict to that case.

(B) If $\operatorname{div} \boldsymbol{u} \equiv 0$ and the flow is in 2D, there is a scalar function $\psi(\boldsymbol{x}, t)$, called the **stream function**, such that

$$\boldsymbol{u} = \nabla \wedge (\psi \boldsymbol{k}). \tag{4.43}$$

This is the case of an incompressible flow, also called a solenoidal or divergence-free vector field. It is much easier to remember (4.43) in component form:

$$u = \psi_y, \quad v = -\psi_x. \tag{4.44}$$

Again, the proof comes from calculus, and the statement is a converse of the easily verified claim that the divergence of a curl is zero.

As an exercise, find the stream function for the flows given in 4.7.1a and 4.7.1b(i).

The relation **(A)** has implications about our vorticity equation (4.24). If a flow satisfies the momentum equation (4.12), it also satisfies (4.24). The latter was derived by taking the curl of (4.12) and using some vector identities. The curl operation annihilated the term ∇p in (4.12). Now (4.24) is equivalent to $\nabla \times \boldsymbol{F} = 0$, where $\boldsymbol{F}$ is the left side of (4.12). Our assertion **(A)** tells us

that $\boldsymbol{F}$ is a gradient of some potential. All that we now have to do is identify the pressure in terms of that potential, and we see that (4.12) in its original form is satisfied. So in this sense, (4.24) implies (4.12); in short, (4.24) is *equivalent* to (4.12)

4.10.2 The Laplace and Poisson equations

This is not a course in PDE's but we are going to use a few basic facts from that field. Right now, it is worthwhile to just state the basic existence theorems for boundary value problems for the Laplace and Poisson equations. As usual, we use the notation V to denote a bounded smooth domain in 3-space or 2-space. Let $f(\boldsymbol{x})$ be a function defined on V, and $g(\boldsymbol{x})$ be a function defined on ∂V. Here are the most important boundary value problems associated with the Laplace operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$:

Dirichlet problem Find a function $\phi(\boldsymbol{x})$ satisfying

$$\nabla^2\phi(\boldsymbol{x}) = f(\boldsymbol{x}) \text{ in } V; \quad \phi(\boldsymbol{x}) = g(\boldsymbol{x}) \text{ on } \partial V. \tag{4.45}$$

Neumann problem Find a function $\phi(\boldsymbol{x})$ satisfying

$$\nabla^2\phi(\boldsymbol{x}) = f(\boldsymbol{x}) \text{ in } V; \quad \frac{\partial\phi}{\partial n}(\boldsymbol{x}) = g(\boldsymbol{x}) \text{ on } \partial V. \tag{4.46}$$

On the right of (4.46), the derivative signifies the directional derivative in the outward normal direction (the "outward normal derivative", for short). The PDE's are called the Poisson equation. If $f = 0$, they are called the Laplace equation.

Basic facts: *The Dirichlet problem always has a unique solution. The Neumann problem has a solution if and only if*

$$\int_V f = \int_{\partial V} g. \tag{4.47}$$

If this is the case, then the solution of the Neumann problem is unique except for an arbitrary additive constant.

This last clause means that you can add any constant c to a solution ϕ and still get a solution. But this is the *only* way you can get another solution.

These theorems are proved in every beginning graduate course in PDE's. You don't have to know the proofs. Saying that a solution exists, as these theorems do, is a far cry from actually finding that solution. If you want an explicit formula for the solution you may be out of luck; that can only be done in special cases. Otherwise you have to resort to numerical methods.

4.10.3 Decompositions

Along with the facts in Section 4.10.1, equally important is the relation between divergence-free vector fields and gradients. On the face of it, these are just two special classes of vector fields; but it turns out that in a sense any vector field can be written uniquely as the sum of a gradient and a divergence-free field.

But first, let me show another connection (or rather disconnection) between these two classes: they are "orthogonal", in the following sense. Let V be a smooth bounded spatial domain, and let $\boldsymbol{G}$ be a smooth divergence-free vector field defined in V such that $\boldsymbol{G}\cdot\boldsymbol{n} = 0$ at every point of the

boundary ∂V ($\boldsymbol{n}$ is the unit outward normal vector). So $\boldsymbol{G}$ points in the direction tangential to ∂V. Also let q be a smooth scalar field defined in V. Then

$$\int_V \nabla q \cdot \boldsymbol{G} = 0. \tag{4.48}$$

The proof of this consists in seeing that $\nabla q \cdot \boldsymbol{G} = \operatorname{div}(q\boldsymbol{G}) - q \operatorname{div} \boldsymbol{G}$. This is essentially the product rule of differentiation, but in a vector setting. Since $\operatorname{div} \boldsymbol{G} = 0$ by assumption, the integral in (4.48) becomes (by the divergence theorem)

$$\int_V \operatorname{div}(q\boldsymbol{G}) = \int_{\partial V} q\boldsymbol{G} \cdot \boldsymbol{n}, \tag{4.49}$$

and this also vanishes by assumption on $\boldsymbol{G}$. So (4.48) follows.

If we denote $\nabla q = \boldsymbol{H}$, then the integral $\int_V \boldsymbol{F} \cdot \boldsymbol{H}$ appearing on the left of (4.48) is a sort of *scalar product* for vector fields defined in V, analogous to the ordinary dot product for ordinary vectors. For that reason, (4.48) is called an *orthogonality relation*.

The decomposition theorem says that if $\boldsymbol{F}$ is *any* smooth vector field defined in V, then it can be written in a unique manner as the following sum:

$$\boldsymbol{F} = \boldsymbol{G} + \nabla q, \tag{4.50}$$

where $\boldsymbol{G}$ is a divergence-free vector field which is tangent to the boundary ∂V at every point on ∂V, and q is a scalar field.

We can prove this on the basis of the theorem about the Neumann problem in Section 4.10.2. With the given field $\boldsymbol{F}$ at hand, we define q to be the solution of

$$\nabla^2 q(\boldsymbol{x}) = \operatorname{div} \boldsymbol{F}(\boldsymbol{x}); \quad \frac{\partial q}{\partial n}(\boldsymbol{x}) = \boldsymbol{F} \cdot \boldsymbol{n}(\boldsymbol{x}) \text{ on } \partial V. \tag{4.51}$$

This has a solution q if (4.47) holds for $f = \operatorname{div} \boldsymbol{F}$, $g = \boldsymbol{F} \cdot \boldsymbol{n}$. But for these choices of f and g, (4.47) is simply the divergence theorem, so we are done. There is a solution q which is unique except for an additive constant.

Once $q(\boldsymbol{x})$ is defined this way, we proceed to define

$$\boldsymbol{G} = \boldsymbol{F} - \nabla q. \tag{4.52}$$

The only thing remaining to do is check that $\boldsymbol{G}$ has the required properties of being divergence-free and $\boldsymbol{G} \cdot \boldsymbol{n} = 0$ on ∂V. As for the first, we have

$$\operatorname{div} \boldsymbol{G} = \operatorname{div} \boldsymbol{F} - \nabla^2 q, \tag{4.53}$$

which is zero, by (4.51). The second is shown just as easily.

So we have that $\boldsymbol{F}$ can be decomposed in the form (4.50) (which is the same as (4.52)). Let us show that there is only one way to do it. If there were two, we would have not only (4.50) but also the same equation with same $\boldsymbol{F}$ but different fields $\boldsymbol{G_1}$ and q_1. Subtracting the two equations, we would find

$$(\boldsymbol{G} - \boldsymbol{G_1}) + \nabla(q - q_1) = 0. \tag{4.54}$$

Take the scalar product of this equation with $(\boldsymbol{G} - \boldsymbol{G_1})$ and integrate over V. Because of the orthogonality relation (4.48), we get $\int_V |\boldsymbol{G} - \boldsymbol{G_1}|^2 = 0$, and this means $\boldsymbol{G} \equiv \boldsymbol{G_1}$. Finally from (4.54), $\nabla(q - q_1) \equiv 0$, and this implies that q and q_1 differ by only a constant. So they are essentially the same.

All this was for a bounded domain V, but a decomposition result similar to (4.50) holds in great generality for unbounded domains as well. That is a good thing, because most flow regions of interest are unbounded.

4.11 Irrotational incompressible flows

4.11.1 The complex potential

Since vorticity cannot be generated inside the flow of an ideal fluid, there are many situations where it is natural to assume that the flow is irrotational:

$$\boldsymbol{\omega} \equiv \nabla \wedge \boldsymbol{u} \equiv 0. \tag{4.55}$$

Thus, for example, if the flow originates far away as a uniform irrotational flow, then it would be irrotational everywhere.

Moreover, if a flow satisfies (4.55), then it satisfies (with proper choice of p) the Euler momentum equation. That equation reduced to (4.24), which is clearly satisfied, since $\boldsymbol{\omega} \equiv 0$. We saw in Section 4.10.1 that in fact (4.24) is equivalent to (4.12), with proper choice of pressure function. Therefore all irrotational flows satisfy (4.12) .

As we saw in Section 4.10.1, any such flow field is the gradient of a scalar field ϕ called a **velocity potential**:

$$\boldsymbol{u} = \nabla\phi. \tag{4.56}$$

Now in addition if the flow is incompressible, i.e. divergence-free or solenoidal, so that the second of Euler's equations (4.13) holds, then

$$\operatorname{div} \boldsymbol{u} \equiv \operatorname{div} \nabla\phi \equiv \nabla^2\phi \equiv \phi_{xx} + \phi_{yy} + \phi_{zz} = 0. \tag{4.57}$$

This is the Laplace equation, and the operator is the Laplace operator.

The surfaces of constant ϕ are called **equipotentials**; they are perpendicular to the gradient curves, i.e. curves directed along the gradient $\nabla\phi$. But by (4.56), those lines are tangent to the velocity field, so are streamlines. In short, the equipotentials are orthogonal to the streamlines.

Similarly, if we are in 2D we have the stream function (Section 4.10.1), and since the lines of constant ψ are perpendicular to the gradient lines of ψ, they are tangent to the vector field $(\psi_y, -\psi_x)$, which in turn by (4.44) is simply $\boldsymbol{u}$:

$$u = \psi_y, \quad v = -\psi_x. \tag{4.58}$$

Thus in 2D the streamlines are lines of constant ψ; and they are perpendicular to the equipotentials.

The connection between ϕ and ψ is found in the following equations, which come from expressing the components (u, v) using either function:

$$\phi_x = \psi_y, \quad \phi_y = -\psi_x. \tag{4.59}$$

These are the "Cauchy-Riemann equations"; they are intimately tied to complex variable theory, in that they express the relation between the real and imaginary parts of an analytic function $\Phi(x, y) = \phi(x, y) + i\psi(x, y)$. Knowing either one of these real functions, the other can be determined up to an additive constant by integrating (4.59); hence $\Phi(x, y)$, called the **complex potential** of the flow, can also be determined. It therefore follows that Φ is an analytic function of $z = x + iy$.

All this means that the tools of complex variables, especially those of conformal mapping, can effectively be used to investigate incompressible irrotational flows in 2D. There are books on this subject alone.

Steady problems in this category typically involve finding a harmonic function ψ for which the boundary of the flow region is a level line (since the boundary should be a streamline; see Section 4.7), and hence finding the velocity field.

It is often useful to map the flow region conformally to a standard domain such as a disk or a half-plane. For the standard domain, the complex potential is known or can easily be found. One then maps back to the original domain. When this is done, the complex potential found for the simple domain is transformed into the desired complex potential for the original domain.

Classical aerofoil theory—see Acheson, Chapter 4—is a branch of this subject.

Although the basic equations (4.12), (4.13) of Euler are certainly nonlinear, so that for instance two given ideal fluid flows cannot in general be added to one another to produce a third ideal fluid flow, nevertheless it is remarkable that the subclass of potential flows in 2D does have this property. After all, they are all given by interpreting some analytic function as a complex potential. And the sum of two analytic functions is still analytic. Of course in doing so you will usually violate the boundary conditions.

4.11.2 The pressure

If one knows the complex potential, one can find the velocity field from (4.56) or alternatively (4.58). These equations can be expressed concisely in the form

$$u - iv = \frac{d\Phi}{dz}, \tag{4.60}$$

where we have the derivative with respect to the complex variable z. Check that this is equivalent to either (4.56) or (4.58).

The pressure along any streamline can be found from Bernoulli's equation (4.10):

$$p(x, y) = -\frac{1}{2}\rho q^2 + C, \tag{4.61}$$

where $q = |\boldsymbol{u}|$. Again, (4.60) would enable us to replace q^2 in this expression by $\left|\frac{d\Phi}{dz}\right|^2$, but we will find a much better thing to do.

4.11.3 Potential flow past an obstacle; lift and drag

Consider an idealized 2D flow $\boldsymbol{u}$ from the left to the right filling all space except for the obstacle, and approaching a steady horizontal flow with speed U at infinity:

$$\boldsymbol{u}(x, y) \to (U, 0) \quad \text{as } (x, y) \to \infty. \tag{4.62}$$

Because of (4.60), we have

$$\frac{d\Phi}{dz}(\infty) = U. \tag{4.63}$$

Since the particle paths at the boundary of the obstacle follow that boundary, we know that it is a streamline, and therefore (4.61) holds all along it, for some fixed C. There will be a point on the front of the obstacle where a streamline splits, one part going over the top and the other over the bottom. And a similar point on the back there they merge again. But the top and bottom parts have the same C, because they come from a common streamline.

The total force on the obstacle (call the latter V) is given by

$$\text{Force} = \boldsymbol{F} = -\int_{\partial V} p\boldsymbol{n}\, ds. \tag{4.64}$$

We will use the expression (4.61) for p here. However, the quantity q^2 in (4.61) is not an analytic function of z. To use the power of complex variables to the fullest, our integrals should be contour integrals of functions which are analytic except for point singularities. The reason is that then they can be evaluated in a second by the method of residues. It is something of a miracle that the goal of working only with analytic functions can be realized by simply using $\overline{\boldsymbol{F}}$ below instead of $\boldsymbol{F}$.

We will identify 2-vectors, such as $\boldsymbol{u}$ and $\boldsymbol{F}$, with complex numbers: $\boldsymbol{F} = (F_x, F_y) = F_x + iF_y$, $\boldsymbol{u} = u + iv$, $\frac{d\Phi}{dz} = u - iv$, $\boldsymbol{n} = n_1 + in_2$. The component F_x is the **drag** on the obstacle, and F_y is the **lift**. For the reason mentioned above, we prefer to work with the complex conjugate $\overline{\boldsymbol{F}}$ instead of $\boldsymbol{F}$. From (4.64) we obtain

$$\overline{\boldsymbol{F}} = -\int_{\partial V} p\overline{\boldsymbol{n}}\, ds. \tag{4.65}$$

If a point z is on ∂V, and the unit tangent vector in the counterclockwise direction is denoted by $e^{i\theta}$, then the vector $-\boldsymbol{n}$ at that point is obtained by a rotation of $\pi/2$, so that $\boldsymbol{n} = -ie^{i\theta}$ and $\overline{\boldsymbol{n}} = ie^{-i\theta}$. Moreover the differential in z when ∂V is traversed in the positive direction is given by $dz = e^{i\theta}ds$, so that in (4.65), $ds = e^{-i\theta}dz$. All in all, (4.65) becomes

$$\overline{\boldsymbol{F}} = F_x - iF_y = -i\int_{\partial V} pe^{-2i\theta}dz, \tag{4.66}$$

where the integral is in the counterclockwise sense.

Finally by Bernoulli (4.61),

$$\overline{\boldsymbol{F}} = \frac{1}{2}i\rho\int_{\partial V}(q^2 - C)e^{-2i\theta}dz. \tag{4.67}$$

It can be checked that the constant C's contribution to this integral vanishes, so with no loss of generality we set $C = 0$.

At points on the upper part of the boundary ∂V, $\boldsymbol{u}$ is directed clockwise and tangential to ∂V; it follows that

$$\boldsymbol{u} = -qe^{i\theta}, \tag{4.68}$$

and

$$\frac{d\Phi}{dz} = \overline{\boldsymbol{u}} = -qe^{-i\theta}; \quad q^2 = e^{2i\theta}\left(\frac{d\Phi}{dz}\right)^2. \tag{4.69}$$

On the lower part, $\boldsymbol{u} = qe^{i\theta}$ and (4.69) still holds. Putting (4.69) into (4.67), we find

$$\overline{\boldsymbol{F}} = \frac{1}{2}i\rho\int_{\partial V}\left(\frac{d\Phi}{dz}\right)^2 dz. \tag{4.70}$$

This is a useful formula named after Blasius.

It is interesting to find the circulation Γ_C about any clockwise curve C encircling the obstacle. It is

$$\Gamma_C = \int_C \boldsymbol{u}\cdot d\boldsymbol{x} = \int_C \nabla\phi\cdot d\boldsymbol{x} = \int_C \frac{\partial\phi}{\partial s} = [\phi], \tag{4.71}$$

where $\frac{\partial\phi}{\partial s}$ means directional derivative with respect to arclength along the curve, and $[\phi]$ is the change that the function ϕ undergoes once around the circle. Of course if ϕ is single-valued, this change is 0.

4.11.4 Flow past a circular cylinder

The obstacle is now supposed to have, as cross section, the disk $\{|z| < a\}$.

Verify that for any real number Γ, the proposed complex potential

$$\Phi(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\log z \tag{4.72}$$

satisfies the following:

(a) (4.62) holds.

(b) $\psi(z) = U\left(y - \frac{a^2 y}{|z|^2}\right) - \frac{\Gamma}{2\pi}\log|z|$.

(c) The boundary $\{|z| = a\}$ is a streamline.

(d) The circulation around the disk is Γ.

(e) $F_x = 0$ and $F_y = -\rho U\Gamma$. (Just find the coefficient of z^{-1} in the expansion of the integrand in (4.70) and use the residue formula.)

As I have mentioned before, this potential flow is a poor description of an actual flow past a circular cylinder, because of the formation of eddies on the downstream side. However, one can use this solution to obtain a potential flow past an aerofoil by conformally mapping the circle to the aerofoil. Those figures have cusps on the trailing edge, and it is possible to choose the circulation Γ so that the merging of the two parts of the boundary streamline occurs exactly at the cusp. In that case, there will be no vortices shed from the tail when the flow rate is moderate, and the flow pattern is reasonably well approximated in the large by the potential flow.

In particular, the formula for the lift is reasonable. However the theory predicts that the drag $F_x = 0$, and this is certainly not reasonable (this is d'Alembert's paradox). In fact any flow past an obstacle exerts a horizontal force on that obstacle. But it is not always due to the pressure distribution. There is also skin friction, for fluids with small viscosity, acting in a thin boundary layer next to ∂V. In that layer, there is such a large gradient in tangential flow velocity that a significant tangential force appears.

This fact, which relies on viscosity, will shortly lead us away from ideal fluids. But first, let us take a closer look at the last term in (4.72).

4.12 Dynamics of point vortices

The potential flow given by the last term of (4.72),

$$\Phi_0(z) = -\frac{i\Gamma}{2\pi}\log z, \tag{4.73}$$

is incompressible and irrotational for $z \neq 0$, but has a singularity at $z = 0$. The streamlines are circles; the flow is a symmetrical eddy centered at the origin. Since the circulation about any circle is Γ, we say this is a point vortex of strength Γ. See again Problem 4.9.3a.

This is not a classical solution of the Euler equations, but rather an idealized solution. We shall be interested in what laws of motion such idealized solutions ought to obey. It will come out that the superposition of two potential flows with singularities like this does not necessarily give a valid flow, despite the comment I made at the end of Section 4.11.1.

Consider first the single stationary point vortex (4.73). By symmetry, the velocity field $\boldsymbol{u}_0$ averages out to 0 in any disk centered at the origin. So in this sense there is no "net" velocity field induced by the point vortex. Unless there is some other flow field involved besides the point vortex, it stands to reason that this point vortex should have very simple dynamics: it should not move.

Now suppose there is another velocity field $\tilde{\boldsymbol{u}}$ imposed on the system. We can expect the point vortex to move in that case. How would it move? We have the important second Helmholtz theorem (Section 4.9.2). It says that vortex tubes, when convected with the fluid flow, retain their identity as vortex tubes, with the same strength. If we think of point vortices in 2D as limits of cylindrical vortex tubes with axes pointing along the third space direction, as the radius goes to zero, the strength remaining constant, then we conclude that these point vortices also retain their identity when convected with the flow. In other words, the logical consequence of the classical second Helmholtz theorem for point vortices is that they ought to move with the fluid flow. But that flow consists of a part $\boldsymbol{u}_0$ that averages out to zero, plus $\tilde{\boldsymbol{u}}$. We should disregard the first and conclude that

Point vortices in 2D move with any extra imposed velocity field, as though they were particles.

The complex potential of such a vortex will be something like $-\frac{i\Gamma}{2\pi}\log(z - z_0(t))$ (to which we have to add the imposed complex potential), where $z_0(t)$ is a particle path resulting from $\tilde{\boldsymbol{u}}$, namely a solution of

$$\frac{dz_0}{dt} = \tilde{\boldsymbol{u}}(z_0, t). \tag{4.74}$$

Notice that even though this complex potential depends on time, it still represents an irrotational (except at the center) incompressible flow; there was no requirement that $\Phi(z)$ be independent of time.

If there are a number of point vortices in the field, we can now consider each of them as contributing a velocity field which will affect the motion of all the others. If these are the only other velocity fields present, we can identify the velocity field $\tilde{\boldsymbol{u}}$ discussed above as the field contributed by all the *other* point vortices.

Each of a set of point vortices moves with the velocity field generated by all the other ones.

4.12.1 Examples.

(a) Consider two point vortices at locations $z_1(t) = \rho e^{i\alpha t}$, $z_2(t) = -\rho e^{i\alpha t}$ with ρ and α constant. So they move around each other, remaining on diametrically opposite sides of the origin. Show that if each has the same strength Γ, this motion will satisfy the stated requirements if Γ, ρ, α satisfy a certain relation. What is it?

(b) At time $t = 0$, two point vortices with opposite strengths Γ and $-\Gamma$ are situated at positions $z_1 = \rho$, $z_2 = -\rho$. Determine their subsequence motion. Come up with an educated guess, and then verify that it works.

(c) Consider an infinite array of vortices with equal strengths initially equally spaced along the x-axis. Show that this is a stationary solution; they don't move. Later we shall see that it is an unstable configuration.

(d) Consider two parallel infinite horizontal arrays of equally spaced vortices. In one array, the locations are $z_n = nh$, $n = 0, \pm 1, \pm 2, \ldots$ (so the vortices are spaced at a distance h). In the other array, the locations are $z'_n = z_n + \alpha + i\beta$, where α and $\beta \neq 0$ are real. Each vortex at z_n has strength Γ, and each in the other array at z'_n has strength $-\Gamma$.

Let u_{mn} be the velocity field at location z_n due to the vortex at z'_m, and u'_{mn} the velocity field at location z'_n due to the vortex at z_m. Show, using symmetry arguments, that

(i) $u_{nm} = u'_{mn}$.

(ii) The velocity field at z_n due to the collection of all vortices at z_m with $m \neq n$ is zero.

(iii) The velocity vield at z_n due to the collection of all vortices in the other array at z'_m (for all m) is independent of n.

(iv) All vortices move with the same velocity (so that the structure of the arrays is preserved; the arrays are just translated).

(v) The velocity is horizontal if $\alpha = 0$ or $\alpha = h/2$.

5 The equations of viscous flow

5.1 A simple model for viscosity

The most direct feature of viscosity is that it causes tangential forces on fluid surfaces when there is a gradient in the tangential velocity component. One of the important physical mechanisms for this phenomenon is the transfer of tangential momentum between the fluids on the two sides of the surface by random fluctuations in the movements of the molecules. To understand this process better, we'll look at a discrete model.

Consider 1D streams of fluid in 2D moving horizontally, at discrete locations $y = \nu h$, where ν is an integer and h is the separation distance between streams.

We suppose the velocity of stream ν is $\alpha\nu$, so that there is a velocity "gradient" in the y direction. Let ρ be the linear density of these streams of fluid, so that the mass contained in an interval of length ℓ will be $\rho\ell$. The momentum in such an interval at vertical location νh is $\rho\ell\alpha\nu$.

In addition to this motion, suppose there is a random motion of the molecules in a vertical direction, so that some escape from stream ν to join the neighboring streams. In so doing, they carry their original horizontal momentum. In unit time, let θ be the average fraction of molecules escaping to the next higher level, and also this same fraction going to the next lower level. So in unit time, the average amount of momentum transferred by these escapees in a space interval ℓ from level ν to level $\nu \pm 1$ is $\theta\rho\ell\alpha\nu$.

We now look at an intermediate level $y = (\nu_0 + \frac{1}{2})h$ and ask, what is the net momentum transfer across that level in a space interval of length ℓ in unit time? The transfer upward is $\theta\rho\ell\alpha\nu_0$, the transfer downward is $\theta\rho\ell\alpha(\nu_0 + 1)$, and the net is -$\theta\rho\ell\alpha$, independent of ν. In other words, the *downward* momentum flux through any intermediate level is $\theta\rho\alpha$ (momentum per unit time per unit length).

Thus if P is the momentum at any level in length ℓ, then the rate of change due to this momentum transfer is $\theta\rho\alpha\ell$:

$$\frac{dP}{dt} = \theta\rho\alpha\ell. \tag{5.1}$$

What is the effect of this rate of change? It is a force; one might think of this force as being caused by the collisions between the existing molecules in a given stream and the incoming ones. The amount of this force, which we call $T_h\ell$ since it will be proportional to ℓ, is given by Newton's law:

$$\frac{dP}{dt} = T_h\ell = \theta\rho\alpha\ell; \tag{5.2}$$

i.e.

$$T_h = \theta\rho\alpha. \tag{5.3}$$

This is the analog, in our discrete model, of the postulated relation (4.1), and serves as a sort of justification of the latter. In this analogy, the role of u_y is played by α/h, so that $\mu = \theta\rho h$. As expected, the "viscosity" μ is proportional to θ, the rate of escape due to random vertical motions. But the important feature of (4.1) and (5.3) is that *the stress depends linearly on the velocity gradient.*

5.2 The Navier-Stokes stresses

For this section, we will use more logical symbols for velocity and position components: $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{x} = (x_1, x_2, x_2)$.

Read Acheson, pages 207–211.

In an ideal fluid, we defined the pressure to be the normal stress on any surface; it was in fact the only stress in that case. But in nonideal fluids, the pressure still exists as the normal component of the stress, and the same argument as before shows that it is independent of orientation of the surface. It is therefore useful to separate out that normal component and write

$$T_{ij} = p\delta_{ij} + T^D_{ij}, \tag{5.4}$$

as in Acheson, Page 209.

The second part T^D_{ij} accounts for the tangential stresses, and these are the ones which we are going to suppose, in view of the above discrete analogy, depend linearly on the velocity gradient.

We interpret this as T^D_{ij} depending linearly on the components of the matrix

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} = \left\{ \frac{\partial u_i}{\partial x_j} \right\}. \tag{5.5}$$

In addition to this requirement, which Acheson gives as (i) on page 209, these are two other reasonable a priori requirements, (ii) and (iii): the viscous stresses should vanish if there is no deformation of the fluid, and the dependence on the velocity gradients should be an isotropic one. Finally, there is the requirement that T_{ij} be symmetric, which comes from an argument involving the balance of torque on a small element of fluid.

All of these requirements dictate, as Acheson argues, that for an incompressible fluid

$$T^D_{ij} = \mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right), \tag{5.6}$$

hence (Acheson (6.9))

$$T_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right). \tag{5.7}$$

5.3 The Navier-Stokes equations

It is a straight-forward extension of the argument in Section 4.4 to derive the analog of (4.9) for an incompressible fluid with general stress tensor. In component form, it is

$$\frac{\partial u_i}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_i = -\frac{1}{\rho}\frac{\partial T_{ij}}{\partial x_j} - \frac{\partial \chi}{\partial x_i}, \tag{5.8}$$

where the repeated index j is summed. This is one of a multitude of equations named after Cauchy.

In the case (5.7), we get the Navier-Stokes momentum equation (or simply the "Navier-Stokes equation")

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nu\nabla^2\boldsymbol{u} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\boldsymbol{f}, \tag{5.9}$$

where

$$\nu = \frac{\mu}{\rho} \tag{5.10}$$

is called the **kinematic viscosity** and $\boldsymbol{f} = -\rho\nabla\chi$. However, we now generalize this to allow that the body force $\boldsymbol{f}$ may be nonconservative, i.e. not necessarily a gradient.

Equation (5.9), coupled with (4.13):

$$\operatorname{div}\boldsymbol{u} = 0, \tag{5.11}$$

constitute the system of Navier-Stokes equations.

These equations are almost universally recognized as an accurate representation, on the macroscopic level, of flows of many standard kinds of incompressible fluids. Generalizations exist in many forms, such as for nonhomogeneous fluids, fluids in which there is temperature variation, for which μ depends on temperature, and fluids for which slight changes in density are allowed due to temperature variation. Then there are nonstandard, non-Newtonian, fluids, for which the stress tensor takes a more complicated form.

The second (nonlinear) term on the left of (5.9) is called the "inertia term", the first term on the right is the "viscosity term". Features of the solutions are typically governed by the order of magnitude of the ratio of these terms.

5.3.1 Existence questions

A well studied mathematical problem asks whether the Navier-Stokes equations are a meaningful model, in the following sense. If, for some domain V and given forcing $\boldsymbol{f}$, one prescribes an arbitrary initial velocity $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(x)$, $\boldsymbol{x} \in V$, with $\operatorname{div}\boldsymbol{u}_0(x) = 0$ and $\boldsymbol{u} = 0$ for $\boldsymbol{x} \in \partial V$, does there exist a unique solution of (5.9), (5.11) for $t \geq 0$, $\boldsymbol{x} \in V$ satisfying the given initial condition and $\boldsymbol{u} = 0$ for $\boldsymbol{x} \in \partial V$? (The uniqueness of p would have to be only up to an additive constant.) This is the initial-boundary value problem. Or V could be the entire space, and no boundary conditions would then be given.

In this theory there arises a natural concept called "weak solution". In fact, there are several kinds of possible weak solutions. They are not necessarily "classical solutions", for which all the derivatives appearing in the equations exist, but are rather functions which satisfy certain integral relations obtained by multiplying (5.9) by a divergence-free test function (C^∞, compact support) vector field, then integrating by parts to transfer all or some of the derivatives to the test function. If the candidate for a solution satisfies this integral relation for all possible test functions in a certain class, it is called a weak solution in the corresponding sense. Classical solutions are weak solutions, but the converse hasn't been established.

It would be nice to have an existence/uniqueness theory for all $t \geq 0$ for classical solutions, assuming the initial data are smooth, say. However, this has only been proven in the case of 2 space dimensions.

In the 3D case, a short-time existence has been proved (i.e. there is a T, depending on the initial data, $\boldsymbol{f}$, and ν, such that a classical solution exists for $0 \leq t \leq T$). Also, global existence (for all t) of weak solutions has been established (but not uniqueness). However, as far as it is now

known, in some cases a weak solution might not be classical; or there might exist multiple weak solutions. This is a famous open problem.

In studying this evolution problem, it is usually convenient to reformulate it in a slightly different fashion. Recall (4.50), where it was shown that any vector field can be decomposed in a unique manner as the sum of a divergence-free field (with 0 normal component on the boundary) plus a gradient. Give a vector field $\boldsymbol{F}$, the divergence-free part can be denoted by $\mathcal{P}\boldsymbol{F}$. This operator $\mathcal{P}$ is the "projection operator onto the subspace of divergence-free vector fields".

Since $\boldsymbol{u}$ is already divergence-free, we have $\mathcal{P}\boldsymbol{u} = \boldsymbol{u}$, and the same with its time derivative and with the viscosity term. As for the gradients on the right of (5.9), they are annihilated by $\mathcal{P}$: $\mathcal{P}\nabla p = 0$.

Therefore applying $\mathcal{P}$ to (5.9), we obtain

$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathcal{P}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nu\nabla^2\boldsymbol{u} + \frac{1}{\rho}\mathcal{P}\boldsymbol{f}. \tag{5.12}$$

This has the advantage that the pressure does not appear. The equation (5.11) is satisfied by simply requiring that $\boldsymbol{u}$ lie in the subspace of divergence-free fields. So we have an evolution problem in that subspace. Mathematically, this is a simpler formulation than (5.9).

5.3.2 Problems

a. Consider the shear flows in Section 4.7.2a. Suppose it is required that the flow satisfy the Navier-Stokes equations in the channel $\{0 < y < Y\}$ with $f = 0$, where Y is some given positive channel width. It is also required that $h(0) = 0,\ h(Y) = U$. Find all possible solutions $h(y),\ p$. In particular, what velocity profile corresponds to the case $p =$ constant?

b. Let $e = \frac{1}{2}|\boldsymbol{u}|^2$, the kinetic energy density per unit mass. Take the scalar product of (5.9) with $\boldsymbol{u}$ to derive the following conservation law for kinetic energy:

$$\frac{De}{Dt} = \nu\nabla^2 e - \text{div}\left(\frac{p}{\rho}\boldsymbol{u}\right) + \frac{1}{\rho}\boldsymbol{f}\cdot\boldsymbol{u} - \nu|\nabla\boldsymbol{u}|^2, \tag{5.13}$$

where $|\nabla\boldsymbol{u}|^2 = \sum_{i,j}\left(\frac{\partial u_j}{\partial x_i}\right)^2$. The first term on the right shows that energy diffuses, and the last one represents the dissipation of energy through the action of viscosity.

c. Acheson, page 218, 6.6.

d. Acheson, page 219, 6.12.

5.3.3 Poiseuille flow

This is simple steady flow of a viscous fluid down a pipe of uniform cross-section S, forced by a given pressure gradient.

Take x to be in the direction of the pipe; then we require $u = u(y,z),\ v = w = 0$. We also require that the velocity vanish on the boundary:

$$u(y,z) = 0, \quad (y,z) \in \partial S. \tag{5.14}$$

Then (5.9), first of all, implies that p is a function only of x, and secondly that u satisfies

$$u_t = \nu\nabla^2 u - \frac{1}{\rho}p_x. \tag{5.15}$$

There is a natural solution under the assumption that u is time-independent and p_x is constant. Then

$$\nabla^2 u = -P, \tag{5.16}$$

where $P = -\frac{1}{\mu}p_x = \text{constant}$.

Given P, there is a unique solution of (5.16), (5.14). I mentioned this PDE problem in (4.45). We get a unique velocity profile for this flow, which is called Poiseuille flow.

Problem Suppose S is the disk $\{r \le a\}$, where $r^2 = y^2 + z^2$. Show that $u = \frac{1}{4}(a^2 - r^2)P$. In this case of a circular pipe, the profile is a paraboloid and the maximum velocity is at the center, where it is $\frac{1}{4}a^2P$. Notice that this maximum is inversely proportional to the viscosity, so the flow has no inviscid limit as $\nu \to 0$, the other parameters being fixed.

If the evolution starts with an arbitrary initial profile (like maybe $u \equiv 0$), then it can be shown that the profile approaches the steady profile given by the problem exponentially in time.

5.3.4 Couette flow

This is shear flow between two moving plates or between two rotating cylinders. Instead of a pressure gradient forcing the motion, is in the moving boundaries. Explicit solutions can easily be found. One of them you have already found in Problem 5.3.2b. A prime question is that of the stability of these solutions. Although the solutions are extremely easy, this question is extremely difficult.

5.3.5 Vorticity formulation

Recall the derivation of (4.24), which involved taking the curl of the basic momentum equation and using vector identities.

If, as basic momentum equation, we use (5.9) with $\boldsymbol{f} = 0$, then the only additional term beyond the previous model is the viscosity term. Taking the curl of this additional term is easy: the curl operator commutes with the Laplace operator. As a result instead of (4.24), we obtain

$$\frac{D\boldsymbol{\omega}}{Dt} = \nu\nabla^2\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}. \tag{5.17}$$

This additional term says that the vorticity not only convects and changes its strength due to stretching, but it also diffuses. The diffusion coefficient is ν.

One immediate consequence of (5.17) is that all the irrotational incompressible flows we studied in Section 4.11 are solutions of the Navier-Stokes equations.

However, point vortices are not solutions. In fact under the action of viscosity, vorticity which at one time may be concentrated on points or filaments immediately begins to diffuse out to cover larger areas. We shall look at the prototypical situation.

Consider 2D flows for which the streamlines are circles centered at the origin. The vorticity vector $\boldsymbol{\omega} = \omega\boldsymbol{k}$ can be considered pointing in the third direction.

Problem. (i) Show that (5.17) reduces to

$$\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega. \tag{5.18}$$

In 2D there is a streamfunction ψ with $\boldsymbol{u} = \nabla^\perp \psi$.
(ii) Show that $\nabla^2 \psi = -\omega$.

In the case of circular streamlines this provides perhaps a possible way to determine (a) the vorticity, (b) the streamfunction by solving this Poisson equation, and finally (c) the velocity field. Knowing the proper boundary conditions for the vorticity is a big hangup, however.

Suppose that initially, ω is concentrated at the origin, so that $\omega(x, y, 0) = \Gamma \delta(x, y)$, where Γ, as before, is the strength. This is simply the point vortex we studied before in connection with ideal fluids. In that case it remained stationary. In the present case of a viscous fluid, however, we see now that the point vortex becomes smoothed out.

The equation (5.18) is the classical diffusion equation, and the solution with this initial condition is Γ times the classical "heat kernel" in 2D:

$$\omega(x, y, t) = \frac{\Gamma}{4\pi\nu t} \exp\{-\frac{r^2}{4\nu t}\}, \tag{5.19}$$

where $r^2 = x^2 + y^2$.

This says that as time progresses the vorticity distribution is bell-shaped and becomes ever flatter. But the integral of ω remains constant.

5.3.6 Reynolds number

Let us picture a flow field in which certain characteristic quantities can be identified:

U	velocity
L	length
T	time
P	pressure

Characteristic quantities such as these may be ill-defined; U for example may just be a velocity which is typical of the flow field. In steady turbulent flow through a tube, it could be taken as the mean velocity along the axis of the tube.

We define the dimensionless Reynolds number

$$Re = \frac{UL}{\nu} \tag{5.20}$$

and decide to set $P = \frac{\rho U^2 \nu}{Re}$. We then define dimensionless variables by setting $\boldsymbol{x} = L\boldsymbol{x}^*$, $\boldsymbol{u} = U\boldsymbol{u}^*$, etc. Then (5.11) remains the same, and if $\boldsymbol{f} = 0$ (5.9) becomes

$$\frac{L}{UT}\frac{\partial \boldsymbol{u}^*}{\partial t^*} + (\boldsymbol{u}^* \cdot \nabla^*)\boldsymbol{u}^* = \frac{1}{Re}\left(\nabla^2 \boldsymbol{u}^* - \nabla^* p^*\right). \tag{5.21}$$

In the case of "well-developed" flows with no t-dependent external forcing action, the characteristic time will be that related to L and U, so that $T = \frac{L}{U}$, and discarding the asterisks in (5.21), we get

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \frac{1}{Re}\left(\nabla^2 \boldsymbol{u} - \nabla p\right), \tag{5.22}$$

which has only one parameter—the Reynolds number.

On the other hand in problems for which there is a time-dependent forcing, there may be another natural choice for T.

In making the above scalings, it is implicitly assumed that the dimensionless variables (those in (5.22)) have magnitudes of the order of unity.

Then the Reynolds number has meaning as the ratio of the orders of magnitude of the inertia to the viscosity terms:

$$Re \sim \frac{\text{inertia terms}}{\text{viscosity terms}}. \tag{5.23}$$

The implication of (5.22) is that there is really only one parameter, Re, which determines the flow. Any solution of the nondimensional equation (5.22) can be scaled to provide solutions of other problems with appropriately scaled geometry and data, as long as Re is the same for the two problems. A prototypical problem is to find the flow past a bounded obstacle, with the velocity prescibed at ∞. If, for example, if we know the solution in a given domain V with imposed characteristic velocity U, then the solution can be scaled to apply to a domain of the same shape but twice as large, with same viscosity, as long as the prescribed U is reduced by one half. To obtain the new solution, one would simply reduce the original velocity field by half and multiply the vector $\boldsymbol{x}$ by two. The new pressure would be obtained from the old by scaling with the dimensional factor P.

5.4 Steady flows with very small Re

We first consider steady flows of a very viscous fluid. The time derivative in (5.22) vanishes, and the inertia terms, being relative small, will be neglected. In case $\boldsymbol{f} = 0$, we obtain the dimensionless homogeneous **Stokes equations**

$$\nabla^2 \boldsymbol{u} = \nabla p, \quad \operatorname{div} \boldsymbol{u} = 0, \tag{5.24}$$

These equations have to be solved in the given flow domain V under the no-slip boundary condition that

$$\boldsymbol{u} = 0 \quad \text{on } \partial V, \tag{5.25}$$

and if V is unbounded, there will generally be a condition "at ∞" to be satisfied, as in the examples below. If, in addition, there is a body force such as gravity acting, then it could be added to $(5.24)_1$. If the body force is conservative, one can do away with this extra term by redefining p to be pressure plus the potential.

The equations (5.24) are enormously simpler than (5.22), mainly because they are linear. It is easily proved, for example, that boundary problems have unique solutions. Also the corresponding evolution problems, in which the time derivative is present and initial conditions are applied, have unique solutions existing globally in time.

Let me mention some immediate properties of solutions of (5.24). Taking the divergence of the first of those equations and using the second, we find

$$\nabla^2 p = 0, \tag{5.26}$$

so that the pressure is a harmonic function.

Take the curl, which annihilates gradients, and we find

$$\nabla^2 \boldsymbol{\omega} = 0. \tag{5.27}$$

If there were a time derivative present, we would get

$$\frac{L\,Re}{UT}\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla^2 \boldsymbol{\omega}, \tag{5.28}$$

which is the ordinary heat equation (satisfied by each component of the vorticity).

Finally, taking the divergence of (5.24) and using (5.26), we have that

$$\nabla^4 \boldsymbol{u} = 0. \tag{5.29}$$

Thus in the homogeneous case, all flow quantities of interest are either harmonic or biharmonic. Next, we show that all this, plus a couple of mathematical ideas, can be used to solve the important problem of a Stokes flow past a sphere.

5.4.1 Digression on inverting the Laplacian in all space.

It will be convenient to use the standard alternative symbol $\Delta = \nabla^2$. We will be dealing with its inverse Δ^{-1}. Don't make the mistake of writing it ∇^{-2}; that may mislead one into thinking that ∇^{-1} has meaning; it does not. However, all powers Δ^α, such as $\Delta^{-\frac{1}{2}}$, do have meaning.

Fact. For a given function $\psi(\boldsymbol{x})$, defined in all of $\mathbf{R}^3$, continuously differentiable, and with compact support ($\psi = 0$ outside a bounded set), the equation

$$\Delta\phi = \psi \tag{5.30}$$

has a unique solution ϕ with $\phi(\boldsymbol{x})\to 0$ as $\boldsymbol{x}\to\infty$. This is a basic result. Moreover, we have the representation

$$\phi(\boldsymbol{x}) = -\int_{\mathbf{R}^3} \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\psi(\boldsymbol{y})d\boldsymbol{y}. \tag{5.31}$$

The kernel $\frac{1}{4\pi|\boldsymbol{x}|}$ is called the *fundamental solution* for the Laplace operator. It is a harmonic function except for $|\boldsymbol{x}| = 0$, where it is singular of course.

The uniqueness can be proved easily with the aid of the maximum principle. The same result is true if we no longer assume that ψ has compact support, but rather decays to 0 fast enough as $\boldsymbol{x}\to\infty$. If ψ does have compact support, it can be seen from (5.31), since the integration is really only over a bounded region—the support of ψ—that the solution ϕ decays at least as fast as r^{-1}, where $r = |\boldsymbol{x}|$.

We call the solution $\phi = \Delta^{-1}\psi$.

The same results hold if ψ is a distribution with compact support. In general, ϕ in this case will not be a classical solution, since it may have singularities.

Note on distributions. A Schwartzian distribution h is a linear functional (mapping to the real numbers) on the space of test functions ϕ, i.e. infinitely differentiable functions in $\mathbf{R^3}$ with compact support. The values of this functional are denoted by $\langle h, \phi\rangle$. Another requirement is that h be continuous in the sense that if $\{\phi^n\}$ is any sequence of test functions with supports contained in a common ball such that for every differential operator D, $\lim_{n\to\infty} D\phi^n = D\phi$ uniformly, then $\langle h, \phi^n\rangle\to\langle h, \phi\rangle$.

Any locally integrable function $h(\boldsymbol{x})$ can be considered a distribution, namely

$$\langle h, \phi\rangle = \int_{\mathbf{R^3}} h(\boldsymbol{x})\phi(\boldsymbol{x}). \tag{5.32}$$

Every distribution may be differentiated any number of times. For example, $\frac{\partial}{\partial x}h$ is defined as though integration by parts were valid:

$$\langle \frac{\partial}{\partial x}h, \phi\rangle = -\langle h, \frac{\partial}{\partial x}\phi\rangle. \tag{5.33}$$

Actually, we will only be dealing with distributions which are regular functions except at the origin. Examples are the delta function δ, defined by $\langle\delta,\phi\rangle = \phi(0)$, and its various derivatives.

For example, it should be clear from (5.31) that

$$\frac{1}{4\pi|\boldsymbol{x}|} = \frac{1}{4\pi r} = -\Delta^{-1}\delta(\boldsymbol{x}), \tag{5.34}$$

in the sense that if ϕ is any test function and $\psi = \Delta\phi$, we may formally integrate by parts and use (5.31) to get

$$\int \Delta\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\phi(\boldsymbol{y})d\boldsymbol{y} = \int \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\Delta\phi(\boldsymbol{y})d\boldsymbol{y} =$$
$$\int \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\psi(\boldsymbol{y})d\boldsymbol{y} = -\phi(\boldsymbol{x}) = -\int \delta(\boldsymbol{x}-\boldsymbol{y})\phi(\boldsymbol{y}). \tag{5.35}$$

Thus $\Delta\frac{1}{4\pi|\boldsymbol{x}|} = -\delta(\boldsymbol{x})$.

We shall need to attach a meaning to the statement "$\int_V h = 0$ for every open domain V containing the origin (in its interior)". It simply means that $\langle h,\phi\rangle = 0$ for all test functions ϕ which are identically equal to 1 in a neighborhood of the origin. The rationale for this definition is that the arbitrariness of ϕ away from the origin means that $\langle h,\phi\rangle$ vanishes if the support of ϕ does not include the origin, so that h should be 0 there. But near the origin $\phi\equiv 1$, so that $\langle h,\phi\rangle$ appears like $\int_V h$.

Finally, even if ψ doesn't decay fast enough at ∞, the equation (5.30) still will have a solution but it won't be unique. In any case, if (5.30) holds and $\phi\to 0$ as $\boldsymbol{x}\to\infty$, then this is the only solution which does that.

Consider the particular case when ψ is homogeneous of degree n, i.e. $\psi(\alpha\boldsymbol{x}) = \alpha^n\psi(\boldsymbol{x})$. Then a standard relation is

$$\boldsymbol{x}\cdot\nabla\psi(\boldsymbol{x}) = n\psi(\boldsymbol{x}) \tag{5.36}$$

(just differentiate the previous equation with respect to α and set $\alpha = 1$).

Fact. If ψ is homogeneous of degree $n \le -3$, and also satisfies $r^2\Delta\psi(\boldsymbol{x}) = \beta\delta(\boldsymbol{x})$, then

$$\Delta^{-1}\psi = \frac{1}{6+4n}\left(r^2\psi + \frac{\beta}{4\pi r}\right). \tag{5.37}$$

Problem: Prove this, by taking the Laplacian of $r^2\psi$ and then applying Δ^{-1}.

5.4.2 Solving the forced Stokes equations in all space

To make things a little clearer, we revert now to dimensional variables in (5.24) and also add a forcing term $\boldsymbol{f}(\boldsymbol{x})$ to get

$$\mu\Delta\boldsymbol{u} = \nabla p - \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{u} = 0, \tag{5.38}$$

We show how to invert this "operator" and find $(\boldsymbol{u}, p)$ in terms of $\boldsymbol{f}$, when all these functions and their derivatives are required to decay to zero at infinity.

Recall that $\boldsymbol{f}$ has the meaning of force density, so that the total external applied force acting on a blob V of fluid is $\int_V \boldsymbol{f}$.

Applying the divergence operator to (5.38) and using the fact that $\boldsymbol{u}$ is divergence-free, we obtain $\Delta p = \nabla\cdot\boldsymbol{f}$, so that

$$p = \Delta^{-1}\nabla\cdot\boldsymbol{f}. \tag{5.39}$$

From (5.38) we now have

$$\boldsymbol{u} = \frac{1}{\mu}\left[\Delta^{-1}\nabla\Delta^{-1}\nabla\cdot\boldsymbol{f} - \Delta^{-1}\boldsymbol{f}\right]. \tag{5.40}$$

The second equation in (5.38) is guaranteed by (5.39) and the required decay of $\boldsymbol{u}$ and its derivatives at ∞.

5.4.3 Motion of a sphere through a viscous medium

The picture we have in this section is that of a sphere of radius a moving at constant velocity U in the x-direction through a Stokes fluid which is at rest at ∞, so that by the no-slip condition,

$$\boldsymbol{u} = (U, 0, 0) \text{ for } r = a. \tag{5.41}$$

The sphere will exert some force $\boldsymbol{F} = (F, 0, 0)$ on the fluid. The important information to predict is the force/velocity relation; more specifically the relation among U, a, and F. Alternatively, we can consider the steady flow of a viscous fluid past a fixed sphere, with prescribed velocity at infinity, and predict the force on the sphere. The two problems are equivalent, because the equations (5.24) are linear. To get one solution from the other, just add or subtract a constant solution.

From the former point of view, we can think of the force as being prescribed; we would then wish to find the velocity of the fluid at the boundary of the object. That velocity should be constant. More generally, we could seek the entire velocity field.

Taking a clue from the potential flow past a cylinder considered in Section 4.11.4, we shall try to construct the solution by supposing that the velocity and pressure fields can be extended, remaining solutions, inside the sphere except for a singular point at the origin. Another approach would be to work with spherical functions, solving the problem without extending the solution to the inside. Looking for solutions with singularities at the origin, however, will lead naturally into our later study of the motion of microorganisms in water.

Our first attempt is very natural: we look at the solution (5.40), supposing that the force $(F, 0, 0)$ is concentrated at the origin. In that case, $\boldsymbol{f} = \boldsymbol{F}\delta(\boldsymbol{x})$, where $\boldsymbol{F} = (F, 0, 0) = F\boldsymbol{e}_1$, $\boldsymbol{e}_1 = (1, 0, 0)$. Then by integrating, we see that $\boldsymbol{F}$ will be the total external force on any blob of fluid containing the origin in its interior.

Call the solution $\boldsymbol{u}_0$. From (5.40), we have, since $\nabla\cdot\boldsymbol{f} = F\partial_1\delta(\boldsymbol{x})$,

$$\boldsymbol{u}_0 = \frac{1}{\mu}\left[\Delta^{-1}\nabla\Delta^{-1}\nabla\cdot(\boldsymbol{F}\delta) - \Delta^{-1}\boldsymbol{F}\delta\right] = \frac{1}{\mu}F\left[\Delta^{-1}\nabla\Delta^{-1}\frac{\partial}{\partial x_1}\delta - \Delta^{-1}\delta\boldsymbol{e}_1\right]. \tag{5.42}$$

Using (5.34), we have

$$\boldsymbol{u}_0 = -\frac{1}{\mu}F\left[\Delta^{-1}\nabla\frac{\partial}{\partial x_1}\frac{1}{4\pi r} - \frac{1}{4\pi r}\boldsymbol{e}_1\right]. \tag{5.43}$$

At this point, let me introduce the notation $\partial_i = \frac{\partial}{\partial x_i}$ and note that

$$\partial_i r^n = n x_i r^{n-2}, \quad \partial_i\partial_j r^n = n(n-2)x_i x_j r^{n-4} + \delta_{ij} n r^{n-2}. \tag{5.44}$$

From this, we see that (5.43), in component form, becomes (using the Kronecker symbol)

$$u_{0,i} = \frac{1}{4\pi\mu} F \left[\Delta^{-1}(-3x_i x_1 r^{-5} + \delta_{1i} r^{-3}) + \frac{1}{r}\delta_{i1} \right]. \tag{5.45}$$

We now apply (5.37) with $\psi = \partial_i \partial_1 \frac{1}{4\pi r}$, so that $n = -3$. We need to calculate $r^2 \Delta\psi = r^2 \partial_i \partial_1 \Delta \frac{1}{4\pi r} = -r^2 \partial_i \partial_1 \delta$.

Problem: Show that

$$r^2 \partial_i \partial_1 \Delta \frac{1}{4\pi r} = -2\delta_{i1}\delta(r) \tag{5.46}$$

(here the first factor on the right is the Kronecker symbol).

Therefore in (5.37), $\beta = -2\delta_{i1}$, so we obtain that $\Delta^{-1}(-3x_i x_1 r^{-5} + \delta_{1i} r^{-3}) = \frac{1}{2}(x_i x_1 r^{-3} - \delta_{1i} r^{-1})$. Hence from (5.45),

$$u_{0,i} = \frac{1}{4\pi\mu} F \left[\frac{1}{2}(x_i x_1 r^{-3} - \delta_{1i} r^{-1}) + \frac{1}{r}\delta_{i1} \right]. \tag{5.47}$$

In vector form,

$$\boldsymbol{u}_0 = \frac{F}{4\pi\mu} \left[\frac{1}{2} x_1 \boldsymbol{x} r^{-3} + \frac{1}{2} r^{-1} \boldsymbol{e}_1 \right]. \tag{5.48}$$

This simple solution $\boldsymbol{u}_0$ resulting from a point force is called a *stokeslet*. In this case, it satisfies all the requirements except that it is not constant on the sphere $\{r = a\}$, nor on any other sphere for that matter.

We are going to supplement $\boldsymbol{u}_0$ with another vector field $\tilde{\boldsymbol{u}}$ to make up for the deficiency, so that $\boldsymbol{u} = \boldsymbol{u}_0 + \tilde{\boldsymbol{u}}$. There is no need to require $\tilde{\boldsymbol{u}}$ to be divergence-free everywhere—just outside the sphere, where the velocity field is our concern (see the caveat below). But since we are dealing with functions with singular behavior only at the origin, we allow the new velocity field to have a sort of creation and annihilation of mass at the origin in a singular way. However, we do require $\int_{r<a} \nabla\cdot\tilde{\boldsymbol{u}} = 0$ (see the definition in Section 5.4.1); otherwise we could expect that $\int_{r=a} \tilde{\boldsymbol{u}}\cdot\boldsymbol{n} = \int_{r=a} \boldsymbol{u}\cdot\boldsymbol{n} = \int_{r=a} U\boldsymbol{e}_1\cdot\boldsymbol{n} = U\int_{r=a} n_1 \neq 0$, which is false and gives a contradiction.

Making such an addition to $\boldsymbol{u}_0$, we get from (5.38) $\mu\Delta(\boldsymbol{u}_0 + \tilde{\boldsymbol{u}}) = \nabla p - (\boldsymbol{f} - \mu\Delta\tilde{\boldsymbol{u}})$, or

$$\mu\Delta\boldsymbol{u}_0 = \nabla p - (\boldsymbol{f} + \tilde{\boldsymbol{f}}), \quad \tilde{\boldsymbol{f}} = -\mu\Delta\tilde{\boldsymbol{u}}. \tag{5.49}$$

Again, we may allow such a $\tilde{\boldsymbol{u}}$ provided that the extra forces $\tilde{\boldsymbol{f}}$ generated by it do not add to the total force. This is true if $\int_{r<a} \Delta\tilde{\boldsymbol{u}} = 0$.

To select the additional term $\tilde{\boldsymbol{u}}$ meeting these requirements, we take a clue from (5.44). We wish $\tilde{\boldsymbol{u}}$ to have terms with $x_1\boldsymbol{x}$ cancelling those in $\boldsymbol{u}_0$ at $r = a$ given in (5.48) (those latter terms are what was preventing the velocity $\boldsymbol{u}_0$ from being constant on the spherical surface). And from (5.44), clearly this can be done if $\tilde{\boldsymbol{u}}$ is of the form $\tilde{\boldsymbol{u}} = A\nabla\partial_1 r^n$ for some n and A. With this choice,

$$\nabla\cdot\tilde{\boldsymbol{u}} = A\Delta\partial_1 r^n = A\partial_1\Delta r^n \tag{5.50}$$

and

$$\Delta\tilde{\boldsymbol{u}} = A\nabla\partial_1\Delta r^n. \tag{5.51}$$

We want

$$\int_V \nabla\cdot\tilde{\boldsymbol{u}} = \int_V \Delta\tilde{\boldsymbol{u}} = 0 \tag{5.52}$$

for all V containing the origin. The easiest way to do this is to have $\Delta r^n = 0$ for $r \neq 0$. This dictates $n = -1$ by (5.34) (unless $n = 0$, which does not interest us). And then $\Delta r^{-1} = -4\pi\delta$, and (5.52) holds since both expressions (5.50) and (5.51) involve the derivative ∂_1. For example, if $\phi \equiv 1$ near 0, we have $\langle \nabla \cdot \tilde{\boldsymbol{u}}, \phi \rangle = A\langle \partial_1 \nabla r^{-1}, \phi \rangle = -A\langle \nabla r^{-1}, \partial_1 \phi \rangle = 4\pi A \langle \delta, \partial_1 \phi \rangle = 4\pi A \partial_1 \phi(0) = 0$.

Thus from (5.44)

$$\tilde{u}_i = A\partial_i \partial_1 r^{-1} = A(3x_1 x_i r^{-5} - \delta_{1i} r^{-3}), \tag{5.53}$$

i.e.

$$\tilde{\boldsymbol{u}} = 3Ax_1 \boldsymbol{x} r^{-5} - Ar^{-3}\boldsymbol{e}_1. \tag{5.54}$$

We may now choose A to cancel the unwanted terms in (5.48): $A = -\frac{F}{24\pi\mu}a^2$, so that (5.41) becomes

$$\boldsymbol{u} = \boldsymbol{u}_0 + \tilde{\boldsymbol{u}} = \frac{F}{8\pi\mu}a^{-1}\boldsymbol{e}_1 + \frac{F}{24\pi\mu}a^{-1}\boldsymbol{e}_1 = U\boldsymbol{e}_1. \tag{5.55}$$

This condition tells us that $U = \frac{F}{6\pi\mu a}$.

In all, we obtain the relation

$$F = 6\pi\mu a U. \tag{5.56}$$

This is the classical expression for the drag caused by flow past a sphere, or alternatively when a sphere is falling through a viscous liquid at a steady velocity under the action of gravity, then this expression is equal to the weight of the sphere, corrected for buoyancy. It has been verified well by experiments. Millikan made the expression famous through his oil-drop experiment.

5.4.4 Bubbles

The motion of a bubble through a very viscous liquid is modeled in a way similar to that of a solid ball, except that the boundary conditions on the surface $\{r = a\}$ are different. Instead of adhering, the fluid particles move tangentially; and there is no viscous stress. All this means the radial component of the velocity is zero, as well as the derivative with respect to r of the tangential components of the velocity at $\{r = a\}$. All this is with respect to a coordinate system attached to the moving bubble. We suppose the bubble is moving with unknown velocity in the x-direction.

To formulate the problem, let $\boldsymbol{v} = \boldsymbol{u} - U\boldsymbol{e}_1$, where U is the unknown velocity. Thus

$$\boldsymbol{v}(\infty) = -U\boldsymbol{e}_1. \tag{5.57}$$

Similarly, we let $\boldsymbol{v}_0 = \boldsymbol{u}_0 - U\boldsymbol{e}_1$, where u_0 is given by (5.48).

Typically, the force F in (5.48) and elsewhere will be the bubble's buoyancy force and $\boldsymbol{x}$ will be the vertical direction.

Let $\boldsymbol{n} = \frac{\boldsymbol{x}}{r}$, the unit vector in the radial direction. The radial component of $\boldsymbol{v}$ is $v_r = \boldsymbol{v} \cdot \boldsymbol{n}$. The tangential component is defined by

$$\boldsymbol{v}_\theta = \boldsymbol{v} - v_r \boldsymbol{n}. \tag{5.58}$$

Problem 1. Rewrite (5.48) to express the stokeslet $\boldsymbol{u}_0$ in terms of r, $\boldsymbol{n}$ and n_1 (x-component of $\boldsymbol{n}$) only. Show that the radial component

$$u_{0r} = \frac{F}{4\pi\mu r} n_1 \tag{5.59}$$

and

$$\boldsymbol{u}_{0\theta} = \frac{F}{8\pi\mu r}(-n_1 \boldsymbol{n} + \boldsymbol{e}_1). \tag{5.60}$$

Find $\frac{\partial}{\partial r}\boldsymbol{u}_{0\theta}$.

Problem 2. Representing $\boldsymbol{v} = \boldsymbol{u}_0 + \tilde{\boldsymbol{u}} - U\boldsymbol{e}_1$ where $\tilde{\boldsymbol{u}}$ is given by (5.54) with unknown A, choose A and U so that for $r = a$,

$$v_r = 0 \tag{5.61}$$

and

$$\frac{\partial}{\partial r}\boldsymbol{v}_\theta = 0. \tag{5.62}$$

From this, write U as a function of the force F and radius a. Compare it to (5.56).

5.4.5 Distributed sources

Besides the particular stokeslet we constructed in (5.48) corresponding to a concentrated force in the x_1 direction, there are two others for the other two orthogonal directions. Changing notation and setting the force magnitude to unity, we define

$$\boldsymbol{U}_j = \frac{1}{4\pi\mu}\left[\frac{1}{2}x_j\boldsymbol{x}r^{-3} + \frac{1}{2}r^{-1}\boldsymbol{e}_j\right], \quad j = 1, 2, 3. \tag{5.63}$$

They are solutions of the Stokes problem (5.38) with $\boldsymbol{f} = \delta(\boldsymbol{x})\boldsymbol{e}_j$. They are quite analogous to the fundamental solution $\frac{1}{4\pi r}$ of the Laplace operator, which satisfies $\nabla\phi = -\delta(\boldsymbol{x})$, and in fact the triple $\{\boldsymbol{U}_j\}$ can be called the fundamental solution of the Stokes operator.

If we have a concentrated force given by $\boldsymbol{F}\delta(\boldsymbol{x})$ where $\boldsymbol{F} = (F_1, F_2, F_3)$, then by linearity the solution is $F_j\boldsymbol{U}_j$, summation being performed.

Suppose we now have a distributed force density field $\boldsymbol{f}(\boldsymbol{x})$. The solution, which was given in another form in (5.40), can now be written more explicitly as

$$\boldsymbol{u}(\boldsymbol{x}) = \int_{\mathbf{R}^n} f_j(\boldsymbol{y})\boldsymbol{U}_j(\boldsymbol{x} - \boldsymbol{y})d\boldsymbol{y}. \tag{5.64}$$

Of course one has to worry about convergence and whether this solution decays as infinity. Certainly the integral converges and the decay occurs if $\boldsymbol{f}$ is differentiable and has compact support. The fundamental solution in this context has the same "$\frac{1}{r}$" singularity as the Laplace equation fundamental solution.

Representations similar to (5.64) can be written if the force field is concentrated on a surface or a line.

5.5 Swimming microrganisms (from Lighthill)

Up until now we have dealt with time-independent flows, but swimming organisms definitely cause time dependence. In the dimensionless form (5.21) of the Navier-Stokes equations, we must therefore worry about the time derivative, which has order of magnitude $\frac{L}{UT}$. Disregarding transient effects occurring only during the start-up of the motion, one may neglect this time derivative, provided that

$$\frac{L}{UT} \ll \frac{1}{Re}. \tag{5.65}$$

While generally not true for fish, (5.65) is usually valid for the motion of microorganisms, and we shall neglect it in our treatment of them. This means that we shall still seek solutions of the "stationary" Stokes equations (5.38) for each instant of time. The solution $(\boldsymbol{u}, p)$, and of course the forcing term $\boldsymbol{f}$, will depend on time. This type of flow is called "pseudo-steady-state".

Bacteria and other microrganisms often propel themselves with a long attached appendage. If they tried wiggling this tail, they would get nowhere. On the other hand, it can be argued

effectively that if the appendage is formed into a helix called a flagellum and rotated like a ship's propeller, that is quite an efficient way to travel. So a flagellum is a helix-like appendage which is stiff, except that it can be rotated about its axis. We shall assume it is not only stiff but rigid. Our basic problem is this: given the motion of such a rigid curved object, how can one predict the force which the motion exerts on the fluid? This is the same as the propelling force which the fluid exerts on the flagellum. This velocity/force relationship would be analogous to (5.56), except that the object being moved is much more complex than a mere sphere.

Note that (5.56) can be viewed as a force balance equation, valid when the motion is steady, so that there is no acceleration. In fact the left side is the external gravitational force on the object and the right side is the viscous force which the fluid applies to it. The two forces balance, so that there is no net force on the ball, and therefore no acceleration. What we really want to do below is to find such a force balance equation for steady motion of the microorganism.

5.5.1 A first attempt at modeling

Since flagellae have very small cross-section, we might think of them as curves in space which move. In that case, the forces will be concentrated on that curve, and one might propose to represent the velocity field in terms of that one-dimensional force distribution as an integral like (5.64):

$$\boldsymbol{u}(\boldsymbol{x}) = \int_C f_j(\boldsymbol{y}) \boldsymbol{U}_j(\boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{y}, \tag{5.66}$$

where f_j is the j-th component of the linear force density on the flagellum $C = C(t)$. One would then evaluate this velocity field on the flagellum's surface and so get a force/velocity relationship.

Let me bring out some difficulties with this procedure.

(1) At each point along the flagellum's axis (centerline) and at each time t, the velocity field resulting from (5.66) ideally should be constant on the boundary of the flagellum's cross-section at that point. The reason is that we will be assuming the flagellum has a given unique velocity at each point along it. In other words, each point of $C(t)$ will have a given velocity. This is similar to our requirement (5.41) of constant velocity on the surface of the falling sphere.

However, the field given by (5.66) generally will not be constant on any cross-section boundary. Taking a lesson from the sphere problem in Section 5.4.3, we can say at least that the field (5.66) should be supplemented by a linear distribution of velocity fields analogous to (5.54). But even then, because of the curvature of the flagellum's axis, it is not clear that any such distribution would satisfy this requirement.

In the model we develop, we will be satisfied with only an approximate constancy of $\boldsymbol{u}$ on the cross-section boundaries.

(2) Even if difficulty (1) were surmounted, what we would have is a relation giving the velocity distribution $\boldsymbol{u}(s,t)$, where s designates position along the flagellum, as it depends on the distribution $\boldsymbol{f}(s,t)$. In other words, one *entire* function depends on another entire function, as opposed to *pointwise* dependence of velocity on force. This is a difficult situation, because it would be necessary to invert this relationship in order to find how the force distribution depends on the velocity distribution. Inverting a functional dependence like this is far more difficult than inverting a pointwise dependence.

In view of these difficulties, we shall resort to a more approximate model, based on certain assumptions. But first, let us examine the special case of a straight rod instead of a curved flagellum.

5.5.2 Resistance of cylinders

Instead of the ball of radius a considered in Section 5.4.3, think now of a long slender cylinder of length ℓ whose axis is on the x_1-axis, extending (when $t = 0$, say) from $-\ell/2$ to $\ell/2$, and whose radius is a. We assume

$$a \ll \ell. \tag{5.67}$$

The same problem as in that section is considered: If the cylinder is propelled in the x_1-direction with a force F and reaches a steady velocity U, what is that velocity? At any instant of time, the Stokes equations hold.

We suppose the propelling force is evenly distributed along the length of the cylinder, and by extending the flow field inside it, to suppose it is concentrated on the cylinder's axis. Let $f = \frac{F}{\ell}$ be the linear force density. We represent the velocity field as in (5.64), except that the integration is along the cylinder's axis. The field $\boldsymbol{f} = f\boldsymbol{e}_1$ is in the x_1-direction, so that $\boldsymbol{f}_2 = \boldsymbol{f}_3 = 0$.

We obtain from (5.66)

$$\boldsymbol{u}(x_1, x_2, x_3) = \int_{-\ell/2}^{\ell/2} f\boldsymbol{U}_1(x_1 - \xi, x_2, x_3)d\xi. \tag{5.68}$$

The integration variable ξ extends from one end of the cylinder to the other.

At $x_1 = 0$, we find from (5.63)

$$\boldsymbol{u}(0, x_2, x_3) = \int_{-\ell/2}^{\ell/2} \frac{f}{4\pi\mu}\left[-\frac{1}{2}\xi\hat{\boldsymbol{x}}r^{-3} + \frac{1}{2}r^{-1}\boldsymbol{e}_1\right] d\xi, \tag{5.69}$$

where now $r^2 = \xi^2 + \sigma^2$, $\sigma^2 = x_2^2 + x_3^2$ and $\hat{\boldsymbol{x}} = (-\xi, x_2, x_3)$.

We require, if possible, that this velocity equal a constant $U\boldsymbol{e}_1$ when $\sigma = a$. Since the second and third components of the first term in the integrand of (5.69) are odd functions of ξ, we see indeed that those components of (5.69) vanish. Also, the first component depends on x_1 and x_2 only in the combination σ. Therefore our requirement is satisfied (when $x_1 = 0$). We will now estimate U.

First, consider the last term in (5.69). There appears an integral like

$$\int_{-\ell/2}^{\ell/2} r^{-1}d\xi = \int_{-\ell/2}^{\ell/2} \frac{d\xi}{(\sigma^2 + \xi^2)^{1/2}}. \tag{5.70}$$

We evaluate this for $\sigma = a$. Changing the integration variable to $\xi' = \frac{\xi}{a}$, we get

$$\int_{-\ell/2a}^{\ell/2a} \frac{d\xi'}{(1 + (\xi')^2)^{1/2}} \tag{5.71}$$

We want to approximate this for large ℓ/a. Since we are integrating an even function, the integral is twice the integral from 0 to $\ell/2a$. For large ξ', the integrand decays like $\frac{1}{\xi'}$, and so we expect the integral to behave like a constant plus $2\ln(\ell/2a)$. In fact, we find

$$\int_{-\ell/2a}^{\ell/2a} \frac{d\xi'}{(1 + (\xi')^2)^{1/2}} = 2\ln(\ell/2a) + 2\int_1^{\ell/2a} \left[\frac{1}{(1 + (\xi')^2)^{1/2}} - \frac{1}{\xi'}\right] d\xi' + 2\int_0^1 \frac{d\xi'}{(1 + (\xi')^2)^{1/2}}. \tag{5.72}$$

The last term in (5.72) is just some constant C_1 and the integrand in the second term on the right is $\frac{\xi' - (1+(\xi')^2)^{1/2}}{\xi'(1+(\xi')^2)^{1/2}}$. Using the inequality $\sqrt{1+\alpha} - 1 \le \frac{1}{2}\alpha$ with $\alpha = (\xi')^{-2}$, we get that this integand

is bounded (for $1 \le \xi' < \infty$) by $C_2(\xi')^{-2}$, so that the integral of it from 1 to ∞ converges, and the second term is bounded independently of $\ell/2a$. In all, we get (since $\ln(\ell/2a) = \ln(\ell/a) - \ln 2$)

$$\int_{-\ell/2}^{\ell/2} r^{-1} d\xi = 2\ln(\ell/a) + S_1, \tag{5.73}$$

where S_1 is bounded independently of ℓ/a.

For very large $\ell/2a$, therefore, the logarithmic term in (5.73) dominates, being much larger than S_1.

If, instead of setting $x_1 = 0$ we set x_1 to be another value not too close to the endpoints $\pm\ell/2$, we would get essentially the same estimate.

Problem 1 Show that if we set $x_1 = \ell/4$ rather than 0, we could shift the variable of integration so that the last term in (5.69) is $\int_{-\ell/4}^{3\ell/4} \frac{f}{8\pi\mu} \hat{r}^{-1} \boldsymbol{e}_1 d\hat{\xi}$, $\hat{r}^2 = \hat{\xi}^2 + \sigma^2$. Show also that the integral analogous to (5.73) becomes

$$\int_{-\ell/4}^{3\ell/4} \hat{r}^{-1} d\hat{\xi} = 2\ln(\ell/a) + S_1', \tag{5.74}$$

where S_1' is another bounded constant. Hint: write the integral as $\int_{-\ell/4}^{\ell/4} + \int_{\ell/4}^{3\ell/4}$; use the above analysis on the first, and show the second is bounded (by what?).

Now consider the first term on the right of (5.69). The x_1-component provides an integral like

$$\int_{-\ell/2}^{\ell/2} \xi^2 r^{-3} d\xi = \int_{-\ell/2}^{\ell/2} \frac{\xi^2 d\xi}{(\sigma^2 + \xi^2)^{3/2}}. \tag{5.75}$$

The same analysis as before gives that this integral

$$= 2\ln(\ell/a) + S_3. \tag{5.76}$$

Combining these results, we find from (5.69) that the x_1-component of the velocity is $\frac{f}{4\pi\mu}\left[\frac{1}{2}(2\ln(\ell/a) + S_1) + \frac{1}{2}(2\ln(\ell/a) + S_3)\right]$. Therefore

$$u_1(0, x_2, x_3) = \frac{f}{2\pi\mu}(\ln(\ell/a) + S_4), \tag{5.77}$$

where $S_4 = \frac{1}{4}(S_1 + S_3)$.

This gives the velocity field on the surface of the cylinder ($\sigma = a$) on the plane perpendicular to the cylinder at the midpoint between its two ends.

[Because of Problem 1, the same formula (5.77) holds if the plane is not located at the midpoint, but rather at any point no further from the midpoint than $\ell/4$ (say).]

Setting $u_1 = U$ in (5.77), we get a force/velocity relation for the cylinder

$$R_\tau = \frac{f}{U} = \frac{2\pi\mu}{\ln(\ell/a) + S_4}, \tag{5.78}$$

which is valid at points not too far from the ends. The subscript "τ" denotes "tangential", because the force and velocity are tangential to the cylinder. This ratio may be termed the "resistance" of the cylinder when its axis is parallel to the force.

This was all for a cylinder oriented in the direction of the force. Suppose we consider one which is oriented perpendicular to $\boldsymbol{f}$. Let its axis be on the x_3 axis. Then we get an expression similar to (5.69), when $x_3 = 0$:

$$\boldsymbol{u}(x_1, x_2, 0) = \int_{-\ell/2}^{\ell/2} \frac{f}{4\pi\mu} \left[\frac{1}{2} x_1 \hat{\boldsymbol{x}} r^{-3} + \frac{1}{2} r^{-1} \boldsymbol{e}_1 \right] d\xi, \tag{5.79}$$

where this time $\hat{\boldsymbol{x}} = (x_1, x_2, -\xi)$ and $r^2 = \xi^2 + \sigma^2$, but now $\sigma^2 = x_1^2 + x_2^2$. The integration variable ξ is now in the x_3 direction.

Previously in (5.69), the integrand depended on x_2 and x_3 only through σ, so that the x_1-component of the velocity was constant on the surface of the cylinder. That is no longer true this time, and we are obliged to add a velocity correction like $\tilde{\boldsymbol{u}}$ in (5.54) in order to achieve that constancy. However, it turns out that this correction contributes a bounded rather than logarithmic term to the approximate formula (5.78). We therefore incorporate it in the quantity S_5 below.

We perform the same approximation procedure as before to the two terms in (5.79). Contrary to the case when the cylinder was parallel to the force, the first term in (5.79) contributes a bounded rather than logarithmic expression. The last term, however, is the same as before. The total logarithmic contribution this time is therefore half that previously. We have, in place of (5.78),

$$R_\nu = \frac{f}{U} = \frac{4\pi\mu}{\ln(\ell/a) + S_5}, \tag{5.80}$$

the subscript "ν" meaning "normal", where S_5 is bounded independently of ℓ/a. In (5.80) and (5.78), the values of S_4 and S_5 at $\ell/a = \infty$ can be calculated and those values may be used there to produce a valid approximation.

Finally if the cylinder is oriented at some other angle to $\boldsymbol{f}$, we may write the force and velocity as sums of parts tangential and normal to the cylinder:

$$\boldsymbol{f} = \boldsymbol{f}_\tau + \boldsymbol{f}_\nu, \quad \boldsymbol{u} = \boldsymbol{u}_\tau + \boldsymbol{u}_\nu. \tag{5.81}$$

By linearity of the problem, $\boldsymbol{f}_\tau$ and $\boldsymbol{u}_\tau$ are related by (5.78), where the ratio is interpreted as follows: $\boldsymbol{f}_\tau$ and $\boldsymbol{u}_\tau$ are in the same direction, say $\boldsymbol{f}_\tau = f_\tau \boldsymbol{\tau}$, $\boldsymbol{u}\tau = u_\tau \boldsymbol{\tau}$. Then $\frac{f}{U}$ in (5.78) means $\frac{f_\tau}{u_\tau}$. Similarly, $\boldsymbol{f}_\nu$ and $\boldsymbol{u}_\nu$ are related by (5.80) with the analogous interpretation of $\frac{f}{U}$.

Ultimately, only the ratio $\mathcal{R}$ of the two resistances will be of primary concern:

$$\begin{aligned} \mathcal{R} = \frac{R_\nu}{R_\tau} &= 2\frac{\ln(\ell/a) + S_4}{\ln(\ell/a) + S_5} = 2\frac{1 + S_4(\ln(\ell/a))^{-1}}{1 + S_5(\ln(\ell/a))^{-1}} \\ &= 2 + O\left((\ln(\ell/a))^{-1}\right) \quad \text{as } \ell/a \to \infty. \end{aligned} \tag{5.82}$$

5.5.3 Lighthill's resistive-force theory

Lighthill proposed the following theory (which will be critiqued in the next section) to determine the force/velocity relation for a flagellum whose centerline we represent as a moving space curve $C(t)$: Let s be the arclength parameter along C. Suppose we knew the velocity $\boldsymbol{u}(s,t)$ of the curve at each (s,t). This would be the velocity of the material portion of the flagellum located at position s. In particular, we would have the parts $\boldsymbol{u}_\tau(s,t)$ and $\boldsymbol{u}_\nu(s,t)$ (as in (5.81) above). The idea is to then find $\boldsymbol{f}_\tau(s,t)$ and $\boldsymbol{f}_\nu(s,t)$ at each point by (5.80) and (5.78) with a "suitable" choice of ℓ, and hence the total force at each point $\boldsymbol{f} = \boldsymbol{f}_\tau + \boldsymbol{f}_\nu$.

For a flagellum being propelled in the x_1-direction, say, the velocity $\boldsymbol{u}(s,t)$, as explained following (5.81), would be composed of two parts: One part is the velocity $\boldsymbol{u}^0(s,t)$ which it would have were it not being propelled, for example if it were out of the water but still rotating (we call this its intrinsic motion). That part is given. The other part is the velocity $U\boldsymbol{e}_1$ by which it is being propelled (but U is not yet known).

We could then determine the unknown velocity U by imposing the simple force balance relation that the integral of the x_1 component of $\boldsymbol{f}$ be zero; this is the condition of no acceleration, and is like the force balance law (5.56). Remember, $\boldsymbol{f}$ has been determined as described above in terms of $\boldsymbol{u}$, hence of U.

Let us go into this procedure in more detail. Suppose the flagellum's position and intrinsic motion is given by the vector function $\boldsymbol{X}^0(s,t)$, where $0 \le s \le L$ is the arclength, the total length of the flagellum if it were stretched out being L. The actual position of the flagellum in motion is

$$\boldsymbol{X}(s,t) = \boldsymbol{X}^0(s,t) + Ut\boldsymbol{e}_1. \tag{5.83}$$

Then the unit tangential vector is

$$\boldsymbol{\tau}(s,t) = \boldsymbol{X}^0_s(s,t) \tag{5.84}$$

and the intrinsic velocity is

$$\boldsymbol{u}^0(s,t) = \boldsymbol{X}_t(s,t). \tag{5.85}$$

We have

$$\boldsymbol{u}_\tau(s,t) = (\boldsymbol{u}\cdot\boldsymbol{\tau})\boldsymbol{\tau} = ((\boldsymbol{u}^0(s,t) + U\boldsymbol{e}_1)\cdot\boldsymbol{\tau}(s,t))\boldsymbol{\tau}(s,t), \qquad \boldsymbol{u}_\nu = \boldsymbol{u} - \boldsymbol{u}_\tau. \tag{5.86}$$

Now suppose for example that the flagellum is in the shape of a rotating circular helix:

$$\boldsymbol{X}^0(s,t) = (\alpha s, A\cos(\beta s + \omega t), A\sin(\beta s + \omega t)), \tag{5.87}$$

where A is the amplitude of the helix and ω is its angular frequency. We have

$$\boldsymbol{X}(s,t) = (\alpha s + Ut, A\cos(\beta s + \omega t), A\sin(\beta s + \omega t)). \tag{5.88}$$

Also

$$\boldsymbol{\tau} = \boldsymbol{X}_s = (\alpha, -\beta A\sin(\beta s + \omega t), \beta A\cos(\beta s + \omega t)). \tag{5.89}$$

For this to be a unit vector, we have the constraint

$$|\boldsymbol{\tau}|^2 = \alpha^2 + \beta^2 A^2 = 1. \tag{5.90}$$

The pitch of the helix will be defined by

$$\boldsymbol{\tau}\cdot\boldsymbol{e}_1 = \alpha < 1. \tag{5.91}$$

From (5.88), we can see that

$$\alpha = \frac{dx_1}{ds}. \tag{5.92}$$

Knowing α and A determines the one remaining parameter β from (5.90).

We now proceed to do the calculations in this case. We have

$$\boldsymbol{u} = (U, -A\omega\sin(\beta s + \omega t), A\omega\cos(\beta s + \omega t)), \tag{5.93}$$

$$\boldsymbol{u}_\tau = (\boldsymbol{u}_\tau \cdot \boldsymbol{\tau})\boldsymbol{\tau} = (\alpha U + \beta A^2 \omega)\boldsymbol{\tau}, \tag{5.94}$$

$$\boldsymbol{u}_\nu = \boldsymbol{u} - \boldsymbol{u}_\tau = (U - \alpha^2 U - \alpha\beta A^2 \omega, \dots) \tag{5.95}$$

where I did not write the 2nd and 3rd components.

Given these velocity components, we now calculate the corresponding frictional forces from (5.78), (5.80):

$$\boldsymbol{f}_\tau = R_\tau \boldsymbol{u}_\tau, \quad \boldsymbol{f}_\nu = R_\nu \boldsymbol{u}_\nu, \quad \boldsymbol{f} = R_\tau \boldsymbol{u}_\tau + R_\nu \boldsymbol{u}_\nu. \tag{5.96}$$

We shall be interested in the component f_1 of $\boldsymbol{f}$ along the x_1-axis, i.e. along the axis of the rotating flagellum. This will be the thrust of the propeller. We have

$$f_1 = \boldsymbol{f} \cdot \boldsymbol{e}_1 = R_\tau(\alpha U + \beta A^2 \omega)\alpha + R_\nu(U - \alpha^2 U - \alpha\beta A^2 \omega). \tag{5.97}$$

Notice that this component does not depend on s. If there is no drag from the body of the organism other than the flagellum (this is often approximately true), then the horizontal force balance equation is simply

$$f_1 = 0. \tag{5.98}$$

This translates into

$$(\alpha^2 U + \alpha\beta A^2 \omega)(R_\tau - R_\nu) + R_\nu U = 0. \tag{5.99}$$

This relation can be written in a more meaningful form if we look at the kinetics of the flagellum's wave motion. In (5.88), the quantity $(\beta s + \omega t)$ is called the phase of the oscillation. The intrinsic motion generates intrinsic waves moving in the x_1 direction. The phase velocity of these waves is the time rate of increase of x_1 when the phase is held constant: $(\beta s + \omega t) =$ constant. We get $\frac{ds}{dt} = -\frac{\omega}{\beta}$, and from (5.92), the phase velocity is

$$V = \frac{dx_1}{dt} = \frac{ds}{dt}\frac{dx_1}{ds} = -\frac{\alpha\omega}{\beta}. \tag{5.100}$$

The second term in parentheses in (5.99) can be written as

$$\alpha\beta A^2 \omega = -V\beta^2 A^2 = -V(1 - \alpha^2), \tag{5.101}$$

where we made use of (5.90).

With (5.101) and a little bit of algebra on (5.99) and (5.90), using (5.82), we now find

$$U = -VK(\alpha), \tag{5.102}$$

where

$$K(\alpha) = \frac{(1 - \alpha^2)(\mathcal{R} - 1)}{\mathcal{R} - \alpha^2(\mathcal{R} - 1)}. \tag{5.103}$$

This gives the steady propulsion velocity U in terms of the flagellum's intrinsic phase velocity V, the ratio $\mathcal{R}$ of resistances, which is a number (5.82) in the neighborhood of 2, and the single geometric parameter α.

Note that the propulsion velocity is nonzero only if $\mathcal{R} \neq 1$, so that propulsion depends on the tangential and normal resistances being different from one another.

Problem If ω and A were fixed and $\mathcal{R} = 2$, how should the organism choose α so as to maximize U?

Now suppose that the head of the bacterium has a drag proportional to its velocity U in the amount of DU (units of force). The thrust must overcome this drag, as well as that of the flagellum. Then (5.98) is not the correct force balance equation. The total propulsive force is $f_1 L$, where recall L is the total arclength of the flagellum. We therefore have

$$f_1 L + DU = 0. \tag{5.104}$$

Verify that the algebra now gives, in place of (5.102),

$$U = -VK_D(\alpha), \tag{5.105}$$

where

$$K_D(\alpha) = \frac{(1-\alpha^2)(\mathcal{R}-1)}{\mathcal{R} - \alpha^2(\mathcal{R}-1) + D/LR_\tau}. \tag{5.106}$$

5.5.4 Critique of the foregoing theory

The most obvious questionable aspect of the theory in Section 5.5.3 is that it applies a force/velocity relation developed for straight cylinders to an object (flagellum) which is not straight. Also, we left the choice of ℓ (first paragraph of that section) open, with no obvious way to make that selection. The only requirement on ℓ was (5.67): $\ell/a \gg 1$, which we now call "requirement 1".

Any reputable applied mathematician would therefore be obliged to give at least some kind of rational criterion for when the theory is expected to be valid, and that is what I'll attempt to do in this section.

Specifically, we will consider only a piece of the flagellum of some length ℓ, and ask

(1) is that piece straight enough to be considered a cylinder? and

(2) is the effect of the rest of the flagellum, not included in that piece, on the resistance of that piece negligible?

Our object will be to choose ℓ such the answers to both are "yes", under specific assumptions on the flagellum.

The straightness of a curve, or rather the lack of it, is gauged by its curvature

$$\kappa = |\boldsymbol{X}_{ss}|, \tag{5.107}$$

which in the case of (5.87) is given by

$$\kappa = A\beta^2 = \frac{1-\alpha^2}{A}, \tag{5.108}$$

use having been made of (5.90).

The curvature is the rate of change of the tangent vector $\boldsymbol{\tau}$ with respect to arclength. The change in direction of $\boldsymbol{\tau}$ in a length ℓ can therefore be estimated by $\kappa\ell$. Any segment of length ℓ will therefore be deemed "approximately straight" if $\kappa\ell \ll 1$, i.e.

$$\frac{\ell(1-\alpha^2)}{A} \ll 1. \tag{5.109}$$

This is our second requirement. It says that ℓ should be small enough so that the cylinder theory can be applied to that segment. The last requirement is that ℓ not be too large, namely that the influence of the rest of the flagellum beyond that cylinder be relatively minor.

When a straight cylinder's length is increased from ℓ_1, say, to ℓ_2, the resistances R_ν and R_τ ((5.78) and (5.80)) are changed by an amount which is the order of $\ln(\ell_2/a) - \ln(\ell_1/a) = \ln(\ell_2/\ell_1)$.

It is reasonable to assume that the rest of the flagellum's influence on the resistance of the segment of length ℓ considered above has this same order of magnitude, where we identify $\ell_2 = L,\ \ell_1 = \ell$. Then the effect has the order $\ln(L/\ell)$.

The *relative* effect of changing from ℓ to L will then be of the order $\frac{\ln(L/\ell)}{\ln(\ell/a)}$ in view of (5.78) and (5.80). We want this to be small; this is our third requirement. It will automatically imply the first requirement, which we therefore drop. In summary the following conditions should ensure the validity of the resistive-force theory: There exists a number ℓ such that

$$\ln(L/\ell) \ll \ln(\ell/a), \quad \frac{\ell(1-\alpha^2)}{A} \ll 1. \tag{5.110}$$

As an example, suppose that α, L and A are fixed. Then ℓ could be chosen to satisfy the last condition, and then a required to be so small that the first condition holds.

Problem Set $S_4 = 1$ and $S_5 = 2$. Write (5.82) in the form $\mathcal{R} = 2 + C(\ln(\ell/a))^{-1} +$ higher order terms. What is C? Neglecting the higher order terms, evaluate this expression for the values $\ell/a = 10,\ 100$ and 1000. In each case also find the exact value from the formula above (5.82). What do you conclude about (a) the sensitivity of $\mathcal{R}$ on $\ln(\ell/a)$, and (b) the accuracy of the above approximation?

Even if the first of (5.110) does not hold, but $\ell/a \gg l$, it is likely that the resistive force theory is a pretty good approximation for some choice of R_ν and R_τ, possibly found from experimental data.

6 Boundary layer theory

Just as the Stokes system is a simplification of the Navier-Stokes equations valid under certain conditions (low Reynolds number), there is another simplification designed for use under certain high Reynolds number conditions. I refer to the Prandtl approximation.

The difference is that whereas essentially no additional assumptions other than $Re \ll 1$ are necessary for the validity of the Stokes approximation, there are stringent ones for the Prandtl system. Mainly, the fluid flow should be well enough "behaved". This concept has no strict definition; however when deriving the Prandtl approximation, one operates under certain explicit or implicit conditions which could be interpreted as conditions of well behavior. Certain the flow should be "laminar" rather than "turbulent", but again these concepts are a little arbitrary. It is difficult to predict whether the solutions of the Navier-Stokes equations with given boundary conditions are going to satisfy the conditions leading to the Prandtl equations, because there are many instabilities which may govern the behavior when Re is large.

The Prandtl theory applies (sometimes) to flow past a plate or past the forward part of a body which obstructs an otherwise simple flow. The conditions for validity nearly always break down at some downstream point, where the flow becomes turbulent.

The Prandtl theory is one of the first "singular perturbation" constructions. The characterization "perturbation" applies because the parameter $\frac{1}{Re}$ in (5.22) is small, so that at least in a formal sense, the equations are a perturbation of the Euler equations. However, in making this comparison, we should rescale the pressure in (5.22) so that the pressure is retained in the limit $\frac{1}{Re} \to 0$. The perturbation is "singular" because if we set $\frac{1}{Re} = 0$ so that the viscosity terms vanish, we have a lower order system which is mathematically of quite a different character. For example, when

$\frac{1}{Re} > 0$, boundary conditions are used which require the fluid particle to stick to the boundaries, but for the Euler system, that boundary condition is generally unmathematical because it results in an overdetermined system.

You should read Acheson, Chapter 8, especially Sections 8.1–8.3. A very interesting historical perspective is given in Section 8.1.

6.1 The concept of boundary layer

Consider the following prototypical scenario: a fluid with constant velocity U in the x direction fills all space. We use the notation $\boldsymbol{x} = (x, y, z)$, $\boldsymbol{u} = (u, v, w)$. The velocity is $U\boldsymbol{e}_1$. A knife blade, infinitely thin and of infinite length, is inserted into the flow parallel to the z-axis. Its cross-section J in the (x, y) plane is the segment $\{0 < x < L,\ y = 0\}$, so that the original flow would be everywhere parallel to the blade. We build a Reynolds number with $U,\ L,\ \nu$ and assume it is very large:

$$Re \gg 1. \tag{6.1}$$

The fluid, which was originally free everywhere to move at velocity U, is now forced by the no-slip condition to slow down to 0 velocity on J. Let us focus attention on the middle of the blade, $x = L/2$. As we move away from J in the (normal) direction of increasing y, u is going going to change from 0 to a neighborhood of U in the space of some characteristic distance which we shall call δ. The layer of fluid next to the boundary with thickness δ will be called a boundary layer. Our first question is, what is δ? We assume it is $< L$.

In the following arguments, we assume that the flow velocity does not depend on the z coordinate, so that essentially it is a planar flow in the (x, y) plane.

6.1.1 The layer's thickness: a physical argument

Consider a tiny blob of fluid with small area A which starts near the left end of J and stays near the blade as it follows the flow. In the course of distance $L/2$, it changes its velocity by an amount of the order of U, so its momentum change is of the order ρUA. It does this in a time of the order $\frac{L/2}{U}$, so the time rate of change of momentum is of order $\frac{\rho UA}{L/U} = \frac{\rho U^2 A}{L}$. I eliminated a factor 2 because that doesn't change the order of magnitude. The force acting on this small blob is given by the viscous force density $\mu(u_{yy} + u_{xx})$ times A. The first term here has magnitude of the order $\frac{\mu UA}{\delta^2}$. In fact u_y has order of magnitude $\frac{U}{\delta}$ and changes by an amount $O(1)$ in a distance $O(\delta)$. Thus $u_{yy} = O(U/\delta^2)$. The second term has the order $\frac{\mu UA}{L^2}$, which is smaller because of our assumption that $\delta < L$. Equating the first term to the rate of change of momentum, we get

$$\frac{\mu UA}{\delta^2} = \frac{\rho U^2 A}{L}, \tag{6.2}$$

i.e.

$$\delta^2 = \frac{\mu L}{\rho U} = \frac{L^2}{Re},$$

or

$$\frac{\delta}{L} = (Re)^{-1/2} \ll 1. \tag{6.3}$$

This gives the correct order of magnitude for the thickness of the boundary layer. It shrinks in size as the Reynolds number tends to infinity.

6.1.2 A scaling argument

Here we give another argument, a little more systematic, leading to the same conclusion. In the boundary layer, we write the Navier-Stokes equations, assumed stationary for simplicity, in rescaled variables. We begin with the dimensionless variables used in (5.21). The asterisks were dropped to obtain (5.22). In the following, the notation without asterisks will signify dimensionless quantities in which space is scaled with our given L and velocity with our given U.

At this point, we make a further rescaling which is appropriate within the boundary layer. The horizontal dimensionless space variable x is not rescaled, because in the direction along the blade, L is the proper characteristic length. However the vertical dimensionless variable y is rescaled according to the dimensionless boundary layer width $\beta = \delta/L$ (although we gave an argument for determining this ratio in the previous section, pretend now that it is unknown). Thus we set $y = \beta\eta$. Likewise, the x-component u of the dimensionless velocity is not rescaled, because the real velocity has characteristic value U in that direction. However, a vertical component v, which was not present before the blade was inserted, is now generated. At this point we don't know its characteristic value, so rescale by $v = \gamma\bar{v}$, where γ is unknown. Finally, the pressure is rescaled by $p = \alpha\bar{p}$, where again α has to be determined.

In summary, we define 3 new variables $\eta,\ \bar{v},\ \bar{p}$ by:

$$y = \beta\eta, \quad v = \gamma\bar{v}, \quad p = \alpha\bar{p}, \tag{6.4}$$

and the variables u, x remain unscaled.

The stationary 2D Navier-Stokes equations then become

$$uu_x + \frac{\gamma}{\beta}\bar{v}u_\eta = \frac{1}{Re}(u_{xx} + \frac{1}{\beta^2}u_{\eta\eta} - \alpha\bar{p}_x), \tag{6.5}$$

$$\gamma u\bar{v}_x + \frac{\gamma^2}{\beta}\bar{v}\bar{v}_\eta = \frac{\gamma}{Re\beta^2}\bar{v}_{\eta\eta} + \frac{1}{Re}\gamma\bar{v}_{xx} + \frac{\alpha}{\beta Re}\bar{p}_\eta, \tag{6.6}$$

$$u_x + \frac{\gamma}{\beta}\bar{v}_\eta = 0. \tag{6.7}$$

We shall use the following guiding principle in choosing the proper scalings β, γ, α. It is a time-honored standard procedure to determine proper rescalings in applied mathematics. This task is more an art than a science, so that one should always check at the end to see whether the results make mathematical and physical sense.

Scaling procedure: In any given equation, try to make as many terms as possible comparable in order of magnitude, and the remaining ones smaller.

As we shall see, when there are several equations the final results, using this "algorithm", may depend on the order in which the equations are subjected to it. This is part of the art. We apply this first to (6.7), which tells us

$$\gamma = \beta.$$

With that settled, we turn to (6.5), which is now

$$uu_x + \bar{v}u_\eta = \frac{1}{Re}(u_{xx} + \frac{1}{\beta^2}u_{\eta\eta} - \alpha\bar{p}_x), \tag{6.8}$$

We disregard the first viscous term on the right, as it is known to be less than the second term since $\beta < 1$. The formal orders of magnitude of the remaining terms are $1, 1, \frac{1}{Re\beta^2}, \frac{\alpha}{Re}$. These can be made the same by choosing $\beta = (Re)^{-1/2}$ and $\alpha = Re$. In summary, we have

$$\beta = \frac{\delta}{L} = (Re)^{-1/2}, \quad \gamma = (Re)^{-1/2}, \quad \alpha = Re. \tag{6.9}$$

Finally, we consider the last equation (6.6). All terms in that equation are formally of order β or smaller except the last, which has the order β^{-1}.

With these scalings, we now turn to examine the terms which were deemed to be of lower order. In (6.5) there is the term $\frac{1}{Re}u_{xx}$, and in (6.6) we have that all terms except the last are lower order. If we now neglect all these lower order terms, we obtain the system

$$uu_x + \bar{v}u_\eta = u_{\eta\eta} - \bar{p}_x, \tag{6.10}$$

$$\bar{p}_\eta = 0, \tag{6.11}$$

$$u_x + \bar{v}_\eta = 0. \tag{6.12}$$

This is the Prandtl system of equations in the stationary 2D case. It has been studied extensively and found to make mathematical and physical sense. Since $\beta = (Re)^{-1/2}$, this scaling agrees with (6.3), found by another means. All this gives considerable credence to the validity of this choice of scales.

6.1.3 The role of the pressure gradient

The arguments given in Sections 6.1.1 and 6.1.2 above tell you what the thickness of the boundary layer should be, provided there is one at all. There are many high Reynolds number flows past boundaries that do not have them. One of the important factors in determining whether there is such a layer or not is the sign of the pressure gradient in the direction of the flow. More or less, the idea is that if the pressure gradient is of such a sign (negative) that it tends to accelerate the flow, then it is favorable for the development and maintenance of a boundary layer. On the other hand if it is positive, then it is doubtful that a boundary layer can persist long.

This is a qualitative statement, and so in the following argument in support of it, I will not be interested in numerical accuracy—only gross estimates. Also I will not be interested in great generality—the scenario will be a very simple one.

Dimensionless variables will be used. Consider a more general situation than the knife blade picture, namely a flow through a 2D rectangle $R = \{0 < x < 3,\ 0 < y < 2\}$ from the left to the right which sticks to the lower boundary $\{0 < x < 3,\ y = 0\}$. This is the only physical boundary; the other edges of R are just artifical lines in the flow. The velocity components are $\boldsymbol{u} = (u, v)$.

For convenience in notation, we use the small parameter $\epsilon^2 = \frac{1}{Re}$ and assume $\epsilon \ll 1$. Let us rescale p in (5.22) so that it incorporates the factor $\frac{1}{Re}$; the right side then becomes $\frac{1}{Re}\nabla^2\boldsymbol{u} - \nabla p$. (This is in accord with the choice $\alpha = Re$ in (6.9).) Then the x-component of the stationary Navier-Stokes momentum equation (5.22) is

$$\epsilon^2\Delta u - uu_x - vu_y = p_x, \tag{6.13}$$

and the continuity equation (5.11) is

$$u_x + v_y = 0. \tag{6.14}$$

For any function $w(x,y)$, we define the linear elliptic partial differential operator L, acting on w, by

$$Lw = \epsilon^2 \Delta w - uw_x - vw_y. \tag{6.15}$$

In this definition, the coefficients $u(x,y)$ and $v(x,y)$ come from the given flow, but w can be any function. In particular, (6.13) can be written

$$Lu = p_x. \tag{6.16}$$

We make the following assumptions about the flow $\boldsymbol{u}$ in R, for some positive constant M:

$$\begin{array}{l} 0 \le u(x,y) \le 1, \\ |u_x| \le M, \\ u(x,0) = v(x,0) = 0. \end{array} \tag{6.17}$$

Under these assumptions plus one about p_x being negative, we will show that in a square domain interior to R, the component u is bounded below by a function of y which rises steeply from the boundary $\{y = 0\}$ until it attains the positive $O(1)$ value in a distance $\approx \epsilon = (Re)^{-1/2}$ from the boundary. This is what I mean by the qualitative development of a boundary layer. A similar upper bound could also be proved. The existence of a boundary layer is not assumed, but rather proved.

From (6.14) and the last two assumptions in (6.17), we know that

$$|v(x,y)| \le My. \tag{6.18}$$

We make use of a special artificial continuous function $g(\xi)$ defined by

$$g(\xi) = \begin{cases} 2\xi - \xi^2, & 0 \le \xi \le 1, \\ 1, & \xi \ge 1. \end{cases} \tag{6.19}$$

You can check that

$$0 \le g \le 1, \quad 0 \le g' \le 2, \quad 0 \ge g'' \ge -2. \tag{6.20}$$

For some positive α which will be chosen later, we define a "subsolution" $\underline{u}$ by

$$\underline{u}(x,y) = \alpha g(x) g(y/\epsilon) g(3-x) g(2-y). \tag{6.21}$$

This function vanishes on ∂R and rises to be equal to α in the interior of R. Our goal is the prove that it is a lower bound for the real velocity component u when $p_x < 0$ and α is chosen appropriately.

We calculate

$$\begin{aligned} L\underline{u} = {} & \epsilon^2 \alpha \left[g''(x) g(y/\epsilon) g(3-x) g(2-y) + g(x) g(y/\epsilon) g''(3-x) g(2-y) \right] + \\ & + \epsilon^2 \alpha \left[\epsilon^{-2} g(x) g''(y/\epsilon) g(3-x) g(2-y) + g(x) g(y/\epsilon) g(3-x) g''(2-y) \right] - \\ & - \alpha u \left[g'(x) g(y/\epsilon) g(3-x) g(2-y) - g(x) g(y/\epsilon) g'(3-x) g(2-y) \right] \\ & - \alpha v \left[\epsilon^{-1} g(x) g'(y/\epsilon) g(3-x) g(2-y) - g(x) g(y/\epsilon) g(3-x) g'(2-y) \right]. \end{aligned} \tag{6.22}$$

Using the estimates (6.18) (which implies $|v| g'(y/\epsilon) \le 2M\epsilon$), (6.20) and the upper bound for u in (6.17), we in turn estimate

$$L\underline{u} \ge -\alpha \left(4 + 6\epsilon^2 + 3M \right) \equiv -\alpha K. \tag{6.23}$$

We also recall (6.16). Now if we define $w = \underline{u} - u$, we have, by subtracting (6.16) from (6.23) that

$$Lw \geq -\alpha K - p_x. \tag{6.24}$$

At this point we assume

$$p_x < -a < 0 \tag{6.25}$$

and finally say what α is. It is a positive constant small enough that

$$-\alpha K + a > 0, \tag{6.26}$$

so that

$$Lw > 0. \tag{6.27}$$

We also know that $w \leq 0$ on ∂R. We show that $w \leq 0$ in all of R by a "maximum principle" argument.

If it were not true that $w \leq 0$, there would be a point $(x_0, y_0) \in R$ at which w has a positive maximum. At such a maximum the first derivatives of w would be zero and the pure second derivatives ≤ 0, so that looking at (6.15) we see that $Lw \leq 0$ at that point. This contradicts (6.27), so it must be true that $w \leq 0$, i.e.

$$u(x, y) \geq \underline{u}(x, y). \tag{6.28}$$

Now let's concentrate on the interior square $\{1 < x < 2,\ 0 < y < 1\}$, where $\underline{u} = \alpha g(y/\epsilon)$, since all the other factors in (6.21) are equal to 1. This is a function which rises steeply from the boundary, and we get such a lower bound for u, as anticipated. Even though there may be no boundary layer where the flow enters R at $x = 0$, one will have developed by the time it reaches the interior square. We note again that a similar upper bound can also be obtained in that square.

The case of an unfavorable pressure gradient It can also be shown that if $p_x > 0$ is large enough, you can't expect to have a boundary layer. We do this by an argument similar to the above, but going in the other direction. We continue to assume (6.17). Instead of R, we use a shorter rectangle $R' = \{0 < x < 3,\ 0 < y < 1\}$.

We propose a "supersolution" in R'

$$\overline{u} = [yg(x)g(3 - x) + 2 - g(x) - g(3 - x)] \tag{6.29}$$

Problem 1 Show that under assumptions (6.17), there is a constant $K > 0$ independent of ϵ such that in R',

$$L\overline{u} \leq K. \tag{6.30}$$

Now assume $p_x > K$ so that $L\overline{u} \leq p_x = Lu$.

Problem 2 Show that on the two lateral sides of R' (where $x = 0$ or $x = 3$), $\overline{u} = 1$. Show the same is on the top side of R'. Show that on the bottom of R', $\overline{u} \geq 0$.

That makes $\overline{u}$ a supersolution. Therefore $\overline{u} \geq u$ on the entire boundary of R'. By essentially the same maximum principle argument as before, we conclude that $\overline{u} \geq u$ everywhere in R. In particular for $1 < x < 2$, $\overline{u} = y$, so that

$$u(x, y) \leq y. \tag{6.31}$$

Therefore u does not rise steeply with y, as it did in the case when there was a favorable pressure gradient.

What this means is that if the flow satisfies (6.17) and $p_x > 0$ is large enough, then no boundary layer can occur in the interior square.

6.1.4 Vorticity layers, skin friction, and layer separation

The "skin friction" at any point on the boundary of a flow domain is the (tangential) stress on the boundary caused by the fluid moving past it. In dimensionless terms, it is $\frac{1}{Re}\frac{\partial u}{\partial n}$, where u is the tangential component of the velocity and n is the normal direction.

Problem Find, by the scaling arguments in Section 6.1.2, the order of magnitude of the skin friction in a boundary layer as $Re \to \infty$. (Give your answer as a power of Re.) Do the same for the horizontal force density $\frac{1}{Re}u_{yy}$ in the boundary layer caused by the y-variation of these stresses.

The order of magnitude of the vorticity $\omega = v_x - u_y$ in a boundary layer is large, so that it is clear that the boundary layer, if it exists, is a region of high vorticity. In fact if the boundary is at $\{y = 0\}$, the second term in the expression for the vorticity is much larger than the first, and is $O(\sqrt{Re})$. The pressure gradient acting throughout the layer, together with the viscous stress, which is greatest next to the boundary and in the opposite direction, serve to impose a torque on the fluid particles; this is the mechanism by which vorticity is generated.

Therefore we may characterize the boundary layer as a layer of high vorticity. Outside that layer, the vorticity is typically low, unless the flow is turbulent.

The arguments in Sections 6.1.2 and 6.1.3 implicitly assumed the existence of a layer of some width adjacent to the boundary, whereas the one in Section 6.1.3 did not. As payment for this less stringent assumption, the pressure gradient was assumed to be favorable.

In fact, when the pressure gradient is not favorable, there exists such a thing as an "interior layer" which has characteristics of the boundary layer, but is not attached. It also can be characterized as a layer of high vorticity, just that it's not attached to the wall. Such an interior vorticity layer occurs when the boundary layer detaches from the wall due to the development of an unfavorable pressure gradient. This is called "boundary layer separation". These interior layers are highly unstable, however, and in typical cases quickly break up into isolated vortex patches.

6.2 Singular perturbations in unbounded domains

In order to understand the reduction of the Navier-Stokes to the Prandtl system better, we present a couple of mathematical examples of analogous phenomena in easier contexts.

6.2.1 An ordinary differential equation

Let $0 < \epsilon \ll 1$ be our basic parameter, let $f(x)$ be a given function, and consider the problem of finding bounded solutions $u^\epsilon(x)$ of

$$\epsilon^2 \frac{d^2}{dx^2}u^\epsilon(x) - u^\epsilon(x) = f(x), \quad x > 0, \quad u^\epsilon(0) = 0. \tag{6.32}$$

We would like to find good simple approximations to the solution, since the explicit formula for the solution is rather complicated. A first naive attempt would be to discard the first term in

(6.32), since ϵ is small. That would leave $u \approx -f$, which is fine except that it doesn't satisfy the boundary condition at $x = 0$ (unless $f(0) = 0$; but let's assume that's not the case).

The remedy for this discrepancy is to recognize that this is an example of a problem with "multiple scales". Multiple scale problems modeling natural phenomena abound. A given model, chosen at random, is more likely to have several scales in space and/or time than not. In the natural world, events occur on different time scales and structures occur on different space scales. This particular problem (6.32) has two natural space scales. Later in the section on turbulence, we will encounter multiple scales on a much grander "scale".

In our problem the two space scales are (1) the scale which is characteristic of the function $f(x)$, which is independent of ϵ, because f does not depend on ϵ, and (2) ϵ itself. In fact, the first term shows that ϵ has dimensions of length. This second scale is much smaller than the first.

But we must expect that the solution of the problem has structures typical of each scale. The popular way to handle the situation is to admit approximations which involve functions of two space variables, x and $\xi = \frac{x}{\epsilon}$. Of course these two variables are related to each other, but to some extent the analysis proceeds as though they were independent.

Since the naive approximation attempted above breaks down because of the boundary condition, we suspect that in a neighborhood of the boundary, the first natural space scale is not the appropriate one to use. We hope to remedy the deficiency in the boundary condition by using ξ as independent variable in place of x.

Our approach is to make two formal expansions of a potential solution in powers of ϵ: (1) the "outer" expansion, in which the coefficients are functions only of x:

$$u^\epsilon = u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \dots, \tag{6.33}$$

and (2) the "inner" expansion, when the coefficients are functions of the "stretched" variable $x/\epsilon = \xi$. Calling $U^\epsilon(\xi) = u^\epsilon(x) = u^\epsilon(\epsilon\xi)$, we propose the formal expansion

$$U^\epsilon = U_0(\xi) + \epsilon U_1(\xi) + \dots. \tag{6.34}$$

To determine the terms in (6.33), we substitute that expansion into (6.32) and equate coefficients of the various powers of ϵ. To determine the $U_i(\xi)$, we substitute (6.34) into the scaled version of (6.32), that is the equation that U^ϵ satisfies, and again equate coefficients. That equation is

$$\frac{d^2}{d\xi^2} U^\epsilon(\xi) - U^\epsilon(\xi) = f(\epsilon\xi). \tag{6.35}$$

Doing this equating alone will not determine the coefficients uniquely, because there are also boundary conditions and matching conditions that need to be satisfied.

The terms of the outer expansion (6.33) in this case are determined by algebraic rather than differential equations:

$$u_0(x) = -f(x), \quad u_1(x) = 0, \quad u_2(x) = u_0''(x), \text{ etc.} \tag{6.36}$$

On the other hand, for the inner terms we have (by setting $f(\epsilon\xi) = f(0) + \epsilon\xi f'(0) + \dots$)

$$\frac{d^2}{d\xi^2} U_0(\xi) - U_0 = f(0), \quad \frac{d^2}{d\xi^2} U_1(\xi) - U_1 = \xi f'(0), \quad \text{etc.} \tag{6.37}$$

With the equations (6.37), we are able to satisfy the boundary condition on the right of (6.32):

$$U_i(0) = 0, \text{ all } i. \tag{6.38}$$

So the expansion should be a good approximate representation near $x = 0$. Solving (6.37), (6.38) for $U_0(\xi)$, we find

$$U_0(\xi) = -f(0)(1 - e^{-\xi}), \quad U_1(\xi) = -\xi f'(0), \text{ etc.} \tag{6.39}$$

(There are other solutions which increase exponentially in ξ as $\xi \to \infty$, which we discard since we are interested in bounded solutions.)

Notice that

$$U_0(\infty) = u_0(0) \quad \text{and} \quad U_1'(\infty) = u_0'(0). \tag{6.40}$$

These conditions, and their higher order analogs, are not just accidental. They are the well-known matching conditions relating the inner to the outer expansions. They are valid not just here, but in great generality in other contexts as well.

6.2.2 An elliptic-to-parabolic singular perturbation

Given a function $f(y,t)$, consider the problem of finding a solution $u^\epsilon(y,t)$, bounded in the quarter-plane $\{y > 0,\ t > 0\}$, of the elliptic equation

$$u_t^\epsilon = \epsilon^2(u_{yy}^\epsilon + u_{tt}^\epsilon) + f(y,t), \tag{6.41}$$

with boundary conditions

$$u^\epsilon(0,t) = u^\epsilon(y,0) = 0. \tag{6.42}$$

The naive suggestion $u \approx \int_0^t f(y,\tau)d\tau$ obtained by setting $\epsilon = 0$ in (6.41) doesn't satisfy the first of (6.42). So, we expect another space scale as before.

Again, we try an outer expansion

$$u^\epsilon(y,t) = u_0(y,t) + \epsilon u_1(y,t) + \dots \tag{6.43}$$

We also define an inner variable $\zeta = y/\epsilon$ and an inner solution $U^\epsilon(\zeta,t)$. The expansion is

$$U^\epsilon(\zeta,t) = U_0(\zeta,t) + \epsilon U_1(\zeta,t) + \dots \tag{6.44}$$

Following the procedure outlined above, we have the outer terms

$$u_0(y,t) = \int_0^t f(y,s)ds, \text{ etc.} \tag{6.45}$$

The inner equation is

$$U_t^\epsilon = U_{\zeta\zeta}^\epsilon + f(\epsilon\zeta,t) + \epsilon^2 U_{tt}^\epsilon. \tag{6.46}$$

We obtain that the lowest order inner term satisfies

$$(U_0)_t = (U_0)_{\zeta\zeta} + f(0,t), \tag{6.47}$$

with boundary conditions (6.42), applied to U_0. This has an explicit solution

$$U_0(\zeta,t) = \int_0^t \left[1 - \frac{\zeta}{2\sqrt{\pi}(t-\tau)^{3/2}} e^{-\frac{\zeta^2}{4(t-\tau)}}\right] f(0,\tau)\, d\tau \tag{6.48}$$

You can check this if you like.

To be sure, it is not a very elegant formula, but at least it is an explicit one, and when an exact solution to a problem in PDE's is obtained, there may be cause for celebration. An explicit expression for the solution of (6.41) is well-nigh impossible, so this approximation has great value.

It can be verified by letting $\zeta \to \infty$ that

$$U_0(\infty, t) = u_0(0, t), \tag{6.49}$$

similar to the matching relation (6.40) found in the previous example.

In summary, the solution of (6.41), (6.42) can be approximated by (6.45) to lowest order in regions where y is not small, and by (6.48) with $\zeta = \frac{y}{\epsilon}$ where y is small. The region of validity of the latter is a layer in the variable y of width $O(\epsilon)$. It is termed a "boundary layer" in analogy to the hydrodynamical boundary layer.

Problem Find $u_1(x,t), u_2(x,t)$ and find the PDE that $U_1(\zeta, t)$ satisfies. No need to solve it.

6.3 The Prandtl reduction again

The reason for introducing the example in Section 6.2.2 is that it illustrates, in a simpler setting, the multiple scale concept of a fluid dynamical boundary layer. In both cases, we have an elliptic-to-parabolic reduction. The elliptic singularly perturbed equation (6.41) should be compared to (6.13), which should be examined in conjunction with (6.14). The variable t in (6.41) is like the variable x in (6.13). If, moreover, we forget about the factor u in the term uu_x in (6.13), so that this term is comparable to u_t in (6.41), and we think of the term $p_x(x,y)$ in (6.13) as a given function $-f(y,t)$, then the problems are indeed seen to be analogous. The term vu_y in (6.13) is a minor complication.

We continue this analogy in a systematic manner. Setting $\frac{1}{Re} = \epsilon^2$ and scaling p as in (6.13), we write the steady-state Navier-Stokes equations (5.22), (5.11) here in component form:

$$uu_x + vu_y = \epsilon^2 \Delta u - p_x, \tag{6.50}$$

$$uv_x + vv_y = \epsilon^2 \Delta v - p_y, \tag{6.51}$$

$$u_x + v_y = 0, \tag{6.52}$$

We consider these equations in the quarter-plane $\{x > 0,\ y > 0\}$, just as we considered (6.41) in $\{y > 0,\ t > 0\}$. Boundary conditions are needed for $x = 0$ and for $y = 0$. We impose the no-slip boundary condition $u(x, 0) = 0$ and leave the specification of $u(0, y)$ until later. We also require $v(x, 0) = 0$.

We employ the same stretched variable $\zeta = y/\epsilon$ as in Section 6.2.2, and the same concept of outer and inner solutions as well. For the lowest order outer solution (u^{out}, v^{out}), we get the equations

$$-uu_x - vu_y = p_x, \quad -uv_x - vv_y = p_y, \quad u_x + v_y = 0, \tag{6.53}$$

which is just the steady-state Euler system. There are a great many solutions, of course. For example, any potential flow $(u^{out}, v^{out}) = \nabla\phi$ for a harmonic velocity potential $\phi(x, y)$ would fill the bill.

Typically, one chooses such an outer solution appropriate for the physics and geometry, and on that basis goes on to examine the inner solution.

The lowest order inner problem, as you can imagine, produces the Prandtl equations (6.10)–(6.12). In fact the scaled variable η used in Section 6.1.2 turns out to be just our familiar stretched variable ζ. As in Section 6.2.2, let us use capitals (U, V, P) for the inner variables. Then from (6.10), (6.11), (6.12),

$$UU_x + VU_\zeta = U_{\zeta\zeta} - P_x, \tag{6.54}$$

$$P_\zeta = 0, \tag{6.55}$$

$$U_x + V_\zeta = 0. \tag{6.56}$$

The equation (6.55) means P is a function only of x. The system therefore becomes the following:

$$UU_x + VU_\zeta = U_{\zeta\zeta} - P_x(x), \tag{6.57}$$

$$U_x + V_\zeta = 0. \tag{6.58}$$

In this derivation we have used rectangular Cartesian coordinates (x, y). Strictly speaking, this would require that the boundary be flat (on the x-axis). However since the boundary layer is thin, the curvature of the boundary is of minor importance, and the asymptotic reduction is still valid for curved boundaries, if we interpret x as being arclength along the boundary and y as perpendicular distance from the boundary. We make that interpretation from now on.

We now replace ζ by our stretched variable y/ϵ and revert to dimensional variables. We use the symbols $\overline{u},\ \overline{v},\ \overline{p}$ for those physical variables. We obtain

$$\overline{u}\,\overline{u}_x + \overline{v}\,\overline{u}_y = \nu\overline{u}_{yy} - \overline{p}_x(x)/\rho, \tag{6.59}$$

$$\overline{u}_x + \overline{v}_y = 0. \tag{6.60}$$

But how does this inner problem depend on the particular outer solution chosen? The answer is, through the function $P(x)$ in (6.57) and the required limiting value of U as $\zeta = y/\epsilon \to \infty$.

The matching conditions (6.40) (just the left hand one, actually) tell us that as $\zeta \to \infty$, the inner pressure should equal the outer pressure $p^{out}(x, 0) \equiv \overline{p}(x)$ at $y = 0$ and similarly the limit of the inner component u should be $u^{out}(x, 0) \equiv \hat{u}(x)$. In summary:

$$\lim_{\zeta\to\infty} U(x, \zeta) = u^{out}(x, 0) \equiv \hat{u}(x), \quad P(x) = p^{out}(x, 0) \equiv \hat{p}(x). \tag{6.61}$$

Using this, we may take the limit $\zeta \to \infty$ in the equation (6.57). Since u approaches the limit $\hat{u}$, depending only on x, then the ζ derivatives should vanish as $\zeta \to \infty$, leaving the relation (at that limit)

$$\hat{u}\hat{u}_x \approx -\hat{p}_x, \tag{6.62}$$

which, when integrated with respect to x, gives us the Bernoulli relation $\frac{1}{2}\hat{u}^2 + \hat{p} =$ constant. This relation we already knew for the Euler equations, which govern the outer solution. It is valid on the streamline "next to the boundary layer".

So the matching conditions between the inner (Prandtl) solution and the outer (Euler) solution make physical sense.

It is extremely significant that in passing this way from the Navier-Stokes to the Prandtl system, the mathematical character of the problem changes drastically. The spatial variable x in this transition assumes the role of an evolution variable, like the time variable t in (6.46). And the elliptic Navier-Stokes momentum equation becomes a parabolic evolution problem. So what is evolving? The y-profile of the tangential velocity component u. A well-posed problem for the Prandtl system (6.57)–(6.58) is to prescribe the input profile $U(0, \zeta)$ (remember I said we would get to this boundary condition), and then find how it evolves in the downstream direction. This problem has a well-developed mathematical theory, with existence and uniqueness theorems and the lot.

In summary, in dimensional terms a typical well posed problem for (6.59), (6.60) would be to specify

$$\overline{u}(x, 0) = \overline{v}(x, 0) = 0, \tag{6.63}$$

$$\overline{u}(0,y) = \overline{u}_0(y) \text{ a given initial profile} \tag{6.64}$$

$$\lim_{y\to\infty} \overline{u}(x,y) = \hat{u}(x), \tag{6.65}$$

where

$$\hat{u}(x)\hat{u}_x(x) + \overline{p}_x(x) = 0. \tag{6.66}$$

Often in practice, the choice for outer solution (of Euler's equations) is quite natural. For example, consider the knife blade boundary mentioned at the beginning of Section 6.1. The outer solution would simply be $\boldsymbol{u}^{out} = (U, 0),\ p^{out} = 0$. The function p_x in (6.10) would be prescribed to be 0, and we would have the matching condition $u(x,\infty) = U$.

For flows around bodies at high Reynolds numbers, the upstream (forward) part of the body would typically have a boundary layer which separates the body from a potential flow around that forward part. Knowing that potential flow provides the term $\overline{p}_x$, which will generally be negative (favorable) on the forward part. At the rear of the body, we will no longer have a boundary layer and there will be no well defined region where the potential flow persists. That is a serious complication for determining the outer solution, even near the forward part.

Problem Take the potential flow (4.72) around a cylinder, with $\Gamma = 0$. Let $p(z)$ be the pressure on the surface of the cylinder. Verify that on the forward part of the cylinder, p decreases with x. In particular, the derivative of p with respect to arclength along the boundary is negative, so that the pressure gradient in favorable. What about the back part? What happens when $\Gamma \neq 0$?

6.4 The Blasius profile

6.4.1 Preliminaries on similarity solutions

Special solutions of PDE's are sometimes found which are functions of a single "similarity" variable, which is some combination of the independent variables, times perhaps a power of one of the variables. These are called similarity solutions. Solving for them is reduced to solving an ODE for that function of the similarity variable.

Similarity solutions are useful and enlightening, but because of their very special nature, you may be lucky if they satisfy the right boundary conditions.

Here is an example of a similarity solution. Consider the problem for $u(x,t)$

$$u_t = u_{xx}, \quad x > 0,\ t > 0, \tag{6.67}$$

$$u(x,0) = 0,\ u(0,t) = 1. \tag{6.68}$$

For any number $\alpha > 0$, let $v_\alpha(x,t) = u(\alpha x, \alpha^2 t)$. It is pretty easily checked that the function v_α also satisfies (6.67) and (6.68). Assuming the solution is unique (which it is if it is required to be bounded), we must have $u(x,t) = v_\alpha(x,t) = u(\alpha x, \alpha^2 t)$. For any point (x,t) in the quarter plane, we set $\alpha = t^{-1/2}$. Thus $u(x,t) = u(xt^{-1/2}, 1)$, which is clearly a function only of the combined variable $\eta = xt^{-1/2}$: $u(x,t) = f(\eta)$.

We have to find what the function f is. Substituting $u(x,t) = f(\eta)$ into (6.67), we find that $f''(\eta) = -\frac{1}{2}\eta f'(\eta)$, which has the general solution $f(\eta) = A\int_0^\eta e^{-s^2/4}ds + B$ for arbitrary $A,\ B$. Since $\int_0^\infty e^{-s^2/4}ds = \sqrt{\pi}$, the boundary conditions will be satisfied if we choose the coefficients so that

$$f(\eta) = 1 - \frac{1}{\sqrt{\pi}}\int_0^\eta e^{-s^2/4}ds. \tag{6.69}$$

This provides $u(x,t) = f(xt^{-1/2})$.

Problem 1 Consider the nonlinear problem

$$uu_t = u_{xx}, \quad x > 0, \ t > 0,$$

$$u(x,0) = 1, \ u(0,t) = 2.$$

Find a similarity variable and find the ODE that the corresponding similarity function f must satisfy. Specify the required boundary conditions on f. Don't try solve that equation.

Problem 2 Find the bounded solution of $\Delta u(x,y) = 0$ in $\{x > 0, \ y > 0\}$ satisfying the boundary conditions $u(x,0) = a, \ u(0,y) = b$, where a and b are any constants. Use a similarity variable.

6.4.2 The infinite plate problem

The picture now is a slight simplification of that of the knife blade in Section 6.1. We expand the blade's cross-section to be the half-line $\{x > 0\}$ and call it a semiinfinite plate. As before, the impinging flow is horizontal with velocity $(U, 0, 0)$. That will also be the outer solution. We shall now construct the boundary layer (inner) solution. It will be a solution of (6.59), (6.60) with p_x chosen in accordance with (6.66). But $\lim_{y\to\infty} \overline{u}(x,y) \equiv U$, which we identify with $\hat{u}$ in (6.66), so that

$$\overline{p}_x = 0. \tag{6.70}$$

By symmetry, we are allowed to look at the solution in the upper half plane $\{y \geq 0\}$ alone; then extend it by reflection to the lower half plane as well. The boundary conditions are

$$\overline{u}(x,0) = 0, \ x > 0 \tag{6.71}$$

$$\overline{v}(x,0) = 0, \tag{6.72}$$

$$\lim_{y\to\infty} \overline{u}(x,y) = U, \tag{6.73}$$

$$\overline{u}(0,y) = U. \tag{6.74}$$

The condition (6.74) represents the constant velocity profile at the leading edge of the plate, where the boundary layer has not had a chance yet to develop.

First, we give a physical argument to show that the thickness of the boundary layer must increase as we go downstream. As a fluid particle travels downstream in a neighborhood of the plate, it is constantly being decelerated by viscosity. That means the y profile of u will become ever flatter, hence the width of the boundary (the distance over which u rises from 0 at the boundary to U in the main stream) will get larger, as claimed.

The actual solution can be found as a similarity solution. Notice that this particular problem does has a lot of symmetry. It is also very significant that it contains no characteristic length. In fact the only object which one might measure is the plate, and that has infinite length. These circumstances are compelling evidence of the existence of a similarity solution, at least to one accustomed to finding these things.

Therefore we shall try to find a solution which is a function of a single variable which is some combination of x and y. We need to find what that combination is. The following is an argument which leads to the correct combination.

Since there is no characteristic length, there is no way to construct a Reynolds number. But actually there is, if we are willing to take a little liberty with the definition of that number. Suppose we wish to examine the solution only in a finite interval $0 \leq x \leq x_0$, disregarding larger values.

Then with this narrower focus, we could identify x_0 as the characteristic length for use in defining an x_0-dependent Reynolds number

$$Re(x_0) \equiv \frac{Ux_0}{2\nu}. \tag{6.75}$$

(I put in a spurious factor 2 in the denominator, which doesn't change the order of magnitude, simply to make the algebra below slightly easier.)

With this, we can estimate the width of the boundary layer at the value x_0. According to the left part of (6.9) with $L = x_0$, that width will be

$$\delta(x_0) \approx x_0 Re(x_0)^{-1/2} = \left(\frac{2\nu x_0}{U}\right)^{1/2} \tag{6.76}$$

This is the scale in y on which significant variations in u should occur. It depends on x_0. If there is a similarity variable, which we shall call $\eta(x, y)$, it should not change when $\frac{y}{\delta(x)}$ doesn't change. Therefore η should be the ratio

$$\eta = \frac{y}{\delta(x)} = \frac{y\sqrt{U}}{\sqrt{2\nu x}}, \tag{6.77}$$

where I have removed the subscript from x_0, since that is an arbitrary variable.

6.4.3 The profile

With the identification of η in hand, at least provisionally, we go on the find the velocity profile. The fact that we will be successful is corroboration of the preceding argument. Since we are in two dimensions and working with an incompressible flow, it turns out to be simpler to use the streamfunction formulation. So let $\psi(x, y)$ be the streamfunction (4.44).

As you can read in Acheson, pages 272 and 273, it is convenient to write $\psi = Ug(x)f(\eta)$, $u = Uf'(\eta_)$ with $g(x) = \frac{y}{\eta} = (2\nu x/U)^{1/2}$. This function g was already determined by the above reasoning; Acheson uses another argument to come to the same conclusion. We are then left with the problem of determining what the function $f(\eta)$ is. It satisfies a third order nonlinear ODE, Acheson's (8.20), together with boundary conditions (8.21). Thus in typical fashion for similarity solutions, the PDE system reduces, by wise choice of variables and representations, to the solution of an ODE.

The ODE for f has been studied and has a complete existence/uniqueness theory. It can of course also be computed. The resulting velocity profile, known as the Blasius profile, is seen on page 274 of Acheson.

6.4.4 Eventual instability

If we are considering an actual flow past a long plate as in Section 6.4.3, we must consider that the effective Reynolds number is a function of distance along the plate, and in fact increases linearly with distance (6.75). In any flow, laminar solutions become unstable at sufficiently high Reynolds number. Therefore we may expect that the boundary layer flow past such a long plate will become unstable sufficiently far from the leading end. This is certainly born out by experiments, which typically show the Blasius profile in a region near the front end being transformed into a turbulent flow further downstream.

6.5 Boundary layer separation and beyond

In high Reynolds number flows past bodies of finite extent, any boundary layers which form will at some point separate from the body. This phenomenon, and the subsequent behavior of the flow behind the body, are only incompletely understood. The incomplete understanding that does exist involves a series of concepts and events, each interesting in its own right: **momentum shadow, eddy formation, backflow, vorticity convection and diffusion, vorticity layer, its instability, curling, vortex patch formation, and vorticity layer indentation.**

We examine each of them in turn. There is little mathematical analysis available; mainly a physical picture.

6.5.1 Momentum shadow

Consider once again a fluid at rest filling all space except for a bounded obstacle. At time 0, the obstacle instantaneously begins to move toward the negative x-axis at a given constant speed $(-U, 0, 0)$. This is called impulsive motion. By using a frame of reference moving at this velocity, we may alternatively think of an impulsive flow to the right past the obstacle, which is fixed in this frame. The flow then attains the value $(U, 0, 0)$ at infinity and vanishes on the body's surface. In front (to the left of) the obstacle, the particle paths diverge in order to pass around it, but our first question is, do the paths come together again behind the obstacle?

Focus attention on a 2D plane through the center of the obstacle parallel to the flow at infinity. In this plane, the obstacle will have a "top" and a "bottom". One may conceive of two extreme situations: (1) Particle paths that begin their journey around the obstacle near its surface leave the obstacle and continue approximately in a horizontal direction after they pass the top or bottom of the object. This would be the case if there were no, or very weak, stresses or pressures forcing the particles to come together behind the object. When they pass by the top of the object, the particles are moving horizontally, so have 0 vertical momentum. By conservation of momentum, without vertical stresses the particles would continue to have no vertical momentum, and would simply continue to move horizontally. The fluid directly behind (to the right of) the object would be little affected by the outside flow. Its particles would originate and remain in that wake region. We call this wake region a **momentum shadow** because the momentum of the particles passing the object shield it from penetration by those particles.

(2) The second situation is when the particles passing close to the object remain near it. The paths wrap closely around the object.

Which occurs in practice? Usually, something in between these two extremes.

At very low Re, (2) is the case. A combination of viscous stresses and pressure gradients conspire to force the particles to change their directions behind the object, so that they re-merge. There is no visible wake. In fact, in the Stokes approximation, the inertia of the particles (which would cause any wake) is completely neglected. (Recall (5.23), where Re is characterised as measuring the relative importance of inertia versus viscosity.)

As Re is increased, however, there will be a threshhold value at which the momentum of the fluid particles in the x direction becomes great enough, relative to the vertical pressure gradient, that the stream of particles cannot be deflected enough in the vertical direction by that gradient to remain near the object. Particles moving past the front of the obstacle above it and close to it therefore break away from the surface at some point. They remain separate from the particles passing below the object for at least some distance downstream. They may or may not eventually reunite with the stream of particles from the other side of the obstacle.

For large enough values of Re, therefore, a momentum shadow is evident. As we shall see, its particles are not always distinct from those originating ahead of the body—there may be mixing, especially at very large Re. In the case when there is such a shadow, we must consider the effect of the flow past the body on the particles in that shadow.

6.5.2 Eddy formation

When Re is not too large, and the flow is not too irregular, even though some form of momentum shadow exists, viscosity will generally cause the flow past the body to entrain eddy-like motions in the shadow region. In this case, the influence of the outer flow on the shadow flow is by viscous transfer of momentum rather than by mixing of particles.

6.5.3 Backflow

When eddies first occur at moderate Re, they do so in pairs. We may speak of them as being the "upper" and the "lower" eddy. The one occupying the upper part of the shadow will be clockwise with negative vorticity, and the lower one will be opposite.

Near the rear surface of the obstacle, the flow due to the eddy will be in the opposite direction from the flow near the front surface. This is called **backflow**. There will be a point on the surface where the backflow meets the forward flow. At that point the flow is stagnant. It is here that the forward moving particle paths depart from the surface. It is called the separation point.

Since we are dealing with a impulsive initial condition, the flow pattern will be time-dependent. The place of separation will therefore generally move in time.

6.5.4 Vorticity transport

The accounting for vorticity—its generation and transport—is a prime consideration for flows, especially at high Re. Recall (Section 6.1.4) that the boundary layer is a layer of high vorticity, and that vorticity is generated at the obstacle's surface by the adherence (no-slip) boundary condition.

What happens to the vorticity which has been generated? We have (5.17). In 2D, the last term vanishes, and we have left

$$\frac{\partial \omega}{\partial t} + (\boldsymbol{u}\cdot\nabla)\omega = \nu\nabla^2\omega. \tag{6.78}$$

Two processes are described in this equation: diffusion and convection. The flow pattern depends largely on which of the two is dominant.

In 3D, we have, in addition to these two modes of transport, the process of vortex stretching. This essentially allows a greater degree of freedom, with its consequent greater complexity, than would occur in 2D; flows in 2D are in fact unrealistic for this reason.

At very small Re, vorticity is generated all around the body, but there is no boundary layer. The vorticity is carried away in all directions by diffusion. The convection terms in (6.78) would in fact be ignored in the Stokes approximation.

At moderate Re, say around 10, vorticity is generated, but the influence of the convective process makes the flow field asymmetric. There is not much of a boundary layer. Due to convection, there will be an accumulation of vorticity behind the body, from which is also diffuses. Such diffusion to the interior of the wake is what sets up a pair of eddies which are about the same size as the body, or smaller.

The streamlines which pass the forward part of the body and which carry the vorticity continue around the eddies and converge downstream from them. By this time, the vorticity they have been carrying has been largely lost through diffusion.

This was for moderate Re. At higher values, the flow is faster and the convection of vorticity becomes more important, relative to its diffusion. This is also when there is a more pronounced boundary layer in the forward part of the body. It is a **layer of high vorticity**. The particle paths in that boundary layer detach from the body and form, at least temporarily, a thin layer of high vorticity streaming behind the body. This layer is a candidate for the boundary of the momentum shadow. We may call it a **free** vorticity layer.

6.5.5 Instability of the vorticity layer

The fact is, the free vorticity layer does not last long, due to its strong instability.

To see why it is unstable, think of a 2D model in which the layer is replaced by a train of equally spaced point vortices, all of the same sign (negative, for concreteness) and magnitude. Suppose also that the flow is irrotational except at the vortices. We can imagine that this situation corresponds to the flow above the vorticity layer being uniform (hence irrotational) and that below the layer in the shadow having only minor vorticity because the latter has not had time to diffuse there from the layer. This was the scenario in Example 4.12.1(c).

Let us travel in a reference frame with respect to which the vortices are fixed. Show by simple diagrams that if one of the point vortices is displaced upward a small amount, all the others remaining where they were, then the displaced one immediately begins to move to the right. That makes it closer to its neighbor on the right than to the other neighbor. Its influence on the right neighbor is therefore greater than the influence of the other vortices on that neighbor.

Show that these two vortices, the displaced one and its right neighbor, interact in such a way as to tend to make then rotate around each other, as in Example 4.12.1(a).

Although far from rigorous, this argument suggests that instabilities in such a train of point vortices will tend to break it up into individual rotating pairs. The analogous behavior of vortex layers turns out to be the curling up (or rolling up) of portions of the layer. That, in fact, is what is observed. Small indentations in the layer cause it to further deform at that location, the layer locally being stretched, carried around in the clockwise direction, and curled up.

6.5.6 Patch formation

In this section, as well as in Sections 6.5.7 and 6.5.8, we describe a 2D flow. The same phenomena happen in 3D, but their description would not be as simplistic.

As the layer deforms and curls at high Re, it approximately retains its original vorticity, because there is minimal diffusion. What develops is a curled up blob of fluid with vorticity. This is a vortex patch. Observations indicate that the shape is typically elliptical, sometimes with pointed ends. The patches continue to rotate. They are a little analogous to the rotating pairs of point vortices in the above discrete model.

At relatively moderate values of Re, it can't be said that the layer rolls up and forms a patch. But in a sense patches do form: they are the eddies mentioned above. They may form at a fair distance from the separation point on the object's surface.

As Re increases, however, the eddies become transformed into patches. There are several differences: (a) They are now formed primarily not by diffusion, but rather by convection and instabilities. (b) Their birth locations move ever closer to the separation point as Re increases. (c)

There is repeated patch formation. (d) After and during their formation, the patches are convected downstream with the fluid.

For a certain range of Re, their birth is quite a regular process with well-defined period and convection along a well-defined trajectory. At higher values of Re, they are still convected downstream, but their motion is much more irregular.

6.5.7 Vorticity layer indentation

At high Re, the vortex patches induce (by slight viscosity) a rotating ring of fluid at their periphery which, although it is moving in the same direction as the nearby part of the patch, nevertheless has vorticity of opposite sign in the region between the patch and the body. This is because the fluid must stick to the surface of the object—the same process by which vorticity is generated in the boundary layer, but opposite.

The vortex patch closest to the separation point (i.e. the one most recently born) therefore induces a layer of fluid impinging on a neighborhood of the separation point, and to the left of the patch. This is not always a major effect—only when that particular patch has become sufficiently large, and before it has moved downstream. When it does happen, the impingement makes an upward indentation of the vorticity layer. This in turn induces another curling instability at that point, which for high Re is quite close to the separation point. This new instability causes a new vortex patch to form by the usual curling process, while in the meantime the previous patch moves on downstream by convection.

This means the vortex patches are formed in a discrete manner. It is termed "vortex shedding" because the origin of the patches is, after all, the vorticity layer which leaves the obstacle at the separation point. The whole process: patch formation, indentation, new vortex being shed, takes a certain amount of time. This is why vortices sometimes appear to be shed in almost a periodic fashion. For some range of values of Re, it is quite accurately periodic, and this accuracy has been used for industrial measurement purposes. There is a strong correlation, in this range, between external flow velocity and the shedding period.

6.5.8 Karman vortex streets

The most regular shedding processes produce a sequence of vortices being shed at regular intervals from the upper separation point, and another from the lower separation point. Up until now, we have pretty much disregarded the interaction between the upper flow and the lower flow. It is likely that this interaction is minimal when Re is very high. But in the range when the process is regular, there is interaction and it produces the vortex streets, in which the upper vortices are spaced halfway between the lower ones. As you showed in Example 4.12.1(d), such a street arrangement will move backward. This is all relative to the basic fluid motion, however, which of course on average is directed to the right and has a definite mean value.

The net result is that the Karman vortex street moves to the right, but not as fast as the fluid flow outside the wake region.

7 Loss of stability, bifurcation, and the onset of thermal convection

In this section, we begin with some simple mathematical examples illustrating the loss of stability of solutions, and how that is accompanied by the appearance of new forms of solutions. These

phenomena are extremely important occurrences in fluid dynamics.

For example, the circular Couette flow alluded to in Section 5.3.4, i.e. the laminar flow between two rotating cylinders, becomes unstable when the Reynolds number surpasses a certain critical value. Simultaneously, a new form of solution, with fewer symmetries than the original Couette flow, appears. The new form consists of "Taylor vortices", characterized by flow cells of specific heights stacked on each other in the axial direction. In each of these cells, there is not only the prevailing circular Couette-like motion, but also circulating motion in another sense: if one takes a cross-section by a plane containing the cylinder axis, each cell appears as two opposite rectangles. In each of these rectangles, the projections of the streamlines have circulation, i.e. form a concentric family of closed loops. This makes for an angular component of vorticity $\boldsymbol{\omega}$. These new circulating motions are called "Taylor vortices".

The new solution forms are patterned, since they are no longer constant in the axial direction. Their appearance is an example of pattern-forming bifurcation.

As the Reynolds number is further increased, a further bifurcation renders the Taylor vortices unstable and converts them into "wavy vortices", another type of patterned flow. At still larger values of Re, the flow in each cell becomes turbulent.

Most of the pattern-forming bifurcation phenomena in fluid dynamics lead to rather involved mathematics. Often this difficulty is just technical, since the main basic ideas are simpler. I will therefore try to convey those ideas with two model problems bearing only a little bit of resemblence in detail with fluid dynamical problems, but rather faithfully illustrating the ideas.

After that, I will concentrate on the accepted model for thermal convection (Boussinesq equations) and set up the bifurcation problem which leads to the onset of patterned convective solutions when a dimensionless number, the "Rayleigh number", surpasses a critical value. The details of the solution of this bifurcation problem will not be given here.

7.1 Model evolution systems exhibiting loss of stability and bifurcation

7.1.1 An ODE

We look at a simple ODE for a function $y(t)$:

$$y' = y(\beta - y). \tag{7.1}$$

We restrict attention to positive solutions $y > 0$. There exist two constant solutions: $y = 0$ and $y = \beta$.

Think of β as being a variable parameter; sometimes called a "control parameter" or "bifurcation parameter".

When $\beta < 0$, the right side of (7.1) < 0, which implies that all positive solutions decrease to 0 as $t \to \infty$. Therefore the "trivial" solution $y \equiv 0$ is "globally stable".

On the other hand when $\beta > 0$, there is an interval on the y-axis, namely $0 < y < \beta$, such that when y is in this interval, the right side of (7.1) > 0, so that the solution increases in time. Such a solution would approach the limit $y \equiv \beta$ as $t \to \infty$. Since this happens even when the solution starts with $y(0) > 0$ very close to 0, the zero solution is no longer stable: small deviations of y evolve into a different state, namely $y \equiv \beta$.

Thus as β increases past 0, the following two events occur simultaneously: The 0 solution loses its stability, and a new stable solution $y \equiv \beta$ appears and is stable.

When one plots the steady state of (7.1) as a function of β, one sees two branches: $y = 0$ and $y = \beta$. They intersect at $\beta = y = 0$. At this intersection of branches, there is a "bifurcation".

There is still a third characteristic occurrence which often happens after bifurcation: the solution evolves on a slower time scale.

This can be seen as follows. Near the bifurcation point $\beta = 0$, the parameter β will of course be small. We rescale the problem (7.1) by setting $y = \beta z$, so that $\frac{dz}{dt} = \beta z(1-z)$. Further introducing a "slow time variable" $\tau = \beta t$, we obtain

$$\frac{dz}{d\tau} = z(1-z). \tag{7.2}$$

In this regime, the solution is small and its evolution is slow. The smallness and slowness in this case are measured by the small parameter β.

7.1.2 A semilinear heat equation

This time, the problem is the following for $u(x,t)$:

$$u_t = \nu u_{xx} + f(u). \tag{7.3}$$

You can think of ν as being a kinematic viscosity. This problem in intended to be a cartoon of the Navier-Stokes equations. Regarding f, we require

$$f(0) = 0, \quad f'(0) = \alpha > 0. \tag{7.4}$$

And the solutions $u(x,t)$ are required to satisfy

$$u(0,t) = u(1,t) = 0. \tag{7.5}$$

Such solutions can be extended as odd periodic functions of x with period 2, and we shall consider the extended versions. We allow the solutions to be complex valued.

Notice that the trivial solution $u(x,t) \equiv 0$ satisfies all these requirements. It turns out that when ν is small enough, other stable patterned solutions arise.

First, consider the linear case $f(u) = \alpha u$. Then we can solve the problem completely by Fourier series. The "basis functions" $\sin n\pi x$ satisfy the periodicity and oddness conditions, and in fact they form a complete set in that every square integrable function $f(x)$ satisfying those conditions can be expanded in a Fourier series

$$f(x) = \sum_{n=1}^{\infty} f_n \sin n\pi x. \tag{7.6}$$

The series will converge in the L_2 sense.

In the present case, we expand the eventual solution $u(x,t)$ in such a series. But since u depends also on t, the coefficients will also depend on t:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin n\pi x. \tag{7.7}$$

Substituting into (7.3), we find

$$u_n'(t) = (-n\pi^2\nu + \alpha)u_n(t). \tag{7.8}$$

Solving, we find

$$u_n(t) = u_n(0)e^{\sigma_n t}, \quad \sigma_n = -n\pi^2\nu + \alpha, \quad n = 1, 2, \ldots \tag{7.9}$$

This functional dependence of the growth rate σ_n on the "wavenumber" n is called the "dispersion relation" for this problem.

If $\alpha < \pi^2\nu$, i.e. $\nu > \frac{\alpha}{\pi^2}$, all the $\sigma_n < 0$, which means that all the u_n decay exponentially in time, and the solution $u(x,t)$ does also. Certainly the trivial solution is stable.

If we consider the viscosity ν as a control parameter, then the 0 solution loses its stability as ν decreases below the critical value

$$\nu_c = \frac{\alpha}{\pi^2}. \tag{7.10}$$

After that point, there are exponentially increasing solutions starting arbitrarily close to 0.

Let us now add a little nonlinearity into the problem by making $f(u) = \alpha u - u^2$. Notice that there is now another constant solution of (7.3): $u \equiv \alpha$; but it does not satisfy the boundary conditions.

In nonlinear problems like this, there is a concept of "linear stability" of a given stationary solution, such as our solution $u \equiv 0$. One simply replaces the equation by its linearization about the given solution, finds the dispersion relation as in (7.9) above (this is sometimes very difficult), and asserts that the solution is "stable according to the linear stability criterion" if every solution σ_n of the dispersion relation (in some cases there are several σ's for a given n) satisfies $\mathcal{R}\sigma_n < 0$ for all n (this means the real part, for often the σ_n can be complex). Moreover, it is unstable according to this criterion if, for at least one n, there is a σ_n with $\mathcal{R}\sigma_n > 0$. This leaves the intermediate case, when we have a σ lying on the imaginary axis. The question of stability in the case is more involved.

It can be proved that for semilinear heat equations, and in fact for a wide class of problems, stability according to this criterion implies stability in a much stricter, more meaningful, sense.

We now apply the criterion to the problem at hand. First, we test the linear stability of the solution $u \equiv 0$. Linearizing, we obtain exactly the same linear problem as we started with when we set $f(u) = \alpha u$. We can use the previous dispersion relation (7.9) and conclude that the trivial solution loses its stability as ν decreases past ν_c given in (7.10).

If the loss of stability follows the pattern set by the previous example, at this point there should appear another stationary nontrivial solution of the problem (7.3), (7.5). In fact, there does. It cannot be written down explicitly, but such a solution does exist. It is also stable; in a sense the stability of the trivial solution has been "transferred" to the new stationary solution. It is a periodic nonconstant, hence patterned, solution.

An approximation to the new nonzero stationary solution $u^*(x)$ when ν is slightly less that ν_c can be found by the following formal perturbation method. It satisfies

$$\nu(u^*)_{xx} + \alpha u^* - (u*)^2 = 0, \quad u^*(0) = u^*(1) = 0. \tag{7.11}$$

Let $\nu = \nu_c - \epsilon = \frac{\alpha}{\pi^2} - \epsilon$, where $0 < \epsilon \ll 1$. It turns out that for this value of ν, $u^*(x)$ has magnitude $O(\epsilon)$. We set $u^* = \epsilon v$. This transforms (7.11) into

$$\frac{\alpha}{\pi^2}\epsilon v_{xx} - \epsilon^2 v_{xx} + \alpha\epsilon v - \epsilon^2 v^2 = 0.$$

In terms of the differential operator $D = \frac{d}{dx}$, this becomes

$$\frac{\alpha}{\pi^2}(D^2 + \pi^2)v = \epsilon\left[v_{xx} + v^2\right]. \tag{7.12}$$

Problem (a) Write $v(x) = v_0(x) + \epsilon v_1(x) + O(\epsilon^2)$. Substitute this into (7.12) to find that

$$v_0(x) = A \sin \pi x \tag{7.13}$$

for some "amplitude" A.

(b) Find that $v_1(x)$ satisfies

$$(D^2 + \pi^2)v_1 = \text{ some function of } x \text{ and } A. \tag{7.14}$$

Find what that function is.

(c) Multiply (7.14) by $\sin \pi x$ and integrate with respect to x from 0 to 1, using the boundary conditions in (7.11). Show that the left side vanishes when this is done, and we get an expression for the number A. Find what A is, and therefore what the nontrivial solution u^* is, to lowest order in ϵ.

7.2 The Boussinesq equations

Thermal convection of an almost incompressible fluid is caused by buoyancy forces, which are in turn caused by slight density differences due to temperature gradients.

You can think of flows governed by the Navier-Stokes equations (5.9), (5.11) with the internal forcing $\boldsymbol{f}$ representing the buoyancy force, and therefore a function of temperature. In addition to these equations, an equation for the transport of heat (measured by temperature) must be adjoined. We will develop a model of this type in a systematic manner.

The details of this project will be formidable, compared to those in the relatively simple illustrations in the previous section 7.1, and they may obscure the main ideas. Please remember that those ideas in essence remain the same as before: there is a control parameter $\mathcal{R}$ (analogous to β and ν in Section 7.1); the stability of an easy solution to the problem is lost when the control parameter passes through a critical value; and simultaneously, new solutions emerge.

We start with an assumed (which is usual) linear dependence of ρ on temperature T:

$$\rho = \rho_0(1 - \alpha T), \tag{7.15}$$

where α is the "thermal expansion coefficient" and ρ_0 is some reference density, namely the density when $T = 0$.

The force density due to gravity is then

$$\boldsymbol{f} = -g\rho \boldsymbol{e}_2 = -g\rho_0(1 - \alpha T)\boldsymbol{e}_2, \tag{7.16}$$

where g is acceleration due to gravity and $\boldsymbol{e}_2$ is the vertical unit vector.

The thermal transport equation is

$$\frac{\partial T}{\partial t} + (\boldsymbol{u}\cdot\nabla)T = \kappa \nabla^2 T. \tag{7.17}$$

Here κ is the thermal diffusion coefficient.

The **Boussinesq model** consists of the Navier-Stokes equations (5.9), (5.11) with $\boldsymbol{f}$ given by (7.16), coupled with (7.17). The quantity ρ in (5.9) is considered constant (except for its implied temperature variation in the expression (7.16)). In fact, this latter forcing term is the only temperature-dependent term in the Navier-Stokes part of this model. Of course an immediate question arises: if we incorporate the temperature dependence of ρ in the forcing term (7.16), why are we allowed to consider it constant elsewhere?

We shall justify this model in the specific context of slow steady-state convective motions in a layer of fluid whose bottom and top are maintained at definite temperatures, the bottom one being larger than the top one. Thus the fluid is heated below. There will be thermal energy transport from the bottom to the top. This transport occurs through conduction (diffusion of heat) and convection. The latter may or may not occur. When it does, it is called **Bénard convection**.

For simplicity, we consider this problem in 2D. The convective patterns that one finds have velocity periodic in x. If one imbeds these solutions in 3D, they would look like (and are called) "rolls". Other possible patterns in 3D, depending on boundary conditions, are hexagons.

For definiteness, our temperature scale is such that $T = 0$ on the bottom. The relevant dimensional parameters will then be $-T_1$, the temperature at the top, the thickness h of the layer, ρ_0, ν, κ, and α. The most convenient combination of parameters with the dimensions of velocity is $\frac{\nu}{h}$, and the one with dimension of time is $\frac{h^2}{\nu}$, so we nondimensionalize $\boldsymbol{u}$ and t using them. However, we also adjoin a small dimensionless factor ϵ to the scaling of $\boldsymbol{u}$, which will represent a measure of the small magnitude of the convection velocity near its onset. At this point ϵ is unknown. It will be determined in terms of the other parameters. This is reasonable since part of the problem would be to determine how fast the convection occurs. In any case we are interested in the onset of convection, so the motion is slow and ϵ is small.

We define dimensionless variables $\boldsymbol{x}^*,\ t^*,\ \boldsymbol{u}^*,\ T^*$ by

$$x = hx^*,\ T = T_1T^*,\ \boldsymbol{u} = \epsilon\frac{\nu}{h}\boldsymbol{u}^*,\ t = \frac{h^2}{\nu}t^*. \tag{7.18}$$

Thus from (7.15)

$$\rho = \rho_0(1 - \alpha T_1 T^*). \tag{7.19}$$

Our first assumption is that the dimensionless parameter

$$\alpha T_1 \ll 1. \tag{7.20}$$

The conservation of mass equation (3.3) becomes

$$\nabla\cdot\boldsymbol{u}^* - \alpha T_1\left[\nabla\cdot\boldsymbol{u}^*T^* + T^*_{t^*} + \nabla T^*\cdot\boldsymbol{u}^*\right] = 0. \tag{7.21}$$

We write (5.9), (7.17) as

$$\frac{\nu^2\rho\epsilon}{h^3}(\boldsymbol{u}^*_{t^*} + \epsilon(\boldsymbol{u}^*\cdot\nabla)\boldsymbol{u}^*) = \frac{\nu\mu\epsilon}{h^3}\nabla^2\boldsymbol{u}^* - \frac{1}{h}\nabla p - \rho_0 g(1 - \alpha T_1 T^*)\boldsymbol{e}_2, \tag{7.22}$$

$$\nabla^2 T^* = \frac{\nu}{\kappa}(T^*_{t^*} + \epsilon(\boldsymbol{u}^*\cdot\nabla)T^*). \tag{7.23}$$

The dimensionless boundary conditions are

$$\boldsymbol{u}^* = 0 \text{ at } y^* = 0, 1;\quad T^* = 0 \text{ at } y^* = 0,\ T^* = -1 \text{ at } y^* = 1. \tag{7.24}$$

There is one easy solution, corresponding to the case of no motion, with a linear temperature profile:

$$\boldsymbol{u}^* \equiv 0,\quad T^* = -y^*,\quad p = -\rho_0 gh(y^* + \frac{1}{2}\alpha T_1(y^*)^2) \equiv \bar{p}. \tag{7.25}$$

This solution is called the **pure conduction solution**. It represents simple conduction of heat from the bottom to the top.

We look for small perturbations from this conduction solution. To do so, we write, for some parameters γ and β that will be selected soon,

$$T^* = -y^* + \gamma\hat{T},\ p = \bar{p} + \beta\hat{p}. \tag{7.26}$$

We want to retain the matching orders of magnitude of the pressure and buoyancy terms on the right of (7.22). This is possible if the orders of magnitude of $\frac{\beta}{h}$ and of $\gamma g\rho_0\alpha T_1$ are the same. The simplest way to guarantee this is to choose $\beta = \rho_0 hg\gamma\alpha T_1$. Assume that $\nu = \mu/\rho_0$ is independent of T. Making the substitution (7.26) in (7.22) and dividing it by $\frac{\nu\mu\epsilon}{h^3}$, we now have

$$\nabla^2\boldsymbol{u}^* - \frac{gh^3\alpha T_1\gamma}{\nu^2\epsilon}(\nabla\hat{p} - \hat{T}\boldsymbol{k}) = \boldsymbol{u}^*_{t^*} + \epsilon(\boldsymbol{u}^*\cdot\nabla)\boldsymbol{u}^*, \tag{7.27}$$

$$\nabla^2\hat{T} = \frac{\nu}{\kappa}(\hat{T}_{t^*} + \epsilon(\boldsymbol{u}^*\cdot\nabla)\hat{T}) - \frac{\nu\epsilon}{\gamma\kappa}v^*, \tag{7.28}$$

$$\nabla\boldsymbol{u}^* + \alpha T_1\gamma\left[\nabla\cdot\boldsymbol{u}^*\hat{T} - \hat{T}_{t^*} + \nabla\hat{T}\cdot\boldsymbol{u}^*\right] = 0, \tag{7.29}$$

where v^* is the vertical component of $\boldsymbol{u}^*$. At this point, there are two parameters which are still undefined: ϵ and γ. As I mentioned before, ϵ is a measure of the magnitude of the convection which is to be determined as part of the problem. So we cannot prescribe it. On the other hand, we can try to select γ, in the usual way, to match orders of magnitude of the various terms in (7.27), (7.28). This involves too many matching conditions for only one parameter: two terms should be matched in (7.27) and two in (7.28) (assuming, as we shall, that $\frac{\nu}{\kappa}$ is an $O(1)$ quantity). One way out of this dilemma is to choose γ so that the coefficient of the last term on the left side of (7.27) is identically equal to 1, and then specify that the **range** of the other parameters in (7.28) is such that the coefficient of the last term on the right is $O(1)$.

We therefore choose γ so that $\frac{gh^3\alpha T_1\gamma}{\nu^2\epsilon} = 1$, and then define a new dimensionless $O(1)$ parameter

$$\mathcal{R} \equiv \frac{g\alpha T_1 h^3}{\nu\kappa}. \tag{7.30}$$

It is called the **Rayleigh number**. The last term in (7.28) then becomes $-\mathcal{R}v^*$. We obtain

$$\nabla^2\boldsymbol{u}^* - \nabla\hat{p} + \hat{T}\boldsymbol{e}_2 = \boldsymbol{u}^*_{t^*} + \epsilon(\boldsymbol{u}^*\cdot\nabla)\boldsymbol{u}^*,\ \nabla^2\hat{T} + \mathcal{R}v^* = \frac{\nu}{\kappa}\hat{T}_{t^*} + \epsilon\frac{\nu}{\kappa}(\boldsymbol{u}^*\cdot\nabla)\hat{T},$$

$$\nabla\cdot\boldsymbol{u}^* + \epsilon\frac{\nu\alpha T_1}{\kappa\mathcal{R}}\left[\nabla\cdot\boldsymbol{u}^*\hat{T} - \hat{T}_{t^*} + \nabla\hat{T}\cdot\boldsymbol{u}^*\right] = 0, \tag{7.31}$$

For appearance's sake we drop the asterisks and hats, and write the complete system as

$$\nabla^2\boldsymbol{u} - \nabla p + T\boldsymbol{e}_2 = \boldsymbol{u}_t + \epsilon(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \tag{7.32}$$

$$\nabla^2 T + \mathcal{R}v = \frac{\nu}{\kappa}T_t + \epsilon\frac{\nu}{\kappa}(\boldsymbol{u}\cdot\nabla)T, \tag{7.33}$$

$$\nabla\cdot\boldsymbol{u} = \epsilon\frac{\nu\alpha T_1}{\kappa\mathcal{R}}\left[\nabla\cdot\boldsymbol{u}T - T_t + \nabla T\cdot\boldsymbol{u}\right] = 0. \tag{7.34}$$

The last term in (7.34) has order of magnitude $O(\epsilon\alpha T_1 \ll \epsilon)$ so it is formally the smallest of all terms in that system. If we drop that term, we would obtain a system which is a non-dimensional version of the Boussinesq model. This, then, is a formal justification of that model.

The boundary conditions are to be

$$\boldsymbol{u} = T = 0 \quad \text{at} \quad y = 0, 1. \tag{7.35}$$

Clearly this system has the 0 solution, which corresponds to the conduction solution, as the symbols $T,\ \boldsymbol{u},\ p$ in (7.32)–(7.34) are actually scaled deviations from that special solution.

We look for other solutions, however. They are the solutions that involve convection, and will exhibit patterns.

7.3 Loss of stability

If we set $\epsilon = 0$ in (7.32)–(7.34), we obtain a system of equations

$$\nabla^2 \boldsymbol{u} - \nabla p + T\boldsymbol{e}_2 = \boldsymbol{u}_t, \tag{7.36}$$

$$\nabla^2 T + \mathcal{R} v = \frac{\nu}{\kappa} T_t, \tag{7.37}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{7.38}$$

which is simply the linearization of the original nonlinear system about the basic conduction solution. It is also the lowest order problem for $(\boldsymbol{u}, T, p)$. The linear stability test consists in finding all solutions of this linear system; and this involves finding the dispersion relation.

We first look for solutions of (7.36)–(7.38) defined in all space $\boldsymbol{x} = (x, y)$. Later, we shall impose restrictions designed partially to account for the boundary conditions (7.35). It turns out that those conditions complicate the analysis considerably without adding to our insight, and we replace them by easier conditions which are not so physical, but whose analysis is more transparent. Since (7.36)–(7.38) is a system of PDE's with constant coefficients, it suffices to look for solutions of the form

$$\boldsymbol{u} = \boldsymbol{U} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{\sigma t},\ T = \tilde{T} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{\sigma t},\ p = P e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{\sigma t}. \tag{7.39}$$

(Here $\boldsymbol{k} = (k_1, k_2)$ is called a vector wavenumber.) The reason is, first of all, that in great generality, functions of $\boldsymbol{x}$ may be expressed as integral combinations of such exponential functions. Specifically, the Fourier transform of a function $f(\boldsymbol{x})$ is a function $\hat{f}(\boldsymbol{k}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} dx_1 dx_2$ such that

$$f(\boldsymbol{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} dk_1 dk_2. \tag{7.40}$$

If $f \in L_2(\boldsymbol{R}^2)$, then $\hat{f}$ is also, and the integral (7.40) converges if it is understood as a certain limit in L_2. In our case, the Fourier transforms will depend also on t. If now we take the Fourier transform of each equation in (7.36)–(7.38), what results is a system of ODE's in t for the Fourier transforms, as functions of t. That system also has constant coefficients, so that solutions exist which depend exponentially (like $e^{\sigma t}$) in time.

So in an integral sense, all solutions can be written as a linear combination of solutions of the type (7.39) (or other functions with essentially the same exponential behavior). Hence there is no loss of generality is seeking solutions of the form (7.39).

The instabilities occur when we find such a solution with exponential growth rate, i.e. real part of $\sigma > 0$. The possible σ's, in their dependence on $\boldsymbol{k}$, constitute the dispersion relation.

Therefore we insert the assumed form (7.39) with $\boldsymbol{U} = (U, V)$ into the system, and obtain the following algebraic system. It is easily obtained by replacing $\frac{\partial}{\partial t}$ by σ and $\frac{\partial}{\partial x_j}$ by ik_j. Here of course

$x_1 = x$, $x_2 = y$. We set $k = |\boldsymbol{k}| = \sqrt{k_1^2 + k_2^2}$ and $r = \frac{\nu}{\kappa}$ (it is called the *Prandtl number*), and assume r to be $O(1)$.

$$-k^2 U - ik_1 P = \sigma U, \tag{7.41}$$

$$-k^2 V - ik_2 P + \tilde{T} = \sigma V, \tag{7.42}$$

$$-k^2 \tilde{T} + \mathcal{R} V = r\sigma \tilde{T}, \tag{7.43}$$

$$k_1 U + k_2 V = 0. \tag{7.44}$$

First, we eliminate P from (7.41), (7.42) by multiplying by k_1 and k_2, respectively and subtracting. Then U is expressed in terms of V (and $\boldsymbol{k}$) by (7.44), and then from (7.43),

$$\tilde{T} = \frac{\mathcal{R}}{k^2 + r\sigma} V. \tag{7.45}$$

These operations provide an equation in V alone:

$$k^2(k^2 + \sigma)V - \frac{\mathcal{R}k_1^2}{k^2 + r\sigma} V = 0. \tag{7.46}$$

Of course we are seeking nonzero solutions, so must require the coefficient of V to vanish:

$$k^2(k^2 + \sigma)(k^2 + r\sigma) = \mathcal{R}k_1^2. \tag{7.47}$$

This is the dispersion relation, providing the σ's in terms of $\boldsymbol{k}$ and $\mathcal{R}$. The linear stability criterion claims stability if the real part of every σ satisfying (7.47) is negative for all possible $\boldsymbol{k}$, and instability if there is a σ with real part positive for some $\boldsymbol{k}$.

At this point we impose a geometrical restriction on k_2. The spatial domain is $\{0 < y < 1\}$ and the solution $\boldsymbol{u}$, T should vanish on the top and bottom. For the simplest convection, typically the velocity components and temperature will be of one sign between. We can extend them as odd functions of y with respect to $y = 0$ and $y = 1$. This way, we would get functions periodic in y with period 2. We therefore restrict the wavenumber k_2 so that the functions in (7.39) are periodic with that period. That implies that k_2 is a multiple of π: $k_2 = n\pi$ for some integer n. We don't want $n = 0$ because we couldn't then have a nontrivial solution satisfying the 0 boundary conditions on top and bottom. Therefore $n \neq 0$.

We rewrite (7.47) in the following form (after dividing by k^2), where for convenience we use the notation $p = k_1^2$, $q = k_2^2 = n^2\pi^2$, so that $k^2 = p + q$.

$$r\sigma^2 + (1 + r)(p + q)\sigma + \left[(p + q)^2 - \frac{p\mathcal{R}}{p + q}\right] = 0. \tag{7.48}$$

Let σ_1 and σ_2 be the two roots of this equation. The coefficients of (7.48) have the property that $\frac{1+r}{r}(p + q) = -(\sigma_1 + \sigma_2) > 0$ and $\frac{1}{r}\left[(p + q)^2 - \frac{p\mathcal{R}}{p+q}\right] = \sigma_1\sigma_2$. Thus the sign of $\sigma_1\sigma_2$ satisfies

$$\text{sign of } \sigma_1\sigma_2 = \text{ sign of } [(p + q)^3 - p\mathcal{R}]. \tag{7.49}$$

By the restriction $n \neq 0$, we know that $q \geq \pi^2$. (There is no such positive lower bound on p because there is no geometrical constraint in the x-direction; we merely have $p \geq 0$.) When $\mathcal{R}$ is small enough, we see from (7.49) that $\sigma_1\sigma_2 > 0$, and since $\sigma_1 + \sigma_2 < 0$, necessarily both $\sigma_j < 0$ for all (p, q). This means that the 0 solution is stable, according to this criterion, for small $\mathcal{R}$.

However, the expression on the right of (7.49) is monotone decreasing in $\mathcal{R}$ and as $\mathcal{R}$ increases there may be a critical point, call it $\mathcal{R} = \mathcal{R}_c$, at which this expression changes sign for some choice of (p, q). After that, there will be at least one root σ with real part positive, so that stability will have been lost. Let us find that point. It satisfies

$$\min\left[(p+q)^3 - p\mathcal{R}_c\right] = 0, \tag{7.50}$$

where the minimum is taken over all $p \geq 0$ and all $q = n^2\pi^2$, $n \geq 1$. Since the expression is an increasing function of q, the minimum must occur at q's minimum, namely π^2. So we set $q = \pi^2$. Then

$$\min\left[(p+\pi^2)^3 - p\mathcal{R}_c\right] = 0. \tag{7.51}$$

The minimum of this expression occurs when

$$p = k_1^2 = \left(\frac{1}{3}\mathcal{R}_c\right)^{1/2} - \pi^2, \tag{7.52}$$

and the minimum equals $\left(\frac{\mathcal{R}_c}{3}\right)^{3/2} - \left(\left(\frac{\mathcal{R}_c}{3}\right)^{1/2} - \pi^2\right)\mathcal{R}_c$. Setting this equal to 0, we find

$$\mathcal{R}_c = \frac{27}{4}\pi^4. \tag{7.53}$$

One physical interpretation is the following. Recall that $\mathcal{R}$ (7.30) is proportional to the temperature difference T_1. Experimentally, one can slowly increase T_1, and in that way increase $\mathcal{R}$. There will be no convective motion until $\mathcal{R}$ reaches the value given in (7.53); after that the conduction solution is unstable. We expect, and can prove, that convection then begins.

Since the product $\nu\kappa$ is in the denominator of $\mathcal{R}$, one can think of a competition between T_1 and the combined effect of viscosity and thermal diffusion. In order for the onset of convection to occur, the temperature gradient must be large enough to overcome the inhibiting effects of ν and κ.

As before, the loss of stability is accompanied by a bifurcation: there are new solutions (convective) which appear, which take over the stability of the previous conduction solution. The form of those convective solutions when $\mathcal{R}$ is slightly higher than $\mathcal{R}_c$ can be found, on the basis of (7.32)–(7.34) with the terms in ϵ retained, by an analysis similar to what you did in the problem at the end of Section 7.1. It turns out that the velocity and temperature deviations have magnitude $O(\epsilon)$, where ϵ is related explicitly to the deviation $\mathcal{R} - \mathcal{R}_c$ of the Rayleigh number from criticality. The velocity and temperature are periodic in x with wavenumber k_1 given approximately by (7.52), (7.53):

$$k_1^2 = \left(\frac{9}{4}\pi^4\right)^{1/2} - \pi^2 = \frac{1}{2}\pi^2. \tag{7.54}$$

Thus, the onset of convection has a naturally occuring wavelength. Its dimensionless value is $\frac{2\pi}{k_1} \approx 2^{3/2}$ (7.54). Therefore its dimensional value is this number times h. The periodic convective solutions provide "convection cells" with this width.

Actually, since k_1 can take on a continuum of values, when $\mathcal{R} > \mathcal{R}_c$ there is a whole interval of values of k_1, including the one given by (7.54), for which there are unstable growth rates σ satisfying (7.48). Each of them generates a solution of the nonlinear problem with that wavenumber. Only some of these new solutions are stable, however. Their stability analysis can be done not by a linear theory, but rather by a "weakly nonlinear" analysis. In this, one looks for solutions which

operate on two space and two time scales: the normal one we have been using up until now, and a longer space and longer time scale, the ratio of the two scales being ϵ.

This regular pattern of convection is an example of the pattern-forming capability of fluid motions. There are many other examples. There follows a discussion of another aspect of patterns: their modulational dynamics. This is a matter of the motion on larger time and space scales, wherein the phase of periodic patterns is slowly modulated. It is an extremely important and well-studied occurrence. The discussion below is in a rather abstract setting, but of course applies to fluid dynamical scenarios.

7.4 Patterns and phase evolutions

(This is from a recent survey I wrote.) Patterns whose amplitude and wavelength vary slowly and on long spatial scales are seen repeatedly in experiments and numerical simulations. Multiscale perturbation methods, going back at least to the work of Eckhaus and Benjamin and Feir, have been useful in deriving approximate evolution laws which govern these modulated solutions of nonlinear PDE's.

For example, small amplitude patterns arising when the control parameter (such as the Rayleigh number in the previous section) is just past its critical value have dynamics given by "amplitude equations". Amplitude equations are derived formally by a weakly nonlinear analysis of the equations with the ansatz that the solutions can be expressed as functions not only of the original space and time variables but also of long range ones. If ϵ is the ratio of small to long space scales, the solution is then expanded formally in powers of ϵ, up to a few orders. This expanded form of the desired solution is put into the original equation and each order considered separately to derive, with the help of solvability conditions belonging to singular operators, equations for the various terms in the expansion. What results is typically a lowest order equation for the evolution of the amplitude in the long space and time variables which is much simpler than the original PDE.

This simplicity is of course paid for by the restrictive assumptions on the form of the solution. Nevertheless, instabilities in the original unmodulated regular pattern are found this way, and it is widely believed that these modulations are an accurate reflection of nature.

Larger amplitude patterns in higher dimensions for values of the control parameter well past criticality are also capable of modulations on such long space-time scales. When these basic patterns are rolls, i.e. periodic in one spatial direction and constant in others (and in the case of traveling waves also periodic in time), one sees slow modulations in the wavelength, orientation and amplitude of the solutions, of which the first two are the most important. These modulations may result in rolls which are now only approximately straight, or whose wavelength is only approximately constant, in local regions of space. The mathematical formalism for this scenario was developed in perhaps the greatest generality by Cross and Newell. Here we briefly describe their procedure.

First, consider a pattern of unmodulated rolls. The state $\boldsymbol{u}$ of the system at a given point in space and time can be specified exactly in terms of the value of the phase θ of the (spatial and/or temporal) oscillation, θ living on the unit circle. With no loss of generality, we may take θ to be $\theta(\boldsymbol{x}, t) = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t \,(\mathrm{mod}\, 2\pi)$ for some constant vector wavenumber $\boldsymbol{k} = \nabla\theta$ pointing in the direction orthogonal to the rolls. Since we shall deal only with derivatives of θ, we shall ignore the "mod 2π" provision. The solution is then given by a periodic function $\boldsymbol{u} = \boldsymbol{u}(\theta)$. For simplicity we consider only the case $\omega = 0$.

For modulations, the wave-form $\boldsymbol{u}(\theta)$ will vary slowly in space and time, and the wavenumber $\boldsymbol{k}$ will do so as well. The first natural ansatz in this case is therefore that the field function $\boldsymbol{u} = \boldsymbol{u}(\theta, \boldsymbol{X}, T)$, which locally is a function only of θ, is also a function of the long space and time

variables $\boldsymbol{X}$ and T. As before, let $\epsilon \ll 1$ be the ratio of short to long space scales: $\boldsymbol{X} = \epsilon \boldsymbol{x}$; it may be taken to be arbitrary.

The second ansatz is designed to express the modulation of $\boldsymbol{k}$, while retaining its meaning as the spatial gradient of θ. In the unmodulated case with $\boldsymbol{k} = \text{const.}$, we have that $\epsilon\theta(\boldsymbol{x}) = \epsilon \boldsymbol{k}\cdot\boldsymbol{x} = \boldsymbol{k}\cdot\boldsymbol{X} \equiv \Theta(\boldsymbol{X})$ is a function only of $\boldsymbol{X}$. In the modulated case, one still assumes that the long-scale phase Θ defined as $\Theta = \epsilon\theta$ is a function only of $\boldsymbol{X}$ and T (it may also depend on ϵ). The slowly varying wavenumber is then given by $\boldsymbol{k}(\boldsymbol{X}, T) = \nabla_{\boldsymbol{X}}\Theta(\boldsymbol{X}, T)$.

When the assumed form $\boldsymbol{u} = \boldsymbol{u}(\theta, \boldsymbol{X}, T)$ is substituted into the basic equations, one identifies $\nabla_{\boldsymbol{x}}\theta = \nabla_{\boldsymbol{X}}\Theta$, so that

$$\nabla_{\boldsymbol{x}}\boldsymbol{u} = \boldsymbol{k}\cdot\boldsymbol{u}_\theta + \epsilon\nabla_{\boldsymbol{X}}\boldsymbol{u} = \nabla_{\boldsymbol{X}}\Theta\cdot\boldsymbol{u}_\theta + \epsilon\nabla_{\boldsymbol{X}}\boldsymbol{u}.$$

Similar expressions are used for time derivatives.

Then one proceeds as before to expand $\boldsymbol{u}$ and Θ in low powers of ϵ, substitute these expansions into the original PDE's rewritten for functions of the long-range as well as the original space/time variables, and examine each order (power of ϵ) separately. The resulting equations are again analyzed with the use of solvability conditions.

The Cross-Newell equation is an equation for the field $\Theta(\boldsymbol{X}, T)$, and is derived as indicated. It is usually written in the form

$$\Theta_T - M_1(\boldsymbol{k})\nabla\cdot\hat{\boldsymbol{k}} - M_2(\boldsymbol{k})(\hat{\boldsymbol{k}}\cdot\nabla)\boldsymbol{k} = 0, \tag{7.55}$$

where $\hat{\boldsymbol{k}}$ is the unit vector in the direction of $\boldsymbol{k}$. Since $\boldsymbol{k} = \nabla_{\boldsymbol{X}}\Theta$, this equation can be written as a second order equation for Θ of the form

$$\Theta_T - \sum_{i,j} M_{i,j}(\nabla\Theta)\frac{\partial^2\Theta}{\partial X_i \partial X_j} = 0 \tag{7.56}$$

for some nonlinear functions $M_{i,j}$ depending on the original field equations and the original roll solutions. Under some special conditions, this equation takes on a divergence form, so that it is a gradient flow for an energy functional whose integrand is a nonlinear function of $\boldsymbol{k}$. This condition makes the analysis easier.

In typical cases, when the original periodic roll solution is stable according to the linear stability criterion, the equation (7.56) is parabolic, so that its initial value problem is well posed for at least a short time interval.

There has been work directed to using the Cross-Newell equation to explain pattern defects such as disclinations (places where the vector $\boldsymbol{k}$ suddenly changes direction or is ill-defined, dislocations (where a roll terminates) or grain boundaries (boundaries between regions with different pattern orientations). It is surmised that near defects, the local wavenumber $\boldsymbol{k}$ is forced outside the domain where the corresponding roll would be stable. At these locations, (7.55) loses its well-posedness, and therefore its physical relevance, unless other interpretive considerations are added to it.

However, a weak solution or a regularized solution of (7.55) can in many cases be defined. The singularities of weak solutions and analogous properties of regularized ones are then interpreted as defects, including domain boundaries between differently oriented regular patterns.

8 Aspects of turbulence

8.1 Dimensional analysis

Every physical quantity has a "dimension", which dictates how that quantity could be measured in terms of a standard set of units, the most common such set being units of length, mass, time,

temperature. So the standard set of units might be the meter, kilogram, second, and Kelvin degree.

In our discussion of turbulence, we will consistently use length (L), mass (M) and time (T) as fundamental dimensions, in terms of which the dimensions of all other quantities can be expressed. These three symbols do not denote numbers, but rather concepts. Quantities with dimensions other than length, mass, or time occur, but in all cases we shall consider, their dimensions will be power law functions of L, M, and T. The dimension of a quantity ϕ is generally denoted by $[\phi]$.

Thus if V, ρ, u, f, a represent some volume, density, velocity, force and acceleration, then

$$[V] = L^3,\ [\rho] = ML^{-3},\ [u] = LT^{-1},\ [f] = MLT^{-2},\ [a] = LT^{-2}. \tag{8.1}$$

Physical laws (such as the Euler equations, for example), or in fact any equation expressing relationships among physical quantities, are such that the dimension of each term must be the same. In fact, the dimension of some physical constants can be found by using this fact. For example, Newton's law of gravitational attraction says that the force between two far away planets is

$$F = \gamma \frac{m_1 m_2}{r^2}, \tag{8.2}$$

where the m's are the masses of the two planets, r is the separation distance, and γ is the gravitation constant. From this, one obtains that

$$[\gamma] = [F]L^2M^{-2} = L^3M^{-1}T^{-2}. \tag{8.3}$$

More a propos to our present context, one can use this method to obtain, for pressure, viscosity, and kinematic viscosity

$$[p] = ML^{-1}T^{-2},\ \ [\mu] = ML^{-1}T^{-1},\ \ [\nu] = L^2T^{-1}. \tag{8.4}$$

Although we have been dealing with some formidable partial differential equations, the bottom line in many investigations into fluid dynamics is the desire to find universal relationships among properties of flows expressible as real numbers. This is where dimensional analysis can simplify the search considerably. We illustrate this by a relatively simple example.

Consider steady flow of a fluid through a circular pipe, under a pressure gradient p_x. Besides the pressure gradient, the only other determining properties of the flow are the diameter D of the pipe, the flow velocity U, the density ρ, and the viscosity μ. We must therefore have, for some function f,

$$p_x = f(D, U, \rho, \mu). \tag{8.5}$$

Now the first three arguments in the function, namely D, U, ρ, have "independent dimensions", in the sense that the dimensions of no one of them can be expressed in terms of the dimensions of the other two.

On the other hand, the dimensions of μ and p_x can be expressed in terms of those of D, U and p:

$$[p_x] = ML^{-2}T^{-2} = [U]^2[D]^{-1}[\rho],\ \ [\mu] = ML^{-1}T^{-1} = [UD\rho]. \tag{8.6}$$

We may therefore make these two quantities dimensionless by defining

$$\Pi = \frac{p_x}{U^2\rho/D},\quad \Pi_1 = \frac{\mu}{UD\rho}. \tag{8.7}$$

Now (8.5) becomes

$$\Pi = f(D, U, \rho, \Pi_1 U\rho D), \tag{8.8}$$

which we call by the function Φ: $\Pi = \Phi(D, U, \rho, \Pi_1)$.

It turns out that Φ is actually independent of the first three arguments (the dimensional quantities). To see this, remember that if we change the units of measurement, dimensionless quantities do not change. Moreover, we can also find a new set of units of length, mass, and time such that the numerical value of D (say) changes but those of U and ρ do not. For example, suppose we start with the units (meter, kilogram, second). We then change from meters to feet. Say 1 foot = α meters. Then the numerical value of D changes by the factor $1/\alpha$. Next, we change the unit of time from second to "third", where 1 third = α seconds. The numerical value of U does not change, since the factor α cancels out. Similarly, one can find a new unit of mass so that the numerical value of ρ does not change. Of course Π_1 does not change. Therefore in (8.8), the only thing on either side which changes is D. Therefore Φ must be independent of D. Similarly it is independent of U and ρ. We are left with

$$\Pi = \Phi(\Pi_1). \tag{8.9}$$

In this case, you can see that $\Pi_1 = \frac{1}{Re}$. Writing (8.9) in dimensional form, we get

$$p_x = \frac{U^2\rho}{D}\Phi\left(\frac{1}{Re}\right). \tag{8.10}$$

True, we don't know what the function Φ is, and in fact due to complication from turbulence it can only be found experimentally, but nevertheless this represents a long step toward understanding the relationship among these four physical quantities: only a single function needs to be determined experimentally.

Now suppose that there were reason to believe that the function $\Phi(\eta)$ approaches some nonzero constant K as $\eta \to 0$: $\Phi(0) = K$. Then for very large Re, one would obtain an approximate law

$$p_x = \frac{U^2\rho}{D}K. \tag{8.11}$$

This method of reducing a multivariable relation such as (8.5) to a simpler one such as (8.9) generalizes easily to problems where there are any number of quantities with independent dimensions and any number whose dimensions depend on the first set.

Problem Use dimensional analysis, as illustrated, to obtain the general form of the force/velocity relation for a sphere moving with constant velocity through a fluid. In the case of very small Re, since the Stokes equations apply, argue that this relation should not depend on ρ. In this case, do another dimensional analysis to determine the general relation. Compare to (5.56).

8.2 Length scales and energy accounting in homogeneous turbulence

Many crucial concepts in the study of turbulence relate to its multitude of length scales. Energy and velocity are two examples of physical properties of a fluid that can exist simultaneously on many spatial scales when the fluid is turbulent. One of our first objects will be to gain an understanding of what this means.

Recall the Navier-Stokes equations (5.9), (5.11) for an incompressible fluid:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nu\nabla^2\boldsymbol{u} - \frac{1}{\rho}\nabla p + \boldsymbol{f}, \tag{8.12}$$

$$\operatorname{div}\boldsymbol{u} = 0. \tag{8.13}$$

Also recall the kinetic energy per unit volume is $\frac{1}{2}\rho|\boldsymbol{u}|^2$, and the kinetic energy (KE) per unit mass

$$e = \frac{1}{2}|\boldsymbol{u}|^2. \tag{8.14}$$

The latter has dimensions of velocity squared, L^2T^{-2}.

The balance of kinetic energy law was obtained in (5.13):

$$\frac{De}{Dt} = \nu\nabla^2 e - \operatorname{div}\left(\frac{p}{\rho}\boldsymbol{u}\right) + \frac{1}{\rho}\boldsymbol{f}\cdot\boldsymbol{u} - \nu|\nabla\boldsymbol{u}|^2, \tag{8.15}$$

The terms in this latter equation have pretty clear meanings. The first one, $\frac{De}{Dt}$, is the rate by which the (kinetic) energy of a particle is changed. This change consists of several parts, namely the other terms in (8.15). The term $\nu\nabla^2 e$ represents, as usual, the diffusion of e due to small random motions of the molecules–the same mechanism by which momentum is diffused. The diffusion coefficient ν is the same in both cases.

The term $\boldsymbol{f}\cdot\boldsymbol{u}$ represents the addition of energy to the fluid by the action of the body force.

The term $\operatorname{div}\left(\frac{p}{\rho}\boldsymbol{u}\right)$ is the divergence of a vector. It therefore represents the transport of energy by means of a flux

$$\boldsymbol{F} = \frac{p}{\rho}\boldsymbol{u}. \tag{8.16}$$

If the fluid occupies a certain spatial region V, then this flux, evaluated on the boundary ∂V, also tells us the rate at which energy is flowing into and out of V by that kind of transport. Let's take a simple example: the flow of a fluid through a straight tube of finite length, with $\boldsymbol{f} = 0$, when the velocity $\boldsymbol{u} = u\boldsymbol{e}_1$ is constant in time and directed along the axis of the tube, which we take to be the x-axis. The flow domain V is the cylindrical interior. The velocity, hence $\boldsymbol{F}$, vanishes on the cylindrical tube walls, so that the only energy flow into V is through the two ends. We suppose the pressure is constant on each of the two ends, the constants being given by p_1 and p_2. Adding the integrals of the normal component of $\boldsymbol{F}$ over the two ends (input end and output end), the get that the total rate of energy input through the boundary is

$$\int_{\text{ends}} \boldsymbol{F}\cdot\boldsymbol{n} = \frac{1}{\rho}(p_2 - p_1)\int_{\text{ends}} u, \tag{8.17}$$

where $p_1 - p_2$ is the pressure difference from one end to the other.

Therefore the energy produced is equal to the pressure difference times the total volume flow through the tube (the integral of u), divided by ρ. In other words, this pressure difference provides a force which generates energy which is added to the fluid. In steady flow, that force is more or less balanced by the frictional drag between the viscous fluid and the walls of the tube.

Finally, the term on the right of (8.15), $-\nu|\nabla\boldsymbol{u}|^2$, is always negative, is not the divergence of a flux, and is proportional to the viscosity ν. It is an energy loss due to the action of the viscosity. Contrary to the energy production, which (except for the action of $\boldsymbol{f}$) occurs at the boundary, this energy loss occurs everywhere in the fluid. That lost energy is converted into heat. This term is called the (negative of the) dissipation ϵ:

$$\epsilon = \nu|\nabla\boldsymbol{u}|^2. \tag{8.18}$$

It has dimensions of energy per unit mass per unit time:

$$[\nu] = L^2T^{-1}, \quad [\epsilon] = L^2T^{-3}. \tag{8.19}$$

The dissipation ϵ is a basic flow quantity depending on space and time, like velocity, pressure, energy, etc., and is often treated on the same footing as the latter quantities.

To recapitulate, the various terms in (8.15) represent the change in KE e due to convective transport, diffusive transport, production of energy at the boundary, and finally frictional dissipation. In steady situations, the total rate of production of such energy is typically known from the conditions of the experiment. This has to balance the total dissipation, which is the integral of ϵ over the flow domain. In situations where most of the flow domain is filled with spatially homogeneous (more or less uniform in space) turbulence, ϵ is also more or less uniform. Then because of this balance, the conditions of the experiment will give us at least an approximatiom to ϵ.

We shall, mainly for this reason, eventually treat ϵ as a more fundamental flow quantity than $\boldsymbol{u}$.

8.2.1 Periodic flows and averaging

For an example of how an apportioning into many spatial scales might come about, we consider a fluid filling all space, so that there are no boundaries, but the fluid properties are required to be periodic with a given 3-dimensional period "cell". This requirement is not likely to disrupt any of the principle concepts of turbulent dynamics. Nothing really happens at the boundary of a period cell (which, by the way, could be set up at an arbirary location in the flow) to restrict the flow. Therefore the boundary of such a cell is really no boundary at all.

Let L be the period of the flow in each of the 3 directions. The periodicity of the velocity is then expressed by

$$\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{n}L,t)=\boldsymbol{u}(\boldsymbol{x},t) \tag{8.20}$$

for every triple of integers $\boldsymbol{n}=(n_1,n_2,n_3)$.

We denote by $\langle f\rangle(t)$ the average of a function $f(\boldsymbol{x},t)$ over a period cell B_L (a cube of volume L^3):

$$\langle f\rangle(t)=\frac{1}{L^3}\int_{B_L} f(\boldsymbol{x},t)d\boldsymbol{x}. \tag{8.21}$$

Problem 1 Show the following are true for any periodic functions f, g, $\boldsymbol{u}$, $\boldsymbol{v}$. Repeated indices are summed.

$$\begin{array}{ll}
(a) & \langle\partial_i f\rangle=0,\\
(b) & \langle(\partial_i f)g\rangle=-\langle f(\partial_i g)\rangle,\\
(c) & \langle(\nabla^2 f)g\rangle=-\langle(\partial_i f)(\partial_i g)\rangle,\\
(d) & \langle(\boldsymbol{u}\cdot(\nabla\wedge\boldsymbol{v})\rangle=\langle(\nabla\wedge\boldsymbol{u})\cdot\boldsymbol{v}\rangle,\\
(e) & \langle\boldsymbol{u}\cdot\nabla^2\boldsymbol{v}\rangle=-\langle(\nabla\wedge\boldsymbol{u})\cdot(\nabla\wedge\boldsymbol{v})\rangle,\quad \text{if } \nabla\cdot\boldsymbol{v}=0.
\end{array} \tag{8.22}$$

First of all, we use (b) and $\nabla\cdot\boldsymbol{u}=0$ (8.13) to find that

$$\langle(\boldsymbol{u}\cdot\nabla)e\rangle=\langle(u_i\partial_i e)\rangle=-\langle(\partial_i u_i)e\rangle=0, \tag{8.23}$$

so that

$$\left\langle\frac{De}{Dt}\right\rangle=\left\langle\frac{\partial e}{\partial t}\right\rangle=\frac{d}{dt}\langle e\rangle\,. \tag{8.24}$$

Next, the averages of the Laplacian and divergence are zero by (8.22)(a) (note, we are also assuming p is periodic). Therefore from (8.15)

$$\frac{d}{dt}\langle e\rangle=\frac{1}{\rho}\langle\boldsymbol{f}\cdot\boldsymbol{u}\rangle-\nu\langle|\nabla\boldsymbol{u}|^2\rangle. \tag{8.25}$$

With some algebra, it can be shown that the last term equals $-\nu\langle|\boldsymbol{\omega}|^2\rangle$. By (8.18), it is also $-\langle\epsilon\rangle$. (By the way, the quantity $\Omega = \frac{1}{2}\langle|\boldsymbol{\omega}|^2\rangle$ is called the "mean enstrophy".) The last term in (8.25), $-\langle\epsilon\rangle$, is of course the **mean dissipation**. The equation (8.25) expresses a balance of energy law. The total rate of change of kinetic energy in a period cell equals the mean rate of production by the force field $\boldsymbol{f}$ working against the velocity, minus the mean rate of dissipation.

Sometimes we use an alternate notation: $\mathcal{E} = \langle e\rangle$, $\mathcal{F} = \langle \boldsymbol{f}\cdot\boldsymbol{u}\rangle$. Then (8.25) reads

$$\frac{d}{dt}\mathcal{E} = \frac{1}{\rho}\mathcal{F} - \langle\epsilon\rangle. \tag{8.26}$$

In steady state we have a balance between the energy added to the fluid by the external force and the energy dissipated:

$$\mathcal{F} = \rho\langle\epsilon\rangle. \tag{8.27}$$

8.2.2 Filters

Let me recall the basics of Fourier series of L-periodic functions in 3D. We allow complex-valued functions. Any such function which is square-integrable: $\int_{B_L}|f(\boldsymbol{x})|^2 d\boldsymbol{x} < \infty$ can be expressed as a series

$$f(\boldsymbol{x}) = \sum \hat{f}_{\boldsymbol{n}} e^{i(2\boldsymbol{n}\pi/L)\cdot\boldsymbol{x}}, \tag{8.28}$$

where the summation is over all integer vectors $\boldsymbol{n} = (n_1, n_2, n_3)$ (positive and negative), and the summation converges in the sense of $L_2(B_L)$. The coefficients are given by

$$\hat{f}_{\boldsymbol{n}} = \frac{1}{L^3}\int_{B_L} f(\boldsymbol{x}) e^{-i(2\boldsymbol{n}\pi/L)\cdot\boldsymbol{x}} d\boldsymbol{x}. \tag{8.29}$$

The sequence $\{\hat{f}_{\boldsymbol{n}}\}$ is called the **spectral representation** of the function f. By using the orthogonality relations

$$\frac{1}{L^3}\left\langle \left(e^{i(2\boldsymbol{n}\pi/L)\cdot\boldsymbol{x}}\right)\left(e^{-i(2\boldsymbol{n}'\pi/L)\cdot\boldsymbol{x}}\right)\right\rangle = \begin{cases} 0, & \boldsymbol{n}\neq\boldsymbol{n}', \\ 1, & \boldsymbol{n}=\boldsymbol{n}' \end{cases}, \tag{8.30}$$

we get the Parseval equation

$$\langle|f(\boldsymbol{x})|^2\rangle = L^3\sum|\hat{f}_{\boldsymbol{n}}|^2. \tag{8.31}$$

Let $\phi(\boldsymbol{n})$ be a function of $\boldsymbol{n}$ satisfying $0 \le \phi(\boldsymbol{n}) \le 1$ for each $\boldsymbol{n}$. We define the ϕ-filtered function

$$f^{(\phi)}(\boldsymbol{x}) = \sum \hat{f}_{\boldsymbol{n}}\phi(\boldsymbol{n}) e^{i(2\boldsymbol{n}\pi/L)\cdot\boldsymbol{x}}, \tag{8.32}$$

where the coefficients $\hat{f}_{\boldsymbol{n}}$ are given by (8.29).

Clearly $f^{(\phi)}(\boldsymbol{x}) + f^{(1-\phi)}(\boldsymbol{x}) = f(\boldsymbol{x})$. We can speak of the projection operator $P^{(\phi)}$, acting on L_2 L-periodic functions, defined by

$$P^{(\phi)}f = f^{(\phi)}. \tag{8.33}$$

Of particular importance are the cutoff filters at level K,

$$\phi_K(\boldsymbol{n}) = \begin{cases} 1, & |\boldsymbol{n}| \le K, \\ 0, & |\boldsymbol{n}| > K \end{cases}. \tag{8.34}$$

Here we use the notation $|\boldsymbol{n}| = \max\{|n_1|, |n_2|, |n_3|\}$. We also use the notation

$$P_K^- = P^{(\phi_K)}; \quad P_K^+ = P^{(1-\phi_K)}, \tag{8.35}$$

$$f_K^{\pm}(\boldsymbol{x}) = P_K^{\pm} f. \tag{8.36}$$

Thus $f_K^-(\boldsymbol{x})$ is the function obtained by truncating the Fourier series of f at the level K, and $f_K^+(\boldsymbol{x})$ is what was discarded in that truncation.

Given an L-periodic velocity field $\boldsymbol{u}$, we may now calculate

$$\langle |\boldsymbol{u}_K^-|^2 \rangle = L^3 \sum_{|\boldsymbol{n}| \le K} |\hat{\boldsymbol{u}}_{\boldsymbol{n}}|^2, \tag{8.37}$$

which follows by the same orthogonality relation (8.30).

Note that although $\boldsymbol{u}$ is a real vector, the components $\hat{\boldsymbol{u}}_{\boldsymbol{n}}$ are in general complex; then for each $\boldsymbol{n}$, $|\hat{\boldsymbol{u}}_{\boldsymbol{n}}|^2$ simply means the sum of the squares of the moduli of the components of the vector $\hat{\boldsymbol{u}}_{\boldsymbol{n}}$.

Recalling that $\frac{1}{2}|\boldsymbol{u}|^2$ is the kinetic energy density per unit mass, we can (as in the previous section) define the mean kinetic energy as

$$\mathcal{E} = \frac{1}{2}\langle |\boldsymbol{u}|^2 \rangle = \langle e \rangle, \tag{8.38}$$

and the "mean kinetic energy due to wavenumbers $\le K$" as

$$\mathcal{E}_K^- = \frac{1}{2}\langle |\boldsymbol{u}_K^-|^2 \rangle. \tag{8.39}$$

Problem Show, for any filter ϕ, that $\langle |f^\phi(\boldsymbol{x})|^2 \rangle = L^3 \sum (\phi(\boldsymbol{n}))^2 |\hat{f}_{\boldsymbol{n}}|^2$.

8.2.3 Transport of energy through the spectrum

Recall (8.26),

$$\frac{d}{dt}\mathcal{E} = \frac{1}{\rho}\mathcal{F} - \langle \epsilon \rangle. \tag{8.40}$$

For any K, we define $\mathcal{E}_K^-$ by (8.39) and make the analogous definition for ϵ_K^-.

Take the projection P_K^- of (8.12), (8.13) to obtain (see (8.33), (8.35), (8.36))

$$\frac{\partial \boldsymbol{u}_K^-}{\partial t} + P_K^-[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}] = \nu\nabla^2 \boldsymbol{u}_K^- - \frac{1}{\rho}\nabla p_K^- + \boldsymbol{f}_K^-, \tag{8.41}$$

$$\operatorname{div} \boldsymbol{u}_K^- = 0. \tag{8.42}$$

Now take the scalar product of (8.41) with $\boldsymbol{u}_K^-$, and then the average over B_L to get

$$\frac{1}{2}\frac{\partial}{\partial t}\langle |\boldsymbol{u}_K^-|^2 \rangle + \langle \boldsymbol{u}_K^- \cdot [(\boldsymbol{u}_K^- + \boldsymbol{u}_K^+)\cdot\nabla)(\boldsymbol{u}_K^- + \boldsymbol{u}_K^+)] \rangle = -\langle \epsilon_K^- \rangle - \frac{1}{\rho}\langle \boldsymbol{u}_K^- \cdot \nabla p_K^- \rangle + \langle \boldsymbol{f}_K^- \cdot \boldsymbol{u}_K^- \rangle. \tag{8.43}$$

This can be simplified quite a bit. First of all, (8.42) and (8.22)(a) imply that $\langle \boldsymbol{u}_K^- \cdot \nabla p_K^- \rangle = 0$. In fact, we obtain the same balance law (8.40) with subscripts K adjoined, except for an extra term coming from the inertial terms in (8.43):

$$\frac{d}{dt}\mathcal{E}_K^- + F_K^- = \frac{1}{\rho}\mathcal{F}_K^- - \langle \epsilon_K^- \rangle, \tag{8.44}$$

where

$$F_K^- = \langle \boldsymbol{u}_K^- \cdot [(\boldsymbol{u}_K^- + \boldsymbol{u}_K^+)\cdot\nabla)(\boldsymbol{u}_K^- + \boldsymbol{u}_K^+)] \rangle. \tag{8.45}$$

This last expression expands out to four terms, but two of them are 0, because of this result:

Problem (a) Show that for any two divergence-free velocity fields $\boldsymbol{u}$ and $\boldsymbol{u}'$,

$$\langle \boldsymbol{u}\cdot[(\boldsymbol{u}'\cdot\nabla)\boldsymbol{u}]\rangle = 0. \tag{8.46}$$

(b) Show, therefore, that

$$F_K^- = \langle \boldsymbol{u}_K^-\cdot[(\boldsymbol{u}_K^-\cdot\nabla)\boldsymbol{u}_K^+]\rangle + \langle \boldsymbol{u}_K^-\cdot[(\boldsymbol{u}_K^+\cdot\nabla)\boldsymbol{u}_K^+]\rangle. \tag{8.47}$$

This quantity is nonzero in general. From its appearance in (8.44), we see that it is an extra contribution to $\frac{d}{dt}\mathcal{E}_K^-$ which was not there in (8.40). It occurs only for the truncated quantities $\mathcal{E}_K^-$, which represent the energy due to wavenumbers up to K. Since the total energy is conserved, we surmise that this term represents the rate of transfer of energy from the energy spectrum with wavenumbers $< K$ to that with wavenumbers $> K$. Thus, it is transport through the spectrum. If so, then F_K^- should depend both on $\boldsymbol{u}_K^-$ and on $\boldsymbol{u}_K^+$, and indeed it does.

This gives a bit more meaning to the notion that in this flow, energy is transferred from some wavenumbers to others. The process by which this is accomplished depends on the inertia (nonlinear) terms. It is definitely a nonlinear phenomenon. In fact if the inertia terms were missing, (8.44) would hold for all K, but without the term F_K^- on the left.

8.3 The Kolmogorov law

In this section, we no longer necessarily think about a periodic flow, but we nevertheless continue to conceive of the flow, at each point in space/time, as being made up of motions associated with a multitude of different wavenumbers, which we now denote by $\boldsymbol{k}$ rather than by $\boldsymbol{n}$ or $2\pi\boldsymbol{n}/L$ as we did before. Conceptually, this is like letting the period $L\to\infty$. We consider here *highly* turbulent flows with Reynolds number $\widehat{Re} = \frac{\hat{U}\hat{L}\rho}{\mu} \gg 1$. The characteristic velocity $\hat{U}$ will be a typical **mean** velocity, and $\hat{L}$ a macrolength.

We consider stationary turbulence, so that averaged quantities are independent of t. In particular, the time derivatives in (8.40) and (8.44) are zero. Below, the symbols for physical quantities will usually denote time averages for the highly variable flow, and we will not always use symbols for averages. For example we use ϵ rather than $\langle\epsilon\rangle$.

Much of past research into turbulence has historically focussed on "energy accounting", that is trying to find out how the kinetic energy of the fluid motion is produced, transported, and dissipated into heat. Some insight into what happens to the energy is obtained from (8.44).

As mentioned, we think of the flow as being composed of motions with many characteristic lengths ℓ. This is like the Fourier series decomposition of velocity fields: (8.28) with $f = \boldsymbol{u}$. The wavenumber $\boldsymbol{k} = \frac{2\boldsymbol{n}\pi}{L}$ designates spatial variation with characteristic length $\ell = \frac{L}{|n|}$. But in the present scenario the parameter ℓ can take on a continuum of values in some range, which will be specified later.

With each ℓ there is a characteristic velocity $U(\ell)$ and a characteristic time $T(\ell) = \ell/U(\ell)$. A cartoon model might consist of a finite number of oscillating particles indexed by $m = 1,\dots,M$, each with an amplitude ℓ_m and maximum velocity U_m, the numbers ℓ_m taking on only a discrete set of values. Then for each value ℓ in that discrete set, we could identify $U(\ell)$ as the average of those U_m with indices m such that $\ell_m = \ell$. The number $T(\ell)$ defined above would be a typical period of the oscillations which have amplitude ℓ.

Of course there is a characteristic wavenumber $k(\ell) = \frac{1}{\ell}$.

Another definition of characteristic velocity, more along the line of the previous two sections, would be to consider the flow as being periodic with very large period L. Then the allowed

wavenumbers $\boldsymbol{k} = \frac{2\boldsymbol{n}\pi}{L}$ are very closely spaced. Let $\hat{\boldsymbol{u}}(\boldsymbol{k}) = \hat{\boldsymbol{u}}_{\boldsymbol{n}}$ denote the Fourier coefficients, for the allowed values of $\boldsymbol{k}$. For positive scalars k, let $\tilde{U}(k)$ be the average of the numbers $|\hat{\boldsymbol{u}}(\boldsymbol{k})|$ over all wavenumbers $\boldsymbol{k}$ with $|\boldsymbol{k}| = k$. Although it is not defined for all values of k, it can be extended continuously so it is defined for all of them. Then it is natural to think of $\tilde{U}(k)$ as being a characteristic velocity (rather speed) associated with wavenumber k, and to identify $U(\ell) = \tilde{U}(\ell^{-1})$.

As in the case of the boundary layer flow past a semiinfinite flat plate, we can define a variable Reynolds number in terms of $U(\ell)$ by

$$Re(\ell) = \frac{U(\ell)\ell}{\nu}, \tag{8.48}$$

the variation being not the spatial location, but rather the space scale being considered. This differs from $\widehat{Re}$ defined above, which will generally be larger than any of the $Re(\ell)$.

Also on the basis of the previous section, there will be an energy flux through the spectrum (like F_K^-), with dimensions of energy per unit mass per time, i.e. velocity squared per time. We denote it by $F_e(\ell)$. We take it to be positive if the flux is from higher to lower values of ℓ, i.e. from lower to higher k. At this point, our basic assumption, to be justified later, is

Hypothesis $F_e(\ell)$ is independent of ℓ, for the range of values of ℓ to be considered.

We also tentatively suppose that F_e has a characteristic value given by naive use of its dimensions:

$$F_e \approx \frac{U(\ell)^2}{T(\ell)} = \frac{U(\ell)^3}{\ell}. \tag{8.49}$$

This, together with the assumption, provides $U(\ell)^3 \approx \ell F_e$, i.e.

$$U(\ell) \approx (\ell F_e)^{1/3} \tag{8.50}$$

for the range of values of ℓ to be considered. This expression will in fact be rederived in a more satisfactory way later.

Since we have a Reynolds number that depends on ℓ, there should be a scale ℓ_0 on which it is $O(1)$. Recalling that Re measures the relative strengths of the convective versus the viscous terms in (8.12), that scale should be where the viscous dissipation of energy is comparable to the transport of energy by convection. Setting $Re(\ell) \approx 1$ for $\ell = \ell_0$, we get from (8.48), (8.50) $\ell_0 \approx \frac{\nu}{U(\ell_0)} \approx \frac{\nu}{(\ell_0 F_e)^{1/3}}$. Solving for ℓ_0, we find

$$\ell_0 \approx \nu^{3/4}(F_e)^{-1/4}. \tag{8.51}$$

It will turn out that we have just used the hypothesis about the constancy of F_e in a range (namely for $\ell = \ell_0$) which is not in the range of validity of that hypothesis. Nevertheless it is reasonable to suppose that the value of F_e at ℓ_0 is the same order of magnitude as that constant value, and that is sufficient for our argument to work.

Problem Show that the right side of (8.51) is the only combination of the quantities ν and F_e with the dimensions of length.

This length ℓ_0 is called the Kolmogorov length. As was mentioned, we expect that motions with characteristic length around ℓ_0 dissipate their energy as much as they convect it. When the Reynolds number $\widehat{Re}$ of the flow, with respect to a macrolength such as the diameter of the

tube through which the fluid passes, is very large, the ratio of that diameter to ℓ_0 is also very large. In short, the Kolmogorov length is very small at high Reynolds numbers $\widehat{Re}$.

The range we wish to consider is

$$\ell_0 \ll \ell \ll \ell_1, \tag{8.52}$$

where ℓ_1 is a length characterized below.

In particular the left hand inequality implies that the scales we consider are far enough away from those for which dissipation of energy due to viscosity is relevant that we may neglect it. In other words, we have no dissipation. We shall select the upper limit ℓ_1 in (8.52) so that there will also be no energy production. Therefore energy is conserved. As energy is transported through the spectrum from large to small length scales, the conservation of energy implies that the flux F_e must be constant, i.e. independent of ℓ. *This justifies our previous hypothesis for this range of length scales.* Moreover, that energy must eventually be dissipated when it reaches smaller scales of size $O(\ell_0)$.

We have been calling the total rate of dissipation ϵ. By energy accounting again, that must equal the constant flux of energy F_e which occurs in the range (8.52):

$$\epsilon = F_e. \tag{8.53}$$

The energy cascade transfers energy down the wavelength scales, and it is dissipated finally when $\ell \sim \ell_0$. But there must be a source at large length scales–what it is? Energy at these scales may typically be generated by the action of an imposed pressure drop, which works together with the viscous drag at any boundaries that are present to inject energy into the flow. This energy will be associated with a length scale much larger than ℓ_0. This process is not too well understood. The upper bound ℓ_1 used in (8.52) will be the scale of the energy generated by the action of external forces. Therefore the range (8.52) includes only scales on which there is neither production nor dissipation of energy. Some experiments trigger turbulence by passing a fast flow through a wire mesh. The spacing of that mesh would then be a natural choice for ℓ_1.

We now address an issue of paramount concern: how is the kinetic energy $\mathcal{E}$ distributed according to length scales? There should be a distribution function $\hat{E}(\ell)$ such that $\mathcal{E} = \int_0^\infty \hat{E}(\ell)d\ell$, and $\int_0^\ell \hat{E}(s)ds$ is the total energy associated with scales up to the value ℓ. We have the dimensions

$$[\hat{E}] = L/T^2 = [\epsilon]^{2/3}L^{-1/3}. \tag{8.54}$$

Instead of distributing with respect to ℓ, it is common to use the wavenumbers $k = \frac{1}{\ell}$. Then if we call the distribution function $E(k)$,

$$\mathcal{E} = \int_0^\infty E(k)dk, \tag{8.55}$$

we have $[E] = L^3T^{-1}$. (Can you see why this has different dimensions from $[\hat{E}]$?)

Besides depending on ℓ, $\hat{E}(\ell)$ should really depend only on ϵ, μ and ρ for ℓ in the above range, because we are far enough away from the larger length scales that the manner in which energy is generated on those scales should not matter.

Therefore we postulate

$$\hat{E}(\ell) = G(\ell, \epsilon, \mu, \rho).$$

Since ρ can be expressed (8.51), (8.53) in terms of the Kolmogorov length ℓ_0, μ, and ϵ, we write

$$\hat{E}(\ell) = H(\ell, \epsilon, \mu, \ell_0). \tag{8.56}$$

We now perform a dimensional analysis as in Section 8.1. We choose $\ell,\ \epsilon,\ \mu$ as our dimensionally independent quantities and form two dimensionless quantities, which can be considered to be a dimensionless $\hat{E}$ and a dimensionless ℓ_0. On the basis of (8.54),

$$\Pi = \frac{\hat{E}\ell^{1/3}}{\epsilon^{2/3}}, \quad \Pi_1 = \frac{\ell_0}{\ell}. \tag{8.57}$$

Note (a) these quantities do not depend on μ, and (b) by (8.51) that $\Pi_1 = \frac{\nu^{3/4}}{\epsilon^{1/4}\ell}$.

Problem Verify that Π and Π_1 are dimensionless.

We therefore must have a relation

$$\Pi = \Phi(\Pi_1). \tag{8.58}$$

For values of ℓ much larger than ℓ_0 (but still much smaller than ℓ_1), Π_1 is very small, so that it can be argued that Π_1 may be set equal to 0. Calling $C = \Phi(0)$ and assuming $C \neq 0$, we then obtain

$$\hat{E}(\ell) = C\epsilon^{2/3}\ell^{-1/3} \tag{8.59}$$

for ℓ in that range.

This is assuming that the function Φ behaves in a regular manner for small values of its argument. This assumption is one of the sources of controversy over the Kolmogorov law.

As mentioned, the length scales are often referenced by wavenumber $k = \ell^{-1}$ rather than by ℓ. We have the spectral density $E(k)$. The dimensions are $[E] = L^3/T$. Doing the above dimensional analysis with k and E in place of ℓ and e, we obtain

$$E = C\epsilon^{2/3}k^{-5/3}. \tag{8.60}$$

This is the famous Kolmogorov 5/3 scaling law. It fits well with experimental data. Think of k as inverse of characteristic length. In a given experiment, ϵ can be considered a constant. Thus $E(k)$ decreases like $k^{-5/3}$ for ℓ in the range considered, namely (8.52).

One can also find the characteristic velocities of the parts of the flow associated with the different values of k. Denote them by $\tilde{U}(k)$. In the same way that we determined $E(k)$, we can set

$$\tilde{U} = G(k, \epsilon, \mu, k_0) \quad (k_0 = \frac{1}{\ell_0}). \tag{8.61}$$

We then manufacture dimensionless quantities $\Pi = \frac{\tilde{U}}{k^{-1/3}\epsilon^{1/3}}$ and $\Pi_1 = \frac{k}{k_0}$ and conclude, as before, that for k in the appropriate range, approximately

$$\tilde{U} \approx Ck^{-1/3}\epsilon^{1/3}. \tag{8.62}$$

Check that this is just (8.50) in another guise. It serves to provide a firmer basis for (8.50). Actually, our only use of (8.50) was to propose a Kolmogorov length ℓ_0 at which dissipation becomes important. If we merely assume that for our range of ℓ dissipation is negligible, we arrive at the same result.

8.4 Reynolds averaging

One common way to conceptualize the qualitative aspects of turbulent flow, as well as to estimate some of its quantitive features, is to suppose that at each spatial location $\boldsymbol{x}$ and time t, the velocity and the other flow quantities have well defined average values. This could be an average over some small (but not too small) region in space-time containing the point in question. In the case of "steady" fully developed turbulence, it could be the time average. The mean (average) values may then depend on space and time, but on a length scale typically larger than any of those considered in the energy cascade.

Thus, we can write the velocity vector as its mean plus its fluctuation about the mean:

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{U}(\boldsymbol{x},t) + \boldsymbol{u}'(\boldsymbol{x},t), \tag{8.63}$$

with a similar decomposition for vorticity, pressure, etc. Averages will be taken of not only primary flow variables, but also of products of them. The averaging operation is denoted by an overbar—for example $\overline{\boldsymbol{u}} = \boldsymbol{U},\ \overline{\partial_i \boldsymbol{u}} = \partial_i \boldsymbol{U},\ \overline{\boldsymbol{u}'} = 0,\ \overline{U_i u_j'} = 0.$

We substitute (8.63) into the Navier-Stokes equations (8.12), (8.13) with $\boldsymbol{f} = 0$ and then take the average of the resulting equations to obtain

$$\partial_t \boldsymbol{U} - \nu\nabla^2\boldsymbol{U} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{U} + \overline{\boldsymbol{u}'\cdot\nabla\boldsymbol{u}'} + \frac{1}{\rho}\nabla P = 0, \tag{8.64}$$

$$\nabla\cdot\boldsymbol{U} = 0. \tag{8.65}$$

Note also that

$$\nabla\cdot\boldsymbol{u}' = 0. \tag{8.66}$$

The i-th component of the 4th term in (8.64) is

$$\overline{u_j'\partial_j u_i'} = \partial_j\overline{(u_i' u_j')},$$

this last by virtue of (8.66). This shows that the 4th term, being a divergence, acts as a pseudo-stress gradient. Compare with the first term on the right of (5.8). Accordingly, we define the "Reynolds stress tensor" $\boldsymbol{\tau}$ by

$$\tau_{ij} = -\overline{u_i' u_j'}, \tag{8.67}$$

which is symmetric, as any stress tensor should be. Therefore (8.64) becomes

$$\partial_t \boldsymbol{U} - \nu\nabla^2\boldsymbol{U} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{U} + \nabla(\frac{1}{\rho}P - \boldsymbol{\tau}) = 0, \tag{8.68}$$

where the last term can also be written $\nabla\cdot\boldsymbol{T}$, where

$$T_{ij} = \frac{1}{\rho}P\delta_{ij} - \tau_{ij}.$$

If we knew the tensor $\boldsymbol{\tau}$, then we could use that knowledge in (8.68) so that (8.68) and (8.65) would constitute a closed system for the determination of $\boldsymbol{U}$ and P. But nature is not so kind, and $\boldsymbol{\tau}$ is not known. Many attempts have been made in the past to remedy this basic defect, by writing down other equations for the determination of $\boldsymbol{\tau}$. These models propose a "closure" mechanism to produce a closed system for the flow components. In all cases, these other equations are at least partly ad hoc, and in most cases partly empirical.

The simplest approach is to specify, algebraically, what the τ_{ij} are in terms of the gradient of the mean velocity $\boldsymbol{U}$. Or to specify what the shear components $\tau_{ij},\ i \neq j$, are in terms of the gradient of $\boldsymbol{U}$, the trace of $\boldsymbol{\tau}$, and the dissipation ϵ. This trace is related to the "turbulent kinetic energy" (TKE), denoted by k. (This is unrelated to the wavenumbers and inverse lengths scales which were also denoted by k.) The turbulent kinetic energy is defined by

$$k = \frac{1}{2}\sum_i \overline{(u_i')^2} = -\frac{1}{2}\mathrm{tr}(\boldsymbol{\tau}). \tag{8.69}$$

In the latter approach, one then has to find the unknowns k and ϵ. In the "$k-\epsilon$" model, this is done by assuming k and ϵ themselves satisfy transport equations (like (8.68)) but with a different diffusivity ν depending on the other flow quantities and a different nonlinearity.

Computationally, the $k-\epsilon$ model is feasible, but it is not always a reliable prediction of reality.

Many other closure techniques and variations on the above have been proposed.

8.5 Structure of fully developed turbulence near a wall

One cannot help being daunted and mystified by the complexity of turbulent flow. But amid the cloudiness of the subject, unexpected light occasionally shines through. One such ray of insight is the Kolmogorov 5/3 law. (There are other scaling laws as well.) Another unexpected flash comes from the study of how the mean velocity profile behaves near a boundary. This behavior is of course much more complex than that of laminar flow, and so we should be surprised that anything can be determined. But we have the law of the wall and the logarithmic law, both remarkable but relatively simple advances. They are explained in this section.

Consider a channel of width $2h$, occupying the region

$$0 < y < 2h, \quad -\infty < x < \infty, \quad -\infty < z < \infty.$$

First, recall the situation in the case of laminar flow with no y or z-velocity components. That is just the 2D version of the Poiseuille flow considered in Section 5.3.3; see also Problem 5.3.2(a) with $U = 0$ and p_x a constant. The velocity is in the direction of the x-axis and can be found exactly as a function of y in about 10 seconds.

The analogous problem for turbulent flow is orders of magnitude more difficult, and can only be solved imprecisely. We will examine it in the framework of Reynolds averaging.

We suppose the turbulent flow to be "fully developed" in the channel. This means that all averaged quantitites except pressure do not depend on x, z, or t; just on y. Moreover, we suppose that the only nonzero component of the mean velocity will be the x-component, which we denote by $U(y)$. (This part is the same as the laminar case.)

At the two walls $\{y = 0,\ y = 2h\}$, both the mean and fluctuating parts of $\boldsymbol{u}$ are 0, so the Reynolds stress $\boldsymbol{\tau}$ vanishes as well.

The PDE's The x and y components of (8.68) are the following, because of the independence of $\boldsymbol{\tau}$ on $x,\ z,\ t$:

$$-\nu U_{yy} + \frac{1}{\rho}P_x - \partial_y \tau_{12} = 0, \tag{8.70}$$

$$\frac{1}{\rho}P_y - \partial_y \tau_{22} = 0. \tag{8.71}$$

We have here two equations for the four unknown functions U, P, τ_{12} and τ_{22}. Actually, we shall assume known the pressure gradient P_x, which will be shown to be constant, so there will be only 3 unknown functions. But still, there are too many unknowns to even think about an exact solution. But we shall nevertheless be able to at least surmise some important information. First of all, (8.71) implies that $\frac{1}{\rho}P - \tau_{22}$ is independent of y; call it $\frac{P_0(x)}{\rho}$. Differentiating with respect to x, we see that

$$P_0'(x) = P_x(x, y), \tag{8.72}$$

since the x-derivatives of components of $\boldsymbol{\tau}$ are zero. Also, since the left side of (8.72) is independent of y, P_x must be also. Let us call it $P'(x)$.

Integrating (8.70), we now find that

$$-\nu U_y + \frac{y}{\rho}P'(x) - \tau_{12} = \text{ const.} \tag{8.73}$$

Since U_y and τ_{12} are independent of x, so is $P'(x)$. We call it P'. This pressure gradient along the channel will in the following be taken as known from the experimental setup.

The friction velocity Let τ_w denote the mean stress exerted on the wall by the fluid flowing past it. Thus

$$\tau_w = \mu\overline{u_y} = \mu[\overline{U_y} + \overline{u_y'}] = \mu U_y,$$

since $\overline{u_y'} = 0$. At $y = 0$, $\nu U_y = \frac{1}{\rho}(\mu U_y) = \frac{1}{\rho}\tau_w > 0$. Now the quantity $\frac{1}{\rho}\tau_w$ has the dimensions of velocity squared; it therefore defines a characteristic velocity u^*, called the *friction velocity*, by

$$\frac{1}{\rho}\tau_w = (u^*)^2.$$

We use this expression, together with the fact that $\tau_{12} = 0$ at $y = 0$, to evaluate the constant in (8.73). We obtain

$$-\nu U_y + \frac{y}{\rho}P' - \tau_{12} = -(u^*)^2. \tag{8.74}$$

In the center of the channel at $y = h$, we invoke the symmetry of the flow to conclude that $\tau_{12} = 0$ and that the y-derivative vanishes there, so that

$$\tau_{12} = 0,\ U_y = 0 \quad \text{at } y = h.$$

Therefore setting $y = h$ in (8.74), we get

$$(u^*)^2 = -\frac{h}{\rho}P'. \tag{8.75}$$

This is an important result which serves to determine the wall stress (represented by $(u^*)^2$) in terms of the pressure gradient and the channel thickness. As expected, the wall stress is proportional to the pressure gradient. This relation simply expresses a force balance.

Setting (8.75) into (8.74), we get

$$\nu U_y + \tau_{12} = (u^*)^2\left(1 - \frac{y}{h}\right). \tag{8.76}$$

At this point we have reduced the problem to a single equation for U and τ_{12}. This is still more unknowns than equations. And we seem to have forgotten about τ_{22}. In fact we won't be able to find out much about τ_{22}.

Dimensionless variables in the interior For better insight, we now seek to nondimensionalize (8.76). There are two natural ways to do this. In any case, we argue that the characteristic velocity in this turbulent flow should be u^*, because the former's magnitude should be directly related to the wall stress, which is what slows the fluid down and in some sense causes the turbulence. So we nondimensionalize U by u^*.

In the interior of the channel, it is pretty clear that y should be scaled by h, and everywhere, the shear stress τ_{12} should be scaled by the shear stress at the boundary, so by $(u^*)^2$. We therefore define

$$u^+ = \frac{U}{u^*}, \quad \eta = \frac{y}{h}, \quad T = \frac{\tau_{12}}{(u^*)^2}.$$

This is the first natural way to nondimensionalize the y-coordinate. Another will be given below. We now have from (8.76)

$$T + \frac{\nu}{u^* h}\frac{du^+}{d\eta} = 1 - \eta.$$

We also use u^* and h to define our Reynolds number $R^* = \frac{u^* h}{\nu}$ to get

$$T + (R^*)^{-1}\frac{du^+}{d\eta} = 1 - \eta. \tag{8.77}$$

As noted earlier, we have an underdetermined problem—a single equation for the two unknowns T and u^+. But there is even more bad news: the boundary condition at $\eta = 0$ is in trouble. Assuming these are the correct scales in the center of the channel, we obtain the following approximation for large R^*:

$$T_{out} = 1 - \eta, \quad 0 < \eta < 2. \tag{8.78}$$

(Here I have used, as on earlier occasions, the subscript denoting "outer", which as you know is a common usage in asymptotic analysis.)

The surface layer, rescaling, and law of the wall Notice from (8.78) that T_{out} vanishes at $\eta = 1$ (the centerline), as it should by symmetry, but not at $\eta = 0$. This stress should vanish there, however, because all components of $\boldsymbol{\tau}$ do. This lack of satisfaction of the boundary conditions for T in (8.78) points to another *boundary layer effect* and suggests the need for a different spatial scaling near the wall.

In fact we do have two space scales in the channel—one of them associated with h and another one close to the wall; we'll get to that shortly. We will have an "outer" and an "inner" solution, and they will have to match. However, we cannot carry out a full-blown asymptotic analysis because too much is unknown. Much of the following proceeds by "reasonable suppositions", which can also be verified to some extent by experiments.

It is natural to choose the inner scaling in such a way that the two terms on the left of (8.77) have the same orders of magnitude. You are familiar by now with this type of criterion. Thus we define

$$y^+ = \eta R^* = \frac{u^*}{\nu} y. \tag{8.79}$$

Then (8.77) becomes

$$T + \frac{du^+}{dy^+} = 1 - (R^*)^{-1} y^+. \tag{8.80}$$

In the lowest order approximation for large R^* we neglect the last term to obtain the inner approximation

$$T_{in} + \frac{du_{in}^+}{dy^+} = 1. \tag{8.81}$$

In this, there is no incompatibility with the boundary conditions at $y^+ = 0$:

$$T = 0 \text{ and } \frac{du^+}{dy^+} = 1 \text{ at } y^+ = 0. \tag{8.82}$$

To summarize, we have an equation for T alone (not u^+) in the outer region, and an equation relating T and u^+ in the inner region.

The region next to the wall where (8.81) holds, i.e. the region where the spatial variations (in y) have characteristic length $\frac{\nu}{u^*}$, is call the **surface layer** (as opposed to the boundary layer).

The choice (8.79) of scaling in this region says that we are treating the scaled Reynolds stress T the same footing as the wall stress or skin friction $(R^*)^{-1}\frac{du^+}{d\eta}$. The former arises from the inertia terms in the Navier-Stokes equations and the latter from the viscosity terms. That does not mean that both effects are equally important everywhere in the surface layer, however. For example at the wall ($y^+ = 0$), the Reynolds stresses vanish but the viscous stress does not. Since both will be smooth functions of y^+, we surmise that the viscous stress dominates for small but $O(1)$ values of y^+. This delineates the *viscous sublayer* of the surface layer. On the other side of the surface layer there will be a region where the inertial effects dominate; we will get to that "inertial sublayer" shortly.

In the surface layer, then, it is reasonable to suppose that u^+ and T are (approximately) functions only of the inner variable y^+. So we set

$$u^+ = f(y^+), \tag{8.83}$$

$$T_{in} = g(y^+). \tag{8.84}$$

From (8.82) we know that $g(0) = 0$; $\quad f'(0) = 1$. In dimensional terms, we have

$$U(y) = u^* u^+(y^+) = u^* f\left(\frac{u^*}{\nu} y\right), \tag{8.85}$$

$$\tau_{12}(y) = (u^*)^2 T_{in}(y^+) = (u^*)^2 g\left(\frac{u^*}{\nu} y\right). \tag{8.86}$$

These relations (8.83), (8.84), i.e. (8.85), (8.86), are called the *law of the wall*. Note that necessarily $f(0) = g(0) = 0$. However, these equations are not valid in the interior of the channel as $R^* \to \infty$; the outer solution is a valid approximation there (but remember we only have T in the outer region; not u^+).

Velocity in the core In a similar vein, the region where the outer approximations hold is called the *core*. That is where (8.78) holds.

Note that the ratio of the thickness of the surface layer to that of the core is $\frac{1}{R^*}$; this contrasts with the laminar hydrodynamic boundary layer (Prandtl theory), where the ratio is $Re^{-1/2}$.

Let $u_0^+ = u^+(\eta = 1)$ be the centerline (which will be the maximal) mean dimensionless velocity (that is one thing we would like to determine). Then we set

$$u^+ - u_0^+ = F(\eta); \quad F(1) = 0. \tag{8.87}$$

There is actually a better reason to suppose that in the interior the quantity u^+ depends only on η; it involves looking at the energy balance equation in the averaging framework. We skip that argument.

Equation (8.87) is called the *velocity defect law* (analogous to the law of the wall).

Intermediate region; inertial sublayer; logarithmic law So far, we have only found general conclusions—not much specific. Now we take up the question of matching the inner and outer solutions, incomplete as they are. When we do that, some magic appears. Our next assumption, therefore, is an important one. It will be that the inner and outer approximations match up in some intermediate region, where both are valid. This assumption is perfectly standard in matched asymptotics; we didn't mention it in previous sections because in the examples we were doing, matching came out automatically. In the present case, however, we will *use* the assumption to gather further information about the velocity which we could not find by any other means. Since the two length scales are related by (8.79), i.e. by a factor R^*, this intermediate region may be where y^+ is large but not too large (say $y^+ \sim (R^*)^\theta$ for some $0 < \theta < 1$), and η is small but not too small ($\eta \sim (R^*)^{\theta-1}$). In the intermediate zone, we shall match the y-derivatives of u^+:

$$\frac{du^+_{in}}{dy^+} = \frac{du^+_{out}}{d\eta}\frac{d\eta}{dy^+}.$$

From (8.87) and (8.83), we get

$$\frac{df}{dy^+} = (R^*)^{-1}\frac{dF}{d\eta}.$$

Multiplying this equation by $y^+ = R^*\eta$, we get

$$y^+\frac{df}{dy^+} = \eta\frac{dF}{d\eta}.$$

In this equation, the left side is a function only of y^+, while the right side is a function only of η. We argue that η and y^+ should be treated here as independent variables; in fact, R^* is to be considered arbitrary, which means that there is no fixed relation between η and y^+. It follows that the two sides of this equation should be a constant, which we call α. Thus

$$y^+\frac{df}{dy^+} = \alpha = \eta\frac{dF}{d\eta}.$$

We integrate these equations to obtain

$$F(\eta) = \alpha \ln \eta + c_1, \tag{8.88}$$

$$f(y^+) = \alpha \ln y^+ + c_2, \tag{8.89}$$

where $\alpha,\ c_1,\ c_2$ are some constants independent of R^*. We don't know what those constants are, but nevertheless this is quite an important result.

These equations (8.88), (8.89) are called the *logarithmic law of the mean velocity profile*, valid in the intermediate region. Its discovery was a major breakthrough in the theory of wall-induced turbulence. The region where it holds is called the *inertial sublayer*. It is still considered a part of the surface layer, but on the outer reaches of the latter, where viscous stresses are no longer important; we shall verify that claim below. The stresses are therefore primarily Reynolds stresses, which are caused by the inertial terms in the Navier-Stokes equations.

Now let us consider the Reynolds shear stress T. The limit as $\eta \to 0$ of the outer expression can be obtained from (8.78): it approaches 1 as $\eta \to 0$. This should match with the limit of T as $y^+ \to \infty$ in the surface layer. For this, we employ (8.81). On the basis of (8.89) and (8.84), we conclude that

$$\frac{du^+}{dy^+} \approx \frac{\alpha}{y^+} \text{ as } y^+ \to \infty,$$

so that from (8.80) we get that $T \to 1$, as it should do, in order to match with the outer limit.

In the intermediate region, $T \sim 1 \gg \frac{du^+}{dy^+}$ which justifies the claim that this zone is dominated by Reynolds stresses, and viscosity plays a very minor role.

The centerline velocity In the inertial sublayer, we have both of the following, from (8.88), (8.89):

$$u^+_{out} - u^+_0 = \alpha \ln \eta + c_1,$$
$$u^+_{in} = \alpha \ln y^+ + c_2,$$

and the two expressions for u^+ should agree. Subtracting one from the other, we obtain the remarkable expression

$$u^+_0 = \alpha \ln \left(\frac{y^+}{\eta} \right) + c_2 - c_1 = \alpha \ln R^* + c, \tag{8.90}$$

where α and $c = c_2 - c_1$ are independent of R^*.

A practical problem would be to find out how the centerline velocity (U_0 in dimensional terms) depends on the imposed pressure gradient $P_x = P'$. In laminar flow, the dependence is linear. In this case we have from (8.75) and the definition of R^* that

$$U_0 = u^+_0 u^*,$$
$$u^* = \sqrt{h/\rho}\sqrt{-P'},$$
$$R^* = \frac{hu^*}{\nu}.$$

Putting this together with (8.90), we find that as $R^* \to \infty$, U_0 increases like $R^* \ln R^*$ and R^* is proportional to $\sqrt{-P'}$. In all, U_0 increases like $\sqrt{|P'|}\ln|P'|$ for large $|P'|$. This is slower than the (linear) laminar rate of increase. What this means is that if the flow is turbulent, the efficiency in forcing a flow through a pipe by imposing a pressure gradient is less than it would be for the corresponding laminar flow.

All of the above has been verified to some extent by experiments, which also provide reasonable values for α and c.

Leveling of the velocity profile The profile of $U(y)$ across the core region (which makes up the bulk of the channel's thickness) can be determined qualitatively from (8.87). In fact, we have

$$\frac{dU}{dy} = \frac{u^*}{h}\frac{du^+}{d\eta} = \frac{u^*}{h}F'(\eta). \tag{8.91}$$

Since F is a regular function of η, this gives a slope of the order of magnitude $\frac{u^*}{h}$. If we increase the pressure gradient P' keeping all other quantities fixed, u^* will increase like $\sqrt{|P'|}$ (see above),

and so will the slope of the mean velocity profile in the core. This is flatter than the corresponding profile for laminar flow, which increases like $|P'|$.

The reason for this relative flatness is not hard to find. In turbulent flow, momentum is transferred laterally (in the y direction) efficiently by the fluctuations, and this transfer tries to erase maxima in the velocity (momentum) profile.

By the same process, one would expect the momentum to be transported closer to the wall (where it is zero) than in the laminar case. This would result in a narrower surface layer. Indeed this is in agreement with what we have already determined: that layer is thinner than the core region by a factor $\frac{1}{R^*}$, rather than the square root of this number, which is characteristic of laminar boundary layers.

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