Math 6610 Spring 2009 Problem Set 6
Solutions

1) Use the Fourier Method to analyze the stability of the Leap-Frog scheme below for the PDE $v_t + cv_x = 0$:

$$u_j^{n+1} = u_j^{n-1} - 2ckD_0 u_j^n$$

For which values of $\alpha = \frac{ck}{h}$ is this scheme stable in the 2-norm? (Yes, this is a three-level scheme.)

Solution:

Using the Fourier method to analyze the stability of the leapfrog scheme we see that

$$\hat{u}^{n+1}(\xi) = \hat{u}^{n-1}(\xi) - 2\frac{ck}{2h}\left(e^{i\xi h} - e^{-i\xi h}\right)\hat{u}^n(\xi) = \hat{u}^{n-1}(\xi) - 2\alpha i \sin(\xi h)\hat{u}^n(\xi).$$

For each value of ξ, we have a linear constant coefficient second order difference equation

$$\hat{u}^{n+1}(\xi) + 2\alpha i \sin(\xi h)\hat{u}^n(\xi) - \hat{u}^{n-1}(\xi) = 0, \tag{1}$$

to solve. Its characteristic polynomial is

$$r^2 + 2i\alpha \sin(\xi h) r - 1 = 0.$$

The roots of this polynomial are

$$r_\pm = -i\alpha \sin(\xi h) \pm (1 - \alpha^2 \sin^2(\xi h))^{1/2}.$$

If $|\alpha| \le 1$, the discrimant in this expression is nonnegative and the magnitude of each root is 1. If $|\alpha| < 1$, then $|r_\pm| = 1$ and the roots are distinct. Hence the roots satisfy the root condition and the solutions of Eq(1) can be bounded uniformly for all ξ.

If $|\alpha| = 1$, the roots have magnitude 1 and are distinct except for values of ξh that are odd multiples of $\pi/2$. Then the two roots are the same and have magnitude 1. So for $|\alpha| = 1$, the root condition is satisfied for all ξh except for ξh which are odd multiples of $\pi/2$. But, even if $\rho(\xi) > 1$ for a countable number of discrete values of ξ, this does not affect the integral which defines $\|\hat{u}^{n+1}\|_2$ and the scheme is stable. This is born out by numerical tests which show no sign of instability with $\alpha = 1$.

If $|\alpha| > 1$, then $r_\pm$ are pure imaginary, and there are intervals of ξ for which $|r_\pm| > 1$, independent of h and k, so the scheme is unstable

2) Consider the PDE $v_t + cv_x = 0$ on the interval $[0, 1]$ with periodic boundary conditions $v(0, t) = v(1, t)$ and the initial condition $v(x, 0) = f(x)$ for the two choices of f given below. Here use $c = .23$. Implement the 'upwind' scheme

$$u^{n+1} = u^n - ckD_- u^n$$

and the Lax-Wendroff scheme

$$u^{n+1} = u^n - ckD_0 u^n + \frac{1}{2} c^2 k^2 D_+ D_- u^n$$

for this problem and apply them for the two cases $f(x) = \sin(4\pi x)$ and $f(x) = 0$ for $0 \le x \le 1/3$, 1 for $1/3 < x < 2/3$ and 0 for $2/3 \le x \le 1$, extended periodically. What is the exact solution for each problem? Plot the exact solution and the approximate solution u_j^n as a function of j for times $t = 0.2, 0.4, 0.6, 0.8, 1.0$ for each method and for several (stable) values of h and k. Comment on the behavior of the two schemes for each of the two initial conditions.

Solution:

Computational.

3) Consider the upwind scheme

$$u^{n+1} = u^n - ckD_- u^n$$

Assume that the function $u(x,t)$ that satisfies this equation is as smooth as you like and derive the terms through order h^2 in the modified equation for this scheme, that is, find the PDE that the numerical solution from the upwind scheme satisfies through terms of second order.

Solution: The modified equation is derived from the scheme written in the form

$$0 = u(x,t+k) - u(x,t) + \frac{ck}{h}(u(x,t) - u(x-h,t)).$$

We start by expanding all terms about the point (x,t).

$$0 = u + ku_t + \frac{k^2}{2}u_{tt} + \frac{k^3}{6}u_{ttt} + O(k^4) - u + \frac{ck}{h}\left(u - u + hu_x - \frac{h^2}{2}u_{xx} + \frac{h^3}{6}u_{xxx} + O(h^4)\right).$$

This simplifies to

$$u_t + cu_x = -\frac{k}{2}u_{tt} + \frac{ch}{2}u_{xx} - \frac{k^2}{6}u_{ttt} - \frac{ch^2}{6}u_{xxx} + h.o.t. \tag{2}$$

For use in eliminating time derivatives on the right-hand-side, we differentiate Eq(2) with respect to t and with respect to x to get the two equations

$$u_{tt} + cu_{xt} = -\frac{k}{2}u_{ttt} + \frac{ch}{2}u_{xxt} - \frac{k^2}{6}u_{tttt} - \frac{ch^2}{6}u_{xxxt} + ... \tag{3}$$

$$u_{tx} + cu_{xx} = -\frac{k}{2}u_{ttx} + \frac{ch}{2}u_{xxx} - \frac{k^2}{6}u_{tttx} - \frac{ch^2}{6}u_{xxxx} + ... \tag{4}$$

Multiplying Eq(4) by c and subtracting the result from Eq(3), we get

$$u_{tt} - c^2u_{xx} = -\frac{k}{2}u_{ttt} + \frac{ch}{2}u_{xxt} + \frac{ck}{2}u_{ttx} - \frac{c^2h}{2}u_{xxx} + ... \tag{5}$$

Solving Eq(5) for u_{tt} and substituting the result into Eq(2), we obtain

$$u_t + cu_x = -\frac{k}{2}\left\{c^2u_{xx} - \frac{k}{2}u_{ttt} + \frac{ch}{2}u_{xxt} + \frac{ck}{2}u_{ttx} - \frac{c^2h}{2}u_{xxx}\right\} + \frac{ch}{2}u_{xx} - \frac{k^2}{6}u_{ttt} - \frac{ch^2}{6}u_{xxx} + ...$$

which implies

$$u_t + cu_x = \frac{ch}{2}(1-\alpha)u_{xx} + \frac{k^2}{12}u_{ttt} - \frac{ck^2}{4}u_{ttx} - \frac{chk}{4}u_{xxt} + \frac{c^2hk}{4}u_{xxx} - \frac{ch^2}{6}u_{xxx} + ... \tag{6}$$

Upon differentiating Eq(3) with respect to t we obtain

$$u_{ttt} + cu_{xtt} = -\frac{k}{2}u_{tttt} + \frac{ch}{2}u_{xxtt} + ... \tag{7}$$

We substitute this into Eq(6) to obtain

$$\begin{aligned} u_t + cu_x &= \frac{ch}{2}(1-\alpha)u_{xx} + \frac{k^2}{12}(-cu_{xtt} + ...) \\ &\quad -\frac{ck^2}{4}u_{ttx} - \frac{chk}{4}u_{xxt} + (\frac{c^2hk}{4} - \frac{ch^2}{6})u_{xxx} + ... \\ &= \frac{ch}{2}(1-\alpha)u_{xx} - ck^2(\frac{1}{12} + \frac{1}{4})u_{ttx} - \frac{chk}{4}u_{txx} + (\frac{c^2hk}{4} - \frac{ch^2}{6})u_{xxx} + ... \end{aligned} \tag{8}$$

Next, differentiate Eq(4) with respect to x and Eq(5) with respect to x we obtain

$$u_{ttx} - c^2 u_{xxx} = O(k) + O(h)$$

$$u_{txx} + c u_{xxx} = O(k) + O(h).$$

Using these to replace the u_{ttx} and u_{txx} terms in Eq(8), we obtain

$$u_t + cu_x = \frac{ch}{2}(1-\alpha)u_{xx} - \frac{c^3k^2}{3}u_{xxx} + \frac{c^2hk}{4}u_{xxx} + \left(\frac{c^2hk}{4} - \frac{ch^2}{6}\right)u_{xxx} + ...$$

which implies

$$u_t + cu_x = \frac{ch}{2}(1-\alpha)u_{xx} - \frac{ch^2}{6}\left(1 - 3\alpha + 2\alpha^2\right)u_{xxx} + ...$$

4) Consider the Forward Euler and Crank-Nicolson schemes for the pure initial value problem for the heat equation $v_t = \beta v_{xx}$. What are the numerical domains of dependence for each of these schemes? Suppose $h \to 0$ and $k \to 0$, with $\lambda = \beta k/h^2$ held constant. Do the schemes satisfy the CFL condition?

Solution:

Forward Euler: For point (x, t) on the grid with spacing h, k, the initial points on which the numerical solution at (x, t) depends are the discrete points $(x - \frac{t}{k}h, x - (\frac{t}{k} - 1)h, \ldots, x + \frac{t}{k}h)$. Suppose that $\lambda = \frac{\beta k}{h^2}$ is held fixed as $h, k \to 0$. Then $k = \frac{\lambda h^2}{\beta}$, and the points in the numerical domain of dependence can be written

$$(x - \frac{t\beta}{\lambda h}, x - \frac{t\beta}{\lambda h} + h, \ldots, x + \frac{t\beta}{\lambda h}).$$

As $h \to 0$, the left and right most points move toward $-\infty$ and ∞, respectively, and the discrete points, which are spaced h apart, become more densely packed. The numerical domain of dependence approaches $\mathbb{R}$ as $h \to 0$.

Crank-Nicolson: The Crank-Nicolson equation can be written

$$-\frac{\lambda}{2}u_{j-1}^{n+1} + (1+\lambda)u_j^{n+1} - \frac{\lambda}{2}u_{j+1}^{n+1} = \frac{\lambda}{2}u_{j-1}^{n} + (1-\lambda)u_j^{n} + \frac{\lambda}{2}u_{j+1}^{n}.$$

At time level $n+1$, these equations couple together the values u_j^{n+1} and couple all of them to all of the values u_i^n. Hence every value at a timestep depends on all the values at the previous timestep, and therefore on all of the initial values. The numerical domain of dependence for fixed h, k, is a discrete set of points that spans the x-axis. As $h \to 0$, these points become more and more densely packed, and the numerical domain of dependence approachs $\mathbb{R}$.

5) Consider the equation $v_t + cv_x = 0$ for $-\infty < x < \infty$, and $t > 0$. Suppose the initial data is as smooth as you want and has compact support (that is, it is 0 outside a closed bounded interval). Show directly by analyzing the equation satisfied by the global error that the Lax-Wendroff scheme converges for this problem provided $|\alpha| \leq 1$ where $\alpha = \frac{ck}{h}$. Do not just quote the Lax-Richtmyer equivalence theorem.

Solution:

First we show that the scheme is stable in the 2-norm using the Fourier method. Proceeding in the usual way, we find that the amplification factor for the Lax-Wendroff scheme for the above PDE is

$$\rho(\xi) = 1 - 2\alpha^2 \sin^2\left(\frac{\xi h}{2}\right) - i2\alpha \sin\left(\frac{\xi h}{2}\right) \cos\left(\frac{\xi h}{2}\right).$$

The square of the modulus of $\rho(\xi)$ is

$$|\rho(\xi)|^2 = 1 + 4\alpha^2(\alpha^2 - 1)\sin^4\left(\frac{\xi h}{2}\right).$$

Since $\max_{0 \leq x \leq 1} x(1-x) = 1/4$, it follows that for $\|\alpha\| \leq 1$, $|\rho(\xi)| \leq 1$, so the scheme is stable, so we know that $\|u^n\|_2 \leq \|u_0\|_2$.

To prepare to analyze the error, write the scheme as

$$u^{n+1} = Cu^n$$

where C is the operator $I - ckD_0 + \frac{c^2k^2}{2}D_+D_-$. From the stability analysis, we have that $\|C\|_2 \leq 1$. Letting $\mathcal{L}^n$ denote the grid function whose values are the location truncation errors made in advancing the system from time level n to time level $n+1$, we have that the solution of the PDE satisfies

$$v^{n+1} = Cv^n + k\mathcal{L}^n.$$

Letting $e^n = v^n - u^n$, it follows from the above two equations that

$$e^{n+1} = Ce^n + k\mathcal{L}^n.$$

This is an equation that describes the evolution of the global error e^n. Iterating this equation we find that

$$e^n = C^n e^0 + k\sum_{j=0}^{n-1} C^{n-j}\mathcal{L}^j.$$

It follows that

$$\|e^n\|_2 \leq \|C^n\|_2 \|e^0\|_2 + k\sum_{j=0}^{n-1} \|C\|^{n-j} \|\mathcal{L}^j\|_2.$$

Hence,

$$\|e^n\|_2 \leq \|e^0\|_2 + k\sum_{j=0}^{n-1} \|\mathcal{L}^j\|_2.$$

Since $\mathcal{L} = O(h^2) + O(k^2)$, and for $nk \leq T$, the last sum goes to 0 as $k, h \to 0$, with α fixed. Provided $\|e^0\|_2 \to 0$, the scheme is convergent.