Math 6620 Problem Set 6
Due May 6, 2009 by 5 PM

1) Use the Fourier Method to analyze the stability of the Leap-Frog scheme below for the PDE $v_t + cv_x = 0$:

$$u_j^{n+1} = u_j^{n-1} - 2ckD_0u_j^n$$

For which values of $\alpha = \frac{ck}{h}$ is this scheme stable in the 2-norm? (Yes, this is a three-level scheme.)

2) Consider the PDE $v_t + cv_x = 0$ on the interval $[0, 1]$ with periodic boundary conditions $v(0, t) = v(1, t)$ and the initial condition $v(x, 0) = f(x)$ for the two choices of f given below. Here use $c = .23$. Implement the 'upwind' scheme

$$u^{n+1} = u^n - ckD_-u^n$$

and the Lax-Wendroff scheme

$$u^{n+1} = u^n - ckD_0u^n + \frac{1}{2}c^2k^2D_+D_-u^n$$

for this problem and apply them for the two cases $f(x) = \sin(4\pi x)$ and $f(x) = 0$ for $0 \leq x \leq 1/3$, 1 for $1/3 < x < 2/3$ and 0 for $2/3 \leq x \leq 1$, extended periodically. What is the exact solution for each problem? Plot the exact solution and the approximate solution u_j^n as a function of j for times $t = 0.2, 0.4, 0.6, 0.8, 1.0$ for each method and for several (stable) values of h and k. Comment on the behavior of the two schemes for each of the two initial conditions.

3) Consider the upwind scheme

$$u^{n+1} = u^n - ckD_-u^n$$

Assume that the function $u(x, t)$ that satisfies this equation is as smooth as you like and derive the terms through order h^2 in the modified equation for this scheme, that is, find the PDE that the numerical solution from the upwind scheme satisfies through terms of second order.

4) Consider the Forward Euler and Crank-Nicolson schemes for the pure initial value problem for the heat equation $v_t = \beta v_{xx}$. What are the numerical domains of dependence for each of these schemes? Suppose $h \to 0$ and $k \to 0$, with $\lambda = \beta k/h^2$ held constant. Do the schemes satisfy the CFL condition?

5) Consider the equation $v_t + cv_x = 0$ for $-\infty < x < \infty$, and $t > 0$. Suppose the initial data is as smooth as you want and has compact support (that is, it is 0 outside a closed bounded interval). Show directly by analyzing the equation satisfied by the global error that the Lax-Wendroff scheme converges for this problem provided $|\alpha| \leq 1$ where $\alpha = \frac{ck}{h}$. Do not just quote the Lax-Richtmyer equivalence theorem.

6) Consider the scheme

$$
\begin{aligned}
(L^h u)_{j,l} \;&\equiv\; \frac{1}{h^2}\Big(-\beta_{j,l-1/2}u_{j,l-1} - \beta_{j-1/2,l}u_{j-1,l} - \beta_{j+1/2,l}u_{j+1,l} - \beta_{j,l+1/2}u_{j,l+1} \\
&+\; (\beta_{j,l-1/2} + \beta_{j-1/2,l} + \beta_{j+1/2,l} + \beta_{j,l+1/2})u_{j,l}\Big) = f_{j,l}
\end{aligned}
$$

for the variable coefficient Poisson equation $-\nabla \cdot (\beta \nabla v) = f$ for $\mathbf{x} \in R \equiv [0,1]^2$ with homogeneous Dirichlet boundary conditions. The local truncation error of this scheme $\mathcal{L} = O(h^2)$, that is $L^h v - f = O(h^2)$ for the solution $v(\mathbf{x})$ of the differential equation problem. Show that the global error $|v_{j,l} - u_{j,l}| = O(h^2)$ for all grid points in R. (Hint: Use a maximum principle argument.) What can you say about the solvability of the discrete equations $L^h u = f$?