Math 6620 Spring 2009 Problem Set 5
Solutions

1) Construct a) an example of a consistent but not zero-stable linear multistep method, and b) an example of a zero-stable but not consistent linear multistep method. What kind of behavior do you expect from the solution to each? (Hint: They fail in very different ways. If you do not see this, implement both of your methods, try them on a suitable initial value problem, and report on the results.)

Solution:

a) A consistent, but not zero stable linear multistep method (LMM) is

$$y_{n+2} - 3y_{n+1} + 2y_n = -kf_n.$$

The characteristic polynomials for this scheme are $\rho(z) = z^2 - 3z + 2$ and $\sigma(z) = -1$. Since $\rho(1) = 0$ and $\rho'(1) = -1 = \sigma(1)$, the scheme is consistent. The roots of $\rho(z) = 0$ are $z = 1$ and $z = 2$ so the root condition is violated and the scheme is not zero stable. For this scheme, numerical solutions will generally grow unboundedly as $k \to 0$, $n \to \infty$, with $nk = t$ fixed.

b) A zero stable, but not consistent LMM is

$$y_{n+2} - \frac{1}{4}y_n = 0.$$

The characteristic polynomials for this scheme are $\rho(z) = z^2 - \frac{1}{4}$ and $\sigma(z) = 0$. Neither consistency condition holds. The roots of $\rho(z) = 0$ are $z = \pm\frac{1}{2}$, so the root condition is satisfied. For any initial condition the numerical solution approaches 0 as $n \to \infty$, independent of the ODE whose solution we wish to compute. The numerical solution does not blow up like in case (a), but it does not converge to the solution of the IVP.

2) a) Find the range of $\alpha \in \mathbf{R}$ for which the method

$$y_{n+2} + (\alpha - 1)y_{n+1} - \alpha y_n = \frac{k}{4}\{(\alpha + 3)f_{n+2} + (3\alpha + 1)f_n\}$$

is consistent and zero-stable.
b) Apply the method with $\alpha = -1$ to the scalar IVP $y' = y$; $y(0) = 1$ and solve **exactly** the resulting difference equation, taking the starting values to be $y_0 = y_1 = 1$. Show that the numerical solution does not converge as $k \to 0$ and $n \to \infty$. (No programming should be done for this problem.)

Solution:

(a) The characteristic polynomials for the scheme are $\rho(z) = z^2 + (\alpha - 1)z - \alpha$ and $\sigma(z) = \frac{1}{4}(\alpha + 3)z^2 + \frac{1}{4}(3\alpha + 1)$. Since $\rho(1) = 0$ and $\rho'(1) = \alpha + 1 = \sigma(1)$, the consistency conditions hold for all values of α. The polynomial ρ factors as $\rho(z) = (z - 1)(z + \alpha)$, so the roots of $\rho(z) = 0$ are $z = 1$ and $z = -\alpha$. To satisfy the root condition and thus have a zero-stable method, we need $-1 < \alpha \leq 1$.

(b) With $\alpha = -1$, the scheme becomes

$$y_{n+2} - 2y_{n+1} + y_n = \frac{k}{2}(f_{n+2} - f_n).$$

Applied to the ODE $y' = y$, this yields the equation

$$\left(1 - \frac{k}{2}\right) y_{n+2} - 2y_{n+1} + \left(1 + \frac{k}{2}\right) y_n = 0.$$

This has characteristic polynomial

$$\left(1 - \frac{k}{2}\right) z^2 - 2z + \left(1 + \frac{k}{2}\right) = 0,$$

which has roots $s_1 = \frac{1+k/2}{1-k/2}$ and $s_2 = 1$. The general solution is $y_n = c_1 s_1^n + c_2$. To match the given starting values $y_0 = y_1 = 1$, we must use $c_1 = 0$ and $c_2 = 1$. So the numerical solution is $y_n = 1$ for all $n \geq 0$ and independent of k. This does not converge to the solution $y(t) = e^t$ of the IVP as $k \to 0$.

3) The classical 4th order Runge-Kutta method is given by:

$$v_1 = kf(x_n, y_n)$$
$$v_2 = kf(x_n \ + \ k/2, y_n \ + \ 1/2v_1)$$
$$v_3 = kf(x_n \ + \ k/2, y_n \ + \ 1/2v_2)$$
$$v_4 = kf(x_n \ + \ k, y_n \ + \ v_3)$$
$$y_{n+1} = y_n + \frac{1}{6}\{v_1 \ + \ 2v_2 \ + \ 2v_3 \ + \ v_4\}$$

The intersection of the region of absolute stability (R_A) with the real axis is the interval [-2.78,0].

Implement the method and apply it to the scalar problem

$$y' \ = \ \sin(x) \ - \ 1000(y \ + \ y^2) \qquad y(0) = 1$$

on the interval [0,0.1].
a) Present numerical evidence that the scheme is 4th order for k small enough.
b) How small should k be so that the calculation is absolutely stable? Show numerical results of what happens when k is not small enough.

Solution:

(a) Computational

(b) The solution of the IVP starts with value 1 and decreases toward 0. The derivative with respect to y of the right-hand-side function f is $-1000(1 + 2 * y)$ can therefore be as small as -3000. We would like $-3000h$ to lie within the region of absolute stability of the 4th order RK method.

4) Consider a linear constant-coefficient r^{th} order difference equation

$$\sum_{j=0}^{r} \alpha_j y_{n+j} = 0, \tag{1}$$

with $\alpha_r \neq 0$ and $\alpha_0 \neq 0$. We showed in class that if s is a root of the polynomial

$$\rho(z) = \sum_{j=0}^{r} \alpha_j z^j,$$

then $y_n = Cs^n$ solves (1) for any constant C.
a) Suppose $\rho(z)$ has r *distinct* roots $s_1, s_2, \ldots, s_r$. Then the linearity of (1) implies that

$$y_n = c_1 s_1^n + c_2 s_2^n + \ldots + c_r s_r^n, \tag{2}$$

solves (1) for any coefficients $c_1, \ldots, c_r$. Prove that *all* solutions of (1) can be written in this form for some choice of the coefficients $c_1, \ldots, c_r$.

b) If $\rho(z)$ has a double root s_1, show that s_1^n and ns_1^n are both solutions of (1). (Hint: $\rho'(s_1) = 0$.)

Solution:

(a) Suppose that $\rho(z)$ has distinct roots $s_1, \ldots, s_r$. Then $y_n = c_1 s_1^n + \ldots + c_r s_r^n$ is a solution of Eq(1) for any $c_1, \ldots, c_r$. Let $\{y_n\}$ denote an arbitrary solution of Eq(1). We want to show that this solution can be written in the form in Eq(2). From Eq(1), we see that

$$y_{n+r} = -\frac{1}{\alpha_r} \left\{ \alpha_0 y_n + \alpha_1 y_{n+1} + \ldots + \alpha_{r-1} y_{n+r-1} \right\},$$

so that the values of the solution are uniquely determined by the r starting values $y_0, \ldots, y_{r-1}$. It therefore suffices to show that we can choose $c_1, \ldots, c_r$ so that $c_1 s_1^j + \ldots + c_r s_k^j$ matches the starting values y_j for $j = 0, 1, \ldots, r-1$. For this to be true, coefficients $c_1, \ldots, c_r$ would have to satisfy the linear system

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ s_1 & s_2 & \ldots & s_r \\ s_1^2 & s_2^2 & \ldots & s_r^2 \\ \vdots & \vdots & & \vdots \\ s_1^{r-1} & s_2^{r-1} & \ldots & s_r^{r-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{r-1}. \end{bmatrix}$$

The matrix here is a vanderMonde matrix which is nonsingular provided that, as is the case here, the numbers $s_1, s_2, \ldots, s_r$ are distinct.

(b) Suppose s_1 is a double root of $\rho(z) = 0$. We show that both s_1^n and ns_1^n are solutions to Eq(1). We already know that s_1^n is a solution. To see that ns_1^n is also a solution, first note that $0 = \rho'(s_1) = \sum_{j=0}^{r} \alpha_j j s_1^{j-1}$. If $s_1 \neq 0$, this can be rewritten as

$$0 = s_1^{-1} \sum_{j=0}^{r} \alpha_j j s_1^j. \tag{3}$$

For ns_1^n to be a solution of Eq(1), the following expression must be zero for all $n \geq 0$,

$$\sum_{j=0}^{r} \alpha_j (n+j) s_1^{n+j} = ns_1^n \sum_{j=0}^{r} \alpha_j s_1^j + s_1^n \sum_{j=0}^{r} \alpha_j j s_1^j$$

If $s_1 = 0$, both terms are trivially 0. If $s_1 \neq 0$, then the first term is 0 because it is the same as $\rho(s_1)$. The second term is zero by Eq(3).

5) a) Determine the 3rd order 3-step BDF method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_k f_{n+k}.$$

b) Use the 2nd order BDF method $y_{n+2} - 4/3y_{n+1} + 1/3y_n = 2h/3f_{n+2}$ to solve the ODE from problem 4:

$$y' \; = \; \sin(x) \; - \; 1000(y \; + \; y^2) \qquad\qquad y(0) = 1.$$

You have $y_0 = y(0)$, but you need y_1. Use a method of at least 2nd order to generate this value (e.g. your 4th order RK method from problem 3), then use the BDF method to solve the ODE on the interval [0,0.1].

Solution:

a) We want to determine values of $\alpha_0, \ldots, \alpha_3$ and β_3 so that

$$\frac{1}{h\beta_3}\sum_{j=0}^{3}\alpha_j y(t+jh) - y'(t+3h) = O(h^3) \tag{4}$$

for any sufficiently smooth function $y(t)$. We begin by expanding the terms $y(\cdot)$ about $t+3h$. In the following, unless indicated otherwise, y and its derivatives are evaluated at $t + 3h$

$$\begin{aligned}
y(t) &= y - y'3h + y''\frac{(3h)^2}{2} - y'''\frac{(3h)^3}{6} + O(h^4)\\
y(t+2h) &= y - y'2h + y''\frac{(2h)^2}{2} - y'''\frac{(2h)^3}{6} + O(h^4)\\
y(t+h) &= y - y'h + y''\frac{(h)^2}{2} - y'''\frac{(h)^3}{6} + O(h^4)\\
y(t+3h) &= y.
\end{aligned}$$

Substituting these into the left-hand-side of Eq(4) and collecting terms, we obtain

$$\frac{1}{h\beta_3}\left\{(\alpha_0+\alpha_1+\alpha_2+\alpha_3)y - (3\alpha_0+2\alpha_1+\alpha_2)hy' + (9\alpha_0+4\alpha_1+\alpha_2)\frac{h^2}{2}y'' - (27\alpha_0+8\alpha_1+\alpha_2)\frac{h^3}{6}y''' + O(h^4)\right\} - y'.$$

For this to be $O(h^3)$, the following equations must hold

$$\begin{aligned}
\alpha_0+\alpha_1+\alpha_2+\alpha_3 &= 0\\
-\frac{1}{\beta_3}(3\alpha_0+2\alpha_1+\alpha_2) &= 1\\
9\alpha_0+4\alpha_1+\alpha_2 &= 0\\
27\alpha_0+8\alpha_1+\alpha_2 &= 0.
\end{aligned}$$

We can normalize by setting $\beta_3 = 1$. Solving the system we find that $\alpha_0 = -1/3$, $\alpha_1 = 3/2$, $\alpha_2 = -3$, and $\alpha_3 = 11/6$. Using these values, the 3rd order BDF scheme is

$$y_{n+3} - \frac{18}{11}y_{n+2} + \frac{9}{11}y_{n+1} - \frac{2}{11}y_n = \frac{6}{11}hf_{n+3}.$$

b) Computational

6) The equation $u'' = -u + \mu(1-u^2)u'$ is known as the Van der Pol equation and represents a simple harmonic oscillator to which has been added a nonlinear term that introduces positive damping for $|u| > 1$ and negative damping for $|u| < 1$. Solutions to the equation approach limit cycles of finite amplitude, and if $\mu >> 1$, the oscillation is characterized by periods of slow change punctuated by short intervals during which $u(t)$ swings rapidly from positive to negative or back again.
a) Reduce the Van der Pol equation to a first order system of ODEs and compute the Jacobian matrix of this system.
b) What can you say about stiffness of this system?
c) Solve the Van der Pol equation with initial values $u(t) = 2$, $u'(t) = 0$ over the interval $0 \le t \le \max(20, 3{*}\mu)$ for each of μ = 1,5,10,50,100,200. Use Matlab's ode solvers ode45 and ode15s (with an analytic Jacobean provided for ode15s) for each of these problems. Report on your experience with each solver for each value of μ. How many timesteps does each solver take on each problem (how long is the output vector t)? You will have to read Matlab's help pages on solving ODEs, and, in particular, the information about the two solvers ode45 and ode15s.

Solution:

(a) Let $v = u'$. Then $v' = -u + \mu(1-u^2)v$. So the vdP equation is equivalent to the system

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ -u + \mu(1-u^2)v \end{bmatrix} = \begin{bmatrix} f(u,v) \\ g(u,v) \end{bmatrix}.$$

The Jacobian matrix is

$$J = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 - 2\mu uv & \mu(1-u^2) \end{bmatrix}.$$

(b) The eigenvalues of J are

$$\lambda_\pm = \frac{1}{2}\left\{\mu(1-u^2) \pm \left(\mu^2(1-u^2)^2 - 4 - 8\mu uv\right)^{1/2}\right\}.$$

or

$$\lambda_\pm = \frac{1}{2}\left\{\mu(1-u^2) \pm |\mu(1-u^2)| \left(1 - \frac{4 + 8\mu uv}{\mu^2(1-u^2)^2}\right)^{1/2}\right\}.$$

If $\mu^2(1-u^2)^2 >> |4 + 8\mu uv|$, then the eigenvalues are approximately 0 and $\mu(1-u^2)$. If $|\mu(1-u^2)| >> 1$, then these eigenvalues are of vastly different size. So if $\mu >> 1$ and u is not too close to ± 1, we expect the system to be stiff.

(c) Computational.

7) This problem concerns similarities and differences between k^{th} order linear constant coefficient ordinary differential equations and k^{th} order linear constant coefficient difference equations.
a) Consider the k^{th} order linear constant coefficient ordinary differential equation

$$y^{(k)}(t) + \alpha_{k-1}y^{(k-1)}(t) + \alpha_{k-2}y^{(k-2)}(t) + ... + \alpha_1 y' + \alpha_0 y(t) = 0. \tag{5}$$

Reduce this to a linear system of first order ODEs of the form

$$\mathbf{v}'(t) = A\mathbf{v}(t). \tag{6}$$

for a vector $\mathbf{v}(t) \in \mathbb{R}^k$. What is $\mathbf{v}(t)$? What is the matrix A? What conditions on the eigenvalues of A ensure that all solutions of (5) remain bounded as $t \to \infty$?
b) Consider the k^{th} order linear constant coefficient difference equation

$$y_{n+k} + \alpha_{k-1}y_{n+k-1} + \alpha_{k-2}y_{n+k-2} + ... + \alpha_0 y_n = 0. \tag{7}$$

Reduce this to a linear system of first order difference equations of the form

$$\mathbf{v}_{n+1} = A\mathbf{v}_n. \tag{8}$$

for a vector $\mathbf{v}_n \in \mathbb{R}^k$. What is the vector $\mathbf{v}_k$? What is the matrix A? What conditions on the eigenvalues of A ensure that all solutions of (7) remain bounded as $n \to \infty$?
c) Recall that for an A-stable linear multistep method, its region of absolute stability includes the entire left $h\lambda$ half plane. Why does the left half plane appear in this definition?

Solution:

(a) For $y^{(k)}(t) + \alpha_{k-1}y^{(k-1)}(t) + \alpha_{k-2}y^{(k-2)}(t) + ... + \alpha_1 y' + \alpha_0 y(t) = 0$, let

$$\mathbf{v} = \begin{bmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(k-1)}(t) \end{bmatrix}.$$

Then,

$$\mathbf{v}' = \begin{bmatrix} y'(t) \\ y''(t) \\ \vdots \\ \vdots \\ y^{(k)}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{k-2} & -\alpha_{k-1} \end{bmatrix} \begin{bmatrix} y(t) \\ y'(t) \\ \vdots \\ \vdots \\ y^{(k-1)}(t) \end{bmatrix}.$$

The k-by-k matrix A is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{k-2} & -\alpha_{k-1} \end{bmatrix}$$

The solution to the ODE remains bounded as $t \to \infty$ if $Re(\lambda) \leq 0$ for each eigenvalue λ of A. This is because solutions of the ODE are linear combinations of terms $e^{\lambda t}$, for eigenvalues λ of A.

(b) For $y_{n+k} + \alpha_{k-1} y_{n+k-1} + \alpha_{k-2} y_{n+k-2} + ... + \alpha_0 y_n = 0$, let

$$\mathbf{v}_n = \begin{bmatrix} y_n \\ y_{n+1} \\ \vdots \\ y_{n+k-2} \\ y_{n+k-1} \end{bmatrix}.$$

Then,

$$\mathbf{v}_{n+1} = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ \vdots \\ \vdots \\ y_{n+k-1} \\ y_{n+k} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{k-2} & -\alpha_{k-1} \end{bmatrix} \begin{bmatrix} y_n \\ y_{n+1} \\ \vdots \\ \vdots \\ y_{n+k-2} \\ y_{n+k-1} \end{bmatrix}.$$

The matrix A is the same as in part (a). For solutions of the difference equation to be bounded as $n \to \infty$, the eigenvalues of A must satisfy the root condition.

(c) For a stiff problem, we expect an eigenvalue with $Re(\lambda) << -1$. To avoid a severe restriction on h, we want λh to be in the region of absolute stability of the method being used. The point λh is well into the left-half-plane for a stiff problem.

8) Prove that any explicit linear multistep method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k-1} \beta_j y_{n+j}$$

has a bounded region of absolute stability.

Solution: Absolute stability of the LMM is determined by the locations of roots of the stability polynomial $\pi_{\overline{h}}(z) = \sum_{j=0}^{k} \alpha_j z^j - \overline{h} \sum_{j=0}^{k-1} \beta_j z^j = \rho(z) - \overline{h}\sigma(z)$. Here, $\overline{h} = \lambda h$. For $\overline{h} \neq 0$, the roots of the stability polynomial are the same as the roots of the polynomial $p(z) = \frac{1}{\overline{h}}\rho(z) - \sigma(z)$. For any finite value of $\overline{h}$, $p(z)$ is a k^{th} degree polynomial and has k roots $r_1(\overline{h})$, $r_2(\overline{h})$,…,$r_k(\overline{h})$. The polynomial $\sigma(z)$ is a $(k-1)^{st}$ degree polynomial and has $k-1$ roots $s_1, \ldots, s_{k-1}$. Roots of a polynomial are continuous functions of the coefficients of the polynomial. As $1/\overline{h} \to 0$, $k-1$ of the roots $r_1(\overline{h}), \ldots, r_k(\overline{h})$ must converge to $s_1, \ldots, s_{k-1}$. The remaining root must to to ∞. So as $\overline{h} \to \infty$, at least one root of $\pi_{\overline{h}}(z) = 0$, must be large and therefore outside of the unit circle.