Math 6620, Spring 2009, Problem Set 5
Due Friday April 17, 2009.

1) Construct a) an example of a consistent but not zero-stable linear multistep method, and b) an example of a zero-stable but not consistent linear multistep method. What kind of behavior do you expect from the solution to each? (Hint: They fail in very different ways. If you do not see this, implement both of your methods, try them on a suitable initial value problem, and report on the results.)

2) a) Find the range of $\alpha \in \mathbb{R}$ for which the method

$$U^{n+2} + (\alpha - 1)U^{n+1} - \alpha U^n = \frac{k}{4}\{(\alpha+3)f^{n+2} + (3\alpha+1)f^n\}$$

is consistent and zero-stable. Here and below, $f^n = f(t_n, U^n)$.
b) Apply the method with $\alpha = -1$ to the scalar IVP $u' = u$; $u(0) = 1$ and solve **exactly** the resulting difference equation, taking the starting values to be $U^0 = U^1 = 1$. Show that the numerical solution does not converge as $k \to 0$ and $n \to \infty$. (No programming should be done for this problem.)

3) The classical 4th order Runge-Kutta method is given by:

$$\begin{aligned}
v_1 &= kf(t_n,\ U^n)\\
v_2 &= kf(t_n + (1/2)k,\ U^n + (1/2)v_1)\\
v_3 &= kf(t_n + (1/2)k,\ U^n + (1/2)v_2)\\
v_4 &= kf(t_n + k,\ U^n + v_3)\\
U^{n+1} &= U^n + \frac{1}{6}\{v_1 + 2v_2 + 2v_3 + v_4\}
\end{aligned}$$

The intersection of the region of absolute stability (R_A) with the real axis is the interval [-2.78,0].

Implement the method and apply it to the scalar problem

$$u' = \sin(x) - 1000(u + u^2) \qquad u(0) = 1$$

on the interval [0,0.1].
a) Present numerical evidence that the scheme is 4th order for k small enough.
b) How small should k be so that the calculation is absolutely stable? Show numerical results of what happens when k is not small enough.

4) Consider a linear constant-coefficient r^{th} order difference equation

$$\sum_{j=0}^{r} \alpha_j U^{(n+j)} = 0, \tag{1}$$

with $\alpha_r \neq 0$ and $\alpha_0 \neq 0$. We showed in class that if s is a root of the polynomial

$$\rho(s) = \sum_{j=0}^{r} \alpha_j s^j,$$

then $U^{(n)} = Cs^n$ solves (1) for any constant C.
a) Suppose $\rho(z)$ has r *distinct* roots $s_1, s_2, \ldots, s_r$. Then the linearity of (1) implies that

$$U^{(n)} = c_1 s_1^n + c_2 s_2^n + \ldots + c_r s_r^n$$

solves (1) for any coefficients $c_1, \ldots, c_r$. Prove that *all* solutions of (1) can be written in this form for some choice of the coefficients $c_1, \ldots, c_r$.

b) If $\rho(z)$ has a double root s_1, show that s_1^n and ns_1^n are both solutions of (1). (Hint: $\rho'(s_1) = 0$.)

5) a) Determine the 3rd order 3-step BDF method

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k\beta_r f^{n+r}.$$

b) Use the 2nd order BDF method $U^{n+2} - (4/3)U^{n+1} + (1/3)U^n = (2k/3)f^{n+2}$ to solve the ODE from problem 3:

$$u' = \sin(x) - 1000(u + u^2) \qquad u(0) = 1.$$

You have $u_0 = u(0)$, but you need u_1. Use a method of at least 2nd order to generate this value (e.g. your 4th order RK method from problem 3), then use the BDF method to solve the ODE on the interval [0,0.1].

6) The equation $u'' = -u + \mu(1-u^2)u'$ is known as the Van der Pol equation and represents a simple harmonic oscillator to which has been added a nonlinear term that introduces positive damping for $|u| > 1$ and negative damping for $|u| < 1$. Solutions to the equation approach limit cycles of finite amplitude, and if $\mu >> 1$, the oscillation is characterized by periods of slow change punctuated by short intervals during which $u(t)$ swings rapidly from positive to negative or back again.
a) Reduce the Van der Pol equation to a first order system of ODEs and compute the Jacobian matrix of this system.
b) What can you say about stiffness of this system?
c) Solve the Van der Pol equation with initial values $u(t) = 2$, $u'(t) = 0$ over the interval $0 \leq t \leq \max(20, 3 * \mu)$ for each of μ = 1,5,10,50,100,200. Use Matlab's ode solvers ode45 and ode15s (with an analytic Jacobean provided for ode15s) for each of these problems. (Write your program so that it keeps track of the number of function evaluations made each time it is run.) Report on your experience with each solver for each value of μ. How many function evaluations does each solver take on each problem? You will have to read Matlab's help pages on solving ODEs, and, in particular, the information about the two solvers ode45 and ode15s.

7) This problem concerns similarities and differences between r^{th} order linear constant coefficient ordinary differential equations and r^{th} order linear constant coefficient difference equations.

a) Consider the r^{th} order linear constant coefficient ordinary differential equation

$$y^{(r)}(t) + \alpha_{r-1}y^{(r-1)}(t) + \alpha_{r-2}y^{(r-2)}(t) + ... + \alpha_1 y' + \alpha_0 y(t) = 0. \tag{2}$$

Reduce this to a linear system of first order ODEs of the form

$$\mathbf{v}'(t) = A\mathbf{v}(t). \tag{3}$$

for a vector $\mathbf{v}(t) \in \mathbb{R}^r$. What is $\mathbf{v}(t)$? What is the matrix A? What conditions on the eigenvalues of A ensure that all solutions of (2) remain bounded as $t \to \infty$?

b) Consider the r^{th} order linear constant coefficient difference equation

$$y_{n+r} + \alpha_{r-1}y_{n+r-1} + \alpha_{r-2}y_{n+r-2} + ... + \alpha_0 y_n = 0. \tag{4}$$

Reduce this to a linear system of first order difference equations of the form

$$\mathbf{v}_{n+1} = A\mathbf{v}_n. \tag{5}$$

for a vector $\mathbf{v}_n \in \mathbb{R}^r$. What is the vector $\mathbf{v}_r$? What is the matrix A? What conditions on the eigenvalues of A ensure that all solutions of (4) remain bounded as $n \to \infty$?

c) Recall that for an A-stable linear multistep method, its region of absolute stability includes the entire left $h\lambda$ half plane. Why does the left half plane appear in this definition?

8) Prove that any explicit linear multistep method

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r-1} \beta_j f^{n+j}$$

has a bounded region of absolute stability.