Math 6620 Spring 2009 Problem Set 4
Solutions

1) Green's Function for Neumann BC

(a) Determine the Green's functions for the two-point boundary value problem $u''(x) = f(x)$ on $0 < x < 1$ with a Neumann boundary condition at $x = 0$ and a Dirichlet condition at $x = 1$, i.e, find the function $G(x, \bar{x})$ solving

$$u''(x) = \delta(x - \bar{x}), \quad u'(0) = 0, \quad u(1) = 0$$

and the functions $G_0(x)$ solving

$$u''(x) = 0, \quad u'(0) = 1, \quad u(1) = 0$$

and $G_1(x)$ solving

$$u''(x) = 0, \quad u'(0) = 0, \quad u(1) = 1.$$

(b) Using this as guidance, find the general formulas for the elements of the inverse of the matrix in equation (2.54). Write out the 5×5 matrices A and A^{-1} for the case $h = 0.25$.

Solution:
a) To solve $u''(x) = \delta(x - \bar{x}), \quad u'(0) = 0, \quad u(1) = 0$, we note that on $(0, \bar{x})$, $u''(x) = 0$, so $u(x) = ax + b$ for some constants a and b. Using the BC at $x = 0$, we see that $a = 0$. Similarly, on $(\bar{x}, 1)$, we get $u(x) = cx + d$, for constants c and d. Using the BC at $x = 1$, we get that $u(x) = c(x - 1)$. Integrating $u''(x) = \delta(x - \bar{x})$ on a vanishingly small interval containing the point $x = \bar{x}$, we see that the jump in $u'(x)$ at this point is 1. Since the slope to the left of $\bar{x}$ is zero, this tells us that $c = 1$. Finally, since $u(x)$ is continuous at $\bar{x}$, we have that $b = \bar{x} - 1$. In summary,

$$G(x, \bar{x}) = \begin{cases} \bar{x} - 1 & 0 \le x \le \bar{x} \\ x - 1 & \bar{x} \le x \le 1. \end{cases}$$

Integrating the ODE $u''(x) = 0$, and using the BC $u'(0) = 1 \quad u(1) = 0$, we find that $G_0(x) = x - 1$. Similarly, $G_1(x) = 1$. Using these facts, we see that the solution to the problem $u''(x) = f(x), \quad u'(0) = \sigma, \quad u(1) = \beta$ is

$$u(x) = \sigma(x - 1) + \int_0^1 G(x, \bar{x}) f(\bar{x}) d\bar{x} + \beta.$$

Now consider the discrete problem $D_+ u_0 = \sigma \quad D_+ D_- u_j = f_j \quad j = 1, \ldots, m \quad u_{m+1} = \beta$. Based on the solution of the continuous problem, we guess that

$$u_i = \sigma(x_i - 1) + \sum_{j=1}^{m} G(x_i, x_j) f(x_j) h + \beta.$$

It is very easy to verify that this is indeed the solution to the discrete problem. The first term satisfies, $D_+ u_0 = \sigma$, has zero second difference, and value 0 at $x = 1$. For $1 \le j \le m$, the grid function $hG(x_i, x_j)$ satisfies both boundary conditions (it is flat near the left boundary) as well as the equation $D_+D_-U_i = h\delta_{ij}$, where δ_{ij} is the Kroenecker delta function, and the last term certainly satisfies the BCs and the discrete ODE. The discrete problem above can be written in matrix vector form as $A\underline{U} = \underline{b}$:

$$\frac{1}{h^2}\begin{bmatrix} -h & h & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & -2 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -2 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & h^2 \end{bmatrix}\begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ U_m \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma \\ f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_m \\ \beta \end{bmatrix}.$$

From the solution to the discrete problem written down above, we can read off the columns of the matrix $B = A^{-1}$. The 0^{th} column has entries $B_{i,0} = x_i - 1$. For $j = 1, \ldots, m$, the j^{th} column has entries $B_{i,j} = hG(x_i, x_j)$, and the $(m+1)^{th}$ column has entries $B_{i,m+1} = 1$.

For the case $m = 3$, $x_j = j/4$, for $j = 0, \ldots, 4$, and

$$A = 16\begin{bmatrix} -1/4 & 1/4 & \cdot & \cdot & \cdot \\ 1 & -2 & 1 & \cdot & \cdot \\ \cdot & 1 & -2 & 1 & \cdot \\ \cdot & \cdot & 1 & -2 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1/16 \end{bmatrix}.$$

and

$$B = \begin{bmatrix} -1 & -3/16 & -1/8 & -1/16 & 1 \\ -3/4 & -3/16 & -1/8 & -1/16 & 1 \\ -1/2 & -1/8 & -1/8 & -1/16 & 1 \\ -1/4 & -1/16 & -1/16 & -1/16 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

2) Boundary conditions in `bvp` codes

(a) Modify the m-file `bvp2.m` so that it implements a Dirichlet boundary condition at $x = a$ and a Neumann condition at $x = b$ and test the modified program.

(b) Make the same modification to the m-file `bvp4.m`, which implements a fourth order accurate method. Again test the modified program.

Solution:

(a) The max-norm errors with 11, 21, 41, and 81 points are, respectively, 1.74886, 0.480811, 0.126086, and 0.0322858. The ratios of successive errors are very close to 1/4 which shows the second order convergence.

(b) The max-norm errors with 11, 21, 41, and 81 points are, respectively, 0.0722610, 0.00521206, 0.000347061, and 0.0000223260. The ratios of successive errors are 13.87, 15.02, and 15.55, which shows (approximately) fourth order convergence.

Computational.

3) Ill-posed boundary value problem

Consider the following linear boundary value problem with Dirichlet boundary conditions:

$$\begin{aligned} &u''(x) + u(x) = 0 \quad \text{for } a < x < b \\ &u(a) = \alpha, \quad u(b) = \beta. \end{aligned}$$

(a) Modify the m-file `bvp2.m` to solve this problem. Test your modified routine on the problem with

$$a = 0, \quad b = 1, \quad \alpha = 2, \quad \beta = 3.$$

Determine the exact solution for comparison.

(b) Let $a = 0$ and $b = \pi$. For what values of α and β does this boundary value problem have solutions? Sketch a family of solutions in a case where there are infinitely many solutions.

(c) Solve the problem with

$$a = 0, \quad b = \pi, \quad \alpha = 1, \quad \beta = -1.$$

using your modified `bvp2.m`. Which solution to the boundary value problem does this appear to converge to as $h \to 0$? Change the boundary value at $b = \pi$ to $\beta = 1$. Now how does the numerical solution behave as $h \to 0$?

(d) You might expect the linear system in part (c) to be singular since the boundary value problem is not well posed. It is not, because of discretization error. Compute the eigenvalues of the matrix A for this problem and show that an eigenvalue approaches 0 as $h \to 0$. Also show that $\|A^{-1}\|_2$ blows up as $h \to 0$ so that the discretization is unstable.

Solution:

(a) The exact solution to the problem

$$\begin{aligned} &u''(x) + u(x) = 0 \quad \text{for } 0 < x < b \\ &u(0) = \alpha, \quad u(b) = \beta. \end{aligned}$$

is

$$u(x) = \alpha\cos(x) + \frac{\beta - \alpha\cos(b)}{\sin(b)}\sin(x)$$

provided $\sin(b) \neq 0$. For the values $\alpha = 2$, $b = 1$, $\beta = 3$, we have

$$u(x) = 2\cos(x) + \frac{3 - 2\cos(1)}{\sin(1)}\sin(x).$$

(b) For $a = 0$ and $b = \pi$, the general solution is $u(x) = A\cos(x) + B\sin(x)$, $u(0) = A = \alpha$ and $u(\pi) = -A$, so a solution exists only if $\beta = -\alpha$. In that case, B is undetermined and the family of solutions is given by $u(x) = \alpha\cos(x) + B\sin(x)$, with B arbitrary. The solutions differ by multiples of the function $\sin(x)$.

With $\alpha = 1$ and $\beta = -1$, the numerical solution shows second order convergence to $\cos(x)$, that is to the solution with $B = 0$. With $\alpha = \beta = 1$, the numerical solution appears to behave like $1 + C(h)\sin(x)$, where the coefficient $C(h)$ becomes increasingly large in magnitude as $h \to 0$.

(c) Computational

(d) For $(m+1)h = \pi$, consider the discrete system

$$A^h \underline{u}^h = \underline{b}^h,$$

where $\underline{u}^h = (u_1, u_2, \ldots, u_m)^T$, $\underline{b}^h = h^{-2}(-1, 0, \ldots, 0, -\beta)^T$, and $A^h = I^h + h^{-2}M^h$ where I^h is the $m \times m$ identity matrix and M^h is the $m \times m$ tridiagonal matrix with diagonal entries -2 and sub- and superdiagonal entries 1. The matrix A^h has eigenvalues $\lambda^{(p)} = 1 - 4h^{-2}\sin^2\left(\frac{ph}{2}\right)$ and eigenvectors $\underline{r}^{(p)}$ with entries $r_j^{(p)} = \sin(jph)$ for $j = 1, \ldots, m$. The eigenvalue $\lambda^{(1)} = 1 - 4h^{-2}\sin^2\left(\frac{h}{2}\right) \approx 1 - 4h^{-2}\left\{\left(\frac{h}{2} + O(h^3)\right)^2\right\} = 1 - 4h^{-2}\left(\frac{h^2}{4} + O(h^4)\right) = O(h^2)$. Hence, $\lim_{h\to 0} \lambda^{(1)} = 0$. We can write down an expression for the solution to $A^h\underline{u}^h = \underline{b}^h$, using the expansion of $\underline{b}^h$ in terms of the eigenvectors $\underline{r}^{(p)}$, $\underline{b}^h = \sum_{p=1}^m c_p \underline{r}^{(p)}$. The solution is

$$\underline{u}^p = \sum_{p=1}^{m} \frac{c_p}{\lambda^{(p)}} \underline{r}^{(p)}.$$

Unless $c_1 \to 0$ as $h \to 0$, the coefficient of $\underline{r}^{(1)}$ will tend to infinity, and the numerical solution will look more and more like the function $\sin(x)$ scaled by an amplitude that becomes infinite as $h \to 0$. The coefficients $c_p = \frac{\underline{b}^h \cdot \underline{r}^{(p)}}{\underline{r}^{(p)} \cdot \underline{r}^{(p)}}$. Hence,

$$c_1 = \frac{-h^{-2}\sin(h) - h^{-2}\beta\sin(mh)}{\sum_{j=1}^m \sin^2(jh)} = -\frac{(1+\beta)\sin(h)}{h^2\sum_{j=1}^m \sin^2(jh)}.$$

With $\beta = -1$, $c_1 = 0$ and there is no component of $\underline{b}^h$ in the direction of the eigenvector corresponding to the eigenvalue that goes to 0. With $\beta = 1$,

$$c_1 = \frac{-2(\sin(h)/h)}{\sum_{j=1}^m h\sin^2(jh)} \to \frac{-2}{\int_0^1 \sin^2(\pi x)dx} \neq 0.$$

So, in this case, the numerical solution blows up as $h \to 0$.

Since A^h and therefore $\left(A^h\right)^{-1}$ are symmetric, the 2-norm of $\left(A^h\right)^{-1}$ is the reciprocal of the smallest eigenvalue of A^h. As we just saw, the smallest eigenvalue of A^h goes to 0 as h goes to 0, so the 2-norm of $\left(A^h\right)^{-1}$ goes to infinity as h goes to 0.

4) Code for Poisson problem

The MATLAB script `poisson.m` solves the Poisson problem on a square $m \times m$ grid with $\Delta x = \Delta y = h$, using the 5-point Laplacian. It is set up to solve a test problem for which the exact solution is $u(x, y) = \exp(x + y/2)$, using Dirichlet boundary conditions and the right hand side $f(x, y) = 1.25\exp(x + y/2)$.

(a) Test this script by performing a grid refinement study to verify that it is second order accurate.

(b) Modify the script so that it works on a rectangular domain $[a_x, b_x] \times [a_y, b_y]$, but still with $\Delta x = \Delta y = h$. Test your modified script on a non-square domain.

(c) Further modify the code to allow $\Delta x \neq \Delta y$ and test the modified script.

Solution:

(a) With $m = 20$, $m = 40$, and $m = 80$, the maximum norm of the error was, respectively, 3.273e-05, 8.601e-06, and 2.205e-06, which shows second order convergence.

(b) Computational.

(c) Computational.

5) 9-point discrete Laplacian

(a) Show that the 9-point Laplacian (3.17) has the truncation error derived in Section 3.5. **Hint:** To simplify the computation, note that the 9-point Laplacian can be written as the 5-point Laplacian (with known truncation error) plus a finite difference approximation that models $\frac{1}{6}h^2 u_{xxyy} + O(h^4)$.

(b) Modify the MATLAB script `poisson.m` to use the 9-point Laplacian (3.17) instead of the 5-point Laplacian, and to solve the linear system (3.18) where f_{ij} is given by (3.19). Perform a grid refinement study to verify that fourth order accuracy is achieved.

Solution:

6) Nonlinear Boundary Value Problem

Consider the following nonlinear boundary value problem with Dirichlet boundary conditions:

$$\begin{aligned} &u''(x) + \lambda e^{u(x)} = 0 \quad \text{for } 0 < x < 1 \\ &u(0) = 0, \quad u(1) = 0. \end{aligned}$$

Here, λ is a positive constant. Write a program to solve this problem using the standard second order approximation to the second derivative, and using Newton's method to solve the nonlinear equations. For $\lambda = 1$, the problem has two isolated solutions. Find both of them.

Solution:

7) Variable coefficient Poisson problem

Consider the scheme

$$
\begin{aligned}
(L^h u)_{j,l} &\equiv \frac{1}{h^2}\big(-\beta_{j,l-1/2}u_{j,l-1} - \beta_{j-1/2,l}u_{j-1,l} - \beta_{j+1/2,l}u_{j+1,l} - \beta_{j,l+1/2}u_{j,l+1} \\
&+ (\beta_{j,l-1/2} + \beta_{j-1/2,l} + \beta_{j+1/2,l} + \beta_{j,l+1/2})u_{j,l}\big) = f_{j,l}
\end{aligned}
$$

for the variable coefficient Poisson equation $-\nabla \cdot (\beta \nabla v) = f$ for $\mathbf{x} \in R \equiv [0,1]^2$ with homogeneous Dirichlet boundary conditions. The local truncation error of this scheme $\mathcal{L} = O(h^2)$, that is $L^h v - f = O(h^2)$ for the solution $v(\mathbf{x})$ of the differential equation problem. Show that the global error $|v_{j,l} - u_{j,l}| = O(h^2)$ for all grid points in R. (Hint: Use a maximum principle argument.) What can you say about the solvability of the discrete equations $L^h u = f$?

Solution:

Suppose that $(L^h u)_{j,l} \geq 0$. Then,

$$
u_{j,l} \leq \frac{\beta_{j,l-1/2}u_{j,l-1} + \beta_{j-1/2,l}u_{j-1,l} + \beta_{j+1/2,l}u_{j+1,l} + \beta_{j,l+1/2}u_{j,l+1}}{\beta_{j,l-1/2} + \beta_{j-1/2,l} + \beta_{j+1/2,l} + \beta_{j,l+1/2}}.
$$

That is, $u_{j,l}$ is no greater than the average of its four neighboring values. This is enough (following the steps used for $-\Delta^h$ in class) to conclude that if $(L^h u)_{j,l} \geq 0$ for all j, l, then u has its maximum at a boundary grid point. From this it follows (again, as with $-\Delta^h$ in class) that for any right-hand-side f, the equation $L^h u = f$ has at most one solution. Since $L^h u = f$ is a finite system of linear algebraic equations with as many equations as unknowns, it follows from this uniqueness result that $L^h u = f$ has a solution for any right hand side.

Let u^h be the solution to $L^h u = f$ with homogeneous Dirichlet boundary conditions, and let v be the solution to $-\nabla \cdot (\beta \nabla v) = f$ with homogeneous Dirichlet boundary conditions. Then, $(L^h v)_{j,l} = f_{j,l} + \tau_{j,l}$ where $|\tau_{j,l}| \leq \tau = O(h^2)$ uniformly for all grid points. Let $e_{j,l} = v_{j,l} - u^h_{j,l}$ denote the global error at (j,l). From the above equations, we see that

$$
(L^h e)_{j,l} = \tau_{j,l}. \tag{E0.0a}
$$

Also, $e_{j,l} = 0$ at boundary points. From Eq(E0.0a) we see that

$$
|(L^h e)_{j,l}| \leq \tau. \tag{E0.0b}
$$

Let ϕ be a grid function such that $(L^h \phi)_{j,l} = 1$ for all non-boundary grid points and $\phi_{j,l} \geq 0$ for all grid points. How do I know that such a ϕ exists? First solve $(L^h \tilde{\phi})_{j,l} = 1$ with homogeneous Dirichlet boundary conditions. We know that this can be done by the arguements above. Let $\tilde{\phi}_{\min}$ be the minimum value of $\tilde{\phi}$ over the grid; this is a constant. Now let $\phi_{j,l} = \tilde{\phi}_{j,l} + |\tilde{\phi}_{\min}|$. Since L^h applied to a constant grid function gives 0, $(L^h \phi)_{j,l} = 1$ and $\phi_{j,l} \geq 0$ as desired.

Now consider the grid function with values $W^{(1)}_{j,l} = e_{j,l} + \tau \phi_{j,l}$. We see that

$$
(L^h W^{(1)})_{j,l} = \tau_{j,l} + \tau \geq 0.
$$

So the maximum of $W^{(1)}_{j,l}$ occurs at a boundary point. Since $e_{j,l} = 0$ on the boundary we have

$$
\max_{j,l}\ e_{j,l} \leq \max_{j,l}\ W^{(1)}_{j,l} \leq \tau \max_{(j,l)\text{ on bdy}} \phi_{j,l}. \tag{E0.0c}
$$

Similarly, consider, $W^{(2)}_{j,l} = e_{j,l} - \tau\phi_{j,l}$. We see that

$$(L^h W^{(2)})_{j,l} = \tau_{j,l} - \tau \leq 0.$$

So the minimum of $W^{(2)}_{j,l}$ occurs at a boundary point. Since $e_{j,l} = 0$ on the boundary we have

$$\min_{j,l}\ e_{j,l} \geq \min_{j,l}\ W^{(2)}_{j,l} \geq \tau \min_{(j,l)\text{ on bdy}} \phi_{j,l}. \tag{E0.0d}$$

The minimum and maximum values of ϕ on the boundary are bounded in magnitude by some number C, so we see that

$$\|e_{j,l}\| \leq C\tau = O(h^2),$$

and so the method is second order .