

1) Green's Function for Neumann BC

- (a) Determine the Green's functions for the two-point boundary value problem $u''(x) = f(x)$ on $0 < x < 1$ with a Neumann boundary condition at $x = 0$ and a Dirichlet condition at $x = 1$, i.e, find the function $G(x, \bar{x})$ solving

$$u''(x) = \delta(x - \bar{x}), \quad u'(0) = 0, \quad u(1) = 0$$

and the functions $G_0(x)$ solving

$$u''(x) = 0, \quad u'(0) = 1, \quad u(1) = 0$$

and $G_1(x)$ solving

$$u''(x) = 0, \quad u'(0) = 0, \quad u(1) = 1.$$

- (b) Using this as guidance, find the general formulas for the elements of the inverse of the matrix in equation (2.54). Write out the 5×5 matrices A and A^{-1} for the case $h = 0.25$.

2) Boundary conditions in bvp codes

- (a) Modify the m-file `bvp2.m` so that it implements a Dirichlet boundary condition at $x = a$ and a Neumann condition at $x = b$ and test the modified program.
- (b) Make the same modification to the m-file `bvp4.m`, which implements a fourth order accurate method. Again test the modified program.

3) Ill-posed boundary value problem

Consider the following linear boundary value problem with Dirichlet boundary conditions:

$$u''(x) + u(x) = 0 \quad \text{for } a < x < b$$

$$u(a) = \alpha, \quad u(b) = \beta.$$

- (a) Modify the m-file `bvp2.m` to solve this problem. Test your modified routine on the problem with

$$a = 0, \quad b = 1, \quad \alpha = 2, \quad \beta = 3.$$

Determine the exact solution for comparison.

- (b) Let $a = 0$ and $b = \pi$. For what values of α and β does this boundary value problem have solutions? Sketch a family of solutions in a case where there are infinitely many solutions.
- (c) Solve the problem with

$$a = 0, \quad b = \pi, \quad \alpha = 1, \quad \beta = -1.$$

using your modified `bvp2.m`. Which solution to the boundary value problem does this appear to converge to as $h \rightarrow 0$? Change the boundary value at $b = \pi$ to $\beta = 1$. Now how does the numerical solution behave as $h \rightarrow 0$?

- (d) You might expect the linear system in part (c) to be singular since the boundary value problem is not well posed. It is not, because of discretization error. Compute the eigenvalues of the matrix A for this problem and show that an eigenvalue approaches 0 as $h \rightarrow 0$. Also show that $\|A^{-1}\|_2$ blows up as $h \rightarrow 0$ so that the discretization is unstable.

4) Code for Poisson problem

The MATLAB script `poisson.m` solves the Poisson problem on a square $m \times m$ grid with $\Delta x = \Delta y = h$, using the 5-point Laplacian. It is set up to solve a test problem for which the exact solution is $u(x, y) = \exp(x + y/2)$, using Dirichlet boundary conditions and the right hand side $f(x, y) = 1.25 \exp(x + y/2)$.

- (a) Test this script by performing a grid refinement study to verify that it is second order accurate.
- (b) Modify the script so that it works on a rectangular domain $[a_x, b_x] \times [a_y, b_y]$, but still with $\Delta x = \Delta y = h$. Test your modified script on a non-square domain.
- (c) Further modify the code to allow $\Delta x \neq \Delta y$ and test the modified script.

5) 9-point discrete Laplacian

- (a) Show that the 9-point Laplacian (3.17) has the truncation error derived in Section 3.5. **Hint:** To simplify the computation, note that the 9-point Laplacian can be written as the 5-point Laplacian (with known truncation error) plus a finite difference approximation that models $\frac{1}{6}h^2 u_{xxyy} + O(h^4)$.
- (b) Modify the MATLAB script `poisson.m` to use the 9-point Laplacian (3.17) instead of the 5-point Laplacian, and to solve the linear system (3.18) where f_{ij} is given by (3.19). Perform a grid refinement study to verify that fourth order accuracy is achieved.

6) Nonlinear Boundary Value Problem

Consider the following nonlinear boundary value problem with Dirichlet boundary conditions:

$$\begin{aligned} u''(x) + \lambda e^{u(x)} &= 0 \quad \text{for } 0 < x < 1 \\ u(0) &= 0, \quad u(1) = 0. \end{aligned}$$

Here, λ is a positive constant. Write a program to solve this problem using the standard second order approximation to the second derivative, and using Newton's method to solve the nonlinear equations. For $\lambda = 1$, the problem has two isolated solutions. Find both of them.

7) Variable coefficient Poisson problem

Consider the scheme

$$\begin{aligned}(L^h u)_{j,l} &\equiv \frac{1}{h^2} \left(-\beta_{j,l-1/2} u_{j,l-1} - \beta_{j-1/2,l} u_{j-1,l} - \beta_{j+1/2,l} u_{j+1,l} - \beta_{j,l+1/2} u_{j,l+1} \right. \\ &\quad \left. + (\beta_{j,l-1/2} + \beta_{j-1/2,l} + \beta_{j+1/2,l} + \beta_{j,l+1/2}) u_{j,l} \right) = f_{j,l}\end{aligned}$$

for the variable coefficient Poisson equation $-\nabla \cdot (\beta \nabla v) = f$ for $\mathbf{x} \in R \equiv [0, 1]^2$ with homogeneous Dirichlet boundary conditions. The local truncation error of this scheme $\mathcal{L} = O(h^2)$, that is $L^h v - f = O(h^2)$ for the solution $v(\mathbf{x})$ of the differential equation problem. Show that the global error $|v_{j,l} - u_{j,l}| = O(h^2)$ for all grid points in R . (Hint: Use a maximum principle argument.) What can you say about the solvability of the discrete equations $L^h u = f$?