Math 6620 Spring 2009 Problem Set 3
Solutions

1) Consider the function $f(x) = \frac{1}{1+x^2}$ on the interval $-5 \leq x \leq 5$. For each $n \geq 1$, define $h = 10/n$, $x_j = -5 + j * h$ for $j = 0, 1, 2, \ldots, n$, and let $p_n(x)$ be the polynomial of degree n which interpolates f at the nodes $x_0, x_1, \ldots, x_n$. Compute p_n for $n = 1, 2, \ldots, 20$, plot $f(x)$ and $p_n(x)$ for each n, and estimate the maximum error $|f(x) - p_n(x)|$ for $x \in (-5, 5)$. Describe what you find.

Solution:

The approximation becomes increasingly bad as n increases especially near the ends of the interval [-5,5].

2) Consider the function $f(x) = \cos(\pi x)$ on the interval $x \in [0, 1]$.
a) Write a program to compute the cubic spline $s(x)$ for f on the interval $[0, 1]$, with the conditions $s''(0) = s''(1) = 0$ and determine the splines for $n = 5$ and $n = 10$ subintervals.
b) What is the linear system that we must solve to determine the cubic periodic spline $s(x)$, that is, the cubic spline that satisfies the conditions $s'(x_0) = s'(x_n)$ and $s''(x_0) = s''(x_n)$? You may assume that $f(x_0) = f(x_n)$.
Solution:
a) Computational

b) For a cubic periodic spline, the two 'extra' conditions are $s''(x_n) = s''(x_0)$ and $s'(x_n) = s'(x_0)$. Recalling the notation used in class that $M_j = s''(x_j)$, the first of these gives us the equation

$$M_0 - M_n = 0. \tag{1}$$

Recalling also the expressions from class for the first derivative of $s(x)$ in each subinterval, we have that

$$s'(x_0) = \frac{-(x_1 - x_0)^2 M_0 + (x_0 - x_0)^2 M_1}{2h_0} + \frac{y_1 - y_0}{h_0} - \frac{(M_1 - M_0)h_0}{6}$$

and

$$s'(x_n) = \frac{-(x_n - x_n)^2 M_{n-1} + (x_n - x_{n-1})^2 M_n}{2h_{n-1}} + \frac{y_n - y_{n-1}}{h_{n-1}} - \frac{(M_n - M_{n-1})h_{n-1}}{6}.$$

The second condition implies that these should be equal, and upon equating them and rearranging terms we obtain the equation

$$\frac{h_0 M_0}{3} + \frac{h_0 M_1}{6} + \frac{h_{n-1} M_{n-1}}{6} + \frac{h_{n-1} M_n}{3} = \frac{y_1 - y_0}{h_0} - \frac{y_n - y_{n-1}}{h_{n-1}}. \tag{2}$$

So the system of equations for the $n+1$ unknowns $M_0, M_1, \ldots, M_n$ consists of equation (1), the equations

$$\frac{h_{j-1}}{6} M_{j-1} + \frac{h_j + h_{j-1}}{3} M_j + \frac{h_j}{6} M_{j+1} = \frac{y_{j+1} - y_j}{h_j} - \frac{y_j - y_{j-1}}{h_{j-1}},$$

for $j = 1, \ldots, n-1$ as in class, and equation (2).

3) Consider the function $f(x) = x^2 e^x$ on the interval $[a, b] = [0, 3]$. Compute the trapezoidal rule approximation to the integral of f over [a,b] with $n = 10$, $n = 20$, and $n = 40$. What is the error in each case? (You can find the exact integral of f using integration by parts.) Use Richardson Extrapolation with these three trapezoidal rule approximations to obtain the best approximation to the true integral that you can. What is the approximate integral and what is the error in using it?

Solution:

Computational. The quadrature errors with n =10, 20, and 40 were approximately 2.25, 0.56, and 0.14, respectively. This shows that the approximations are second order in the subinterval size $(b - a)/n$. Richardson extrapolation combined the three approximations to get an approximate integral with error 4.3682e-07.

4) Let P_j denote the set of polynomials of degree j with leading coefficient 1. Let $w(x)$ be a nonnegative and continuous weight function on the finite interval $[a,b]$, and define an inner product $(f,g) = \int_a^b f(x)g(x)w(x)dx$ for any f and g continuous on $[a,b]$. Prove the following theorem (hint: induction)

Theorem: There exist polynomials $p_j(x) \in P_j$, $j = 0, 1, 2, \ldots$, such that

$$(p_i, p_k) = 0 \quad \text{for i} \neq \text{k.}$$

These polynomials are uniquely defined by the recursions

$$p_0(x) \equiv 1$$

$$p_{i+1}(x) \equiv (x - \delta_{i+1})p_i(x) - \gamma_{i+1}^2 p_{i-1}(x)$$

for $i \geq 0$, where $p_{-1}(x) \equiv 0$ and

$$\delta_{i+1} = \frac{(xp_i, p_i)}{(p_i, p_i)}$$

for $i \geq 0$, and

$$\gamma_{i+1}^2 = \begin{cases} 0 & \text{for i} = 0 \\ \frac{(p_i, p_i)}{(p_{i-1}, p_{i-1})} & \text{for i} \geq 1. \end{cases}$$

Solution:

We prove the claims by induction. By definition

$$p_1(x) = (x - \delta_1)p_0(x) - \gamma_1^2 p_{-1}(x) = x - \frac{(xp_0, p_0)}{(p_0, p_0)}.$$

Clearly, $p_1(x) \in P_1$. Consider the inner product (p_1, p_0):

$$(p_1, p_0) = (x, p_0) - \frac{(xp_0, p_0)}{(p_0, p_0)}(1, p_0) = (1, x) - (x, 1) = 0.$$

So the claims are true for $k = 0$. Now suppose that the claims are true for all $k \leq n$. We want to show that $p_{n+1} \in P_{n+1}$ and that $(p_{n+1}, p_k) = 0$ for all $k \leq n$. By its definition

$$p_{n+1}(x) = xp_n(x) - \delta_{n+1}p_n(x) - \gamma_{n+1}^2 p_{n-1}(x).$$

The first term on the right is a polynomial of degree $n+1$ with leading coefficient 1. The other terms are polynomials of lower degree and so do not affect the leading term. Hence $p_{n+1} \in P_{n+1}$.

Now consider (p_{n+1}, p_k). By the definitions of $p_{n+1}(x)$, δ_{n+1}, and γ_{n+1}^2, we have that

$$(p_{n+1}, p_k) = (xp_n, p_k) - \frac{(xp_n, p_n)}{(p_n, p_n)}(p_n, p_k) - \frac{(p_n, p_n)}{(p_{n-1}, p_{n-1})}(p_{n-1}, p_k). \tag{3}$$

We consider first the case $k < n-1$. The last two terms on the right hand side are 0 by the induction hypothesis. The first term is $(xp_n, p_k) = (p_n, xp_k)$. Note that $xp_k \in P_{k+1}$. Also note that $p_0, p_1, \ldots, p_{k+1}$ are linearly independent (since they are orthogonal) and so every polynomial in P_{k+1} can be written as a linear combination of $p_0, p_1, \ldots, p_{k+1}$. So for some coefficients $xp_k(x) = \sum_{j=0}^{k+1} c_j p_j(x)$. By the induction hypothesis, $(p_n, p_j) = 0$ for all j in the sum, so $(p_n, xp_k) = 0$ and orthogonality is proved for $k < n-1$. Note that the definitions of δ_{n+1} and γ_{n+1}^2 played no role in this argument. They are used to show that p_{n+1} is orthogonal to p_n and p_{n-1}.

Consider

$$(p_{n+1}, p_n) = (xp_n, p_n) - \frac{(xp_n, p_n)}{(p_n, p_n)}(p_n, p_n) - \gamma_{n+1}^2(p_{n-1}, p_n).$$

The first two terms on the right hand side cancel and the last term is 0 by the induction hypothesis.

Consider

$$(p_{n+1}, p_{n-1}) = (xp_n, p_{n-1}) - \frac{(xp_n, p_n)}{(p_n, p_n)}(p_n, p_{n-1}) - \frac{(p_n, p_n)}{(p_{n-1}, p_{n-1})}(p_{n-1}, p_{n-1}).$$

The second term on the right hand side is 0 by the induction hypothesis and the remaining terms simplify to give

$$(p_{n+1}, p_{n-1}) = (xp_n, p_{n-1}) - (p_n, p_n) = (p_n, xp_{n-1}) - (p_n, p_n).$$

From its definition, $p_n(x) = (x - \delta_n)p_{n-1}(x) - \gamma_n^2 p_{n-2}(x)$, so $xp_{n-1}(x) = p_n(x) + \delta_n p_{n-1}(x) + \gamma_n^2 p_{n-2}(x)$. Using this in the equation just above, we have that

$$(p_{n+1}, p_{n-1}) = (p_n, p_n) + \delta_n(p_n, p_{n-1}) + \gamma_n^2(p_n, p_{n-2}) - (p_n, p_n).$$

The second and third terms on the right hand side are 0, and the first and fourth terms cancel, so we have $(p_{n+1}, p_{n-1}) = 0$, and the induction step is proved.

5) Consider the interval $[a, b] = [0, 1]$ and the weight function $w(x) \equiv 1$.
a) Determine the orthogonal polynomials $p_n(x)$ of degree $n = 0, 1, 2$ for this interval and weight function, and normalized so that the coefficient of x^n in $p_n(x)$ is 1.
b) For $n = 2$, determine the best least-squares polynomial approximation of degree n to $f(x) = e^x$.
Solution:
a) We use the recursion from problem 4 to do this problem with weight function $w(x) \equiv 1$. We know $p_0 = 1$. Thus

$$\delta_1 = \frac{(x,1)}{(1,1)} = \frac{\int_0^1 x dx}{\int_0^1 1 dx} = \frac{1}{2} \qquad \text{and} \qquad \gamma_1^2 = 0,$$

so

$$p_1(x) = (x - \frac{1}{2})1 - 0 = x - \frac{1}{2}.$$

For $n = 2$, we have that

$$\delta_2 = \frac{\int_0^1 x(x-1/2)^2 dx}{\int_0^1 (x-1/2)^2 dx} = \frac{1}{2},$$

and

$$\gamma_2^2 = \frac{\int_0^1 (x-1/2)^2 dx}{\int_0^1 1^2 dx} = \frac{1}{12}.$$

Hence,

$$p_2(x) = x(x - \frac{1}{2}) - \frac{1}{2}x - \frac{1}{12}.$$

b) We seek a polynomial $a_2 p_2(x) + a_1 p_1(x) + a_0 p_0(x)$ that best approximates $f(x)$ in the least squares sense. That is, we seek to minimize

$$A = \|f - \sum_{i=0}^{2} a_i p_i\|^2 = (f - \sum_{i=0}^{2} a_i p_i, f - \sum_{j=0}^{2} a_j p_j),$$

where the inner product is $(f, g) = \int_0^1 f(x) g(x) dx$. We expand the expression we seek to minimize, exploiting the fact that p_0, p_1, and p_2 are orthogonal, to get the equivalent expression

$$A = (f, f) - 2 \sum_{i=0}^{2} \big(a_i (f, p_i) + a_i^2 (p_i, p_i) \big).$$

To find the minimum, we differentiate with respect to a_k for $k = 0, 1, 2$, to get the three equations

$$-2(f, p_k) + 2a_k (p_k, p_k) = 0$$

from which follows that $a_k = (f, p_k)/(p_k, p_k)$.

6) Suppose that $\{p_n(x)\}$ are a family of orthogonal polynomials for the weight function $w(x)$ on the interval $[a, b]$. Let $x_1, x_2, \ldots, x_n$ be the roots of $p_n(x)$.
a) Consider the matrix

$$A = \begin{pmatrix} p_0(x_1) & p_0(x_2) & \ldots & p_0(x_n) \\ p_1(x_1) & p_1(x_2) & \ldots & p_1(x_n) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ p_{n-1}(x_1) & p_{n-1}(x_2) & \ldots & p_{n-1}(x_n) \end{pmatrix}.$$

a) Show that A is nonsingular.

b) Show that there is no set of nodes $x_1, x_2, \ldots, x_n$ and coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that the quadrature rule

$$\sum_{j=1}^{n} \alpha_j f(x_j)$$

exactly gives the integral $\int_a^b f(x)w(x)dx$ for all polynomials of degree less than or equal to $2n$.

Solution:
a) For the matrix A defined above, suppose that $A\mathbf{w} = 0$ for some vector $\mathbf{w} = (w_1, w_2, ..., w_n)^T$. Consider the first row of the equation $0 = A\mathbf{w}$:

$$0 = \sum_{j=1}^{n} w_j p_0(x_j) = \sum_{j=1}^{n} w_j.$$

Here I have assumed without loss of generality that the polynomials are normalized so that each has leading coefficient 1. Hence $p_0(x) \equiv 1$. Next consider the second row of $0 = A\mathbf{w}$, and note that $p_1(x) = x + B$ for some constant B:

$$0 = \sum_{j=1}^{n} w_j p_1(x_j) = \sum_{j=1}^{n} w_j x_j.$$

where I used the result for row 1. Similarly, $p_2(x) = x^2 + Bx + C$ for some constants B and C, and the third row of the equation $0 = A\mathbf{w}$ is:

$$0 = \sum_{j=1}^{n} w_j p_2(x_j) = \sum_{j=1}^{n} w_j(x_j^2 + Bx_j + C) = \sum_{j=1}^{n} w_j x_j^2,$$

where I have made use of the results from rows 1 and 2. Proceeding in this way for each row of $0 = A\mathbf{w}$, we see that for $k = 0, 1, ..., n-1$,

$$0 = \sum_{j=1}^{n} w_j x_j^k.$$

This set of n equations can be written in matrix vector form as $V\mathbf{w} = 0$, where V is the matrix

$$V = \begin{pmatrix} x_1^0 & x_2^0 & \dots & x_n^0 \\ x_1^1 & x_2^1 & \dots & x_n^1 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ x_1^n & x_2^n & \dots & x_n^n \end{pmatrix}.$$

The matrix V is the transpose of the standard Vandermonde matrix, and since $x_1, x_2, ..., x_n$ are distinct it is nonsingular. Hence its only null vector is the zero vector which implies that $\mathbf{w} = 0$. Hence, if $\mathbf{w}$ is a null vector of the matrix A, then $\mathbf{w} = 0$, and therefore A is nonsingular.

b) Let $x_1, x_2, ..., x_n$ be distinct but otherwise arbitrary points in $[a, b]$, and let $\alpha_1, \alpha_2, ..., \alpha_n$ be arbitrary complex numbers. Consider the quadrature rule $I_n(f) = \sum_{j=1}^{n} \alpha_j f(x_j)$ as an approximation to $\int_a^b f(x)w(x)dx$. Consider the polynomial $f(x) = (x - x_1)^2(x - x_2)^2 \cdots (x - x_n)^2$ of degree $2n$. For this function $I_n(f) = 0$. But $f(x) > 0$ on $[a, b]$ except at a finite number of discrete points, and $w(x) > 0$ on $[a, b]$, so $\int_a^b f(x)w(x)dx > 0$, so the quadrature rule is not exact for this polynomial.

7) Consider symmetric $n \times n$ tridiagonal matrices of the form

$$J_n = \begin{pmatrix} \delta_1 & \gamma_2 & 0 & 0 & 0 & \dots & 0 \\ \gamma_2 & \delta_2 & \gamma_3 & 0 & 0 & \dots & 0 \\ 0 & \gamma_3 & \delta_3 & \gamma_4 & 0 & \dots & 0 \\ 0 & 0 & \gamma_4 & \delta_4 & \gamma_5 & \dots & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & \gamma_{n-1} & \delta_{n-1} & \gamma_n \\ 0 & 0 & 0 & 0 & \dots & \gamma_n & \delta_n \end{pmatrix}.$$

Define $p_{-1}(z) \equiv 0$, and $p_0(z) \equiv 1$, and let $p_n(z) = \det(zI - J_n)$ be the characteristic polynomial of J_n. Show that, for $n \geq 1$, these characteristic polynomials satisfy the three-term recursion relation

$$p_{n+1}(z) = (z - \delta_{n+1})p_n(z) - \gamma_{n+1}^2 p_{n-1}(z).$$

These polynomials are therefore the same as the orthogonal polynomials in problem 4. So the roots of each of the orthogonal polynomials can be found by solving the eigenvalue problem for the above symmetric matrix.

Solution:
The recursion relation will be established by induction on n. $J_1 = (\delta_1)$ has characteristic polynomial $det(zI - J_1) = z - \delta_1 = (z - \delta_1)p_0(z)$. The matrix J_2

$$J_2 = \begin{pmatrix} \delta_1 & \gamma_2 \\ \gamma_2 & \delta_2 \end{pmatrix}$$

has characteristic polynomial $det(zI - J_2) = (z-\delta_1)(z-\delta_2) - \gamma_2^2 = (z-\delta_2)p_1(z) - \gamma_2^2 p_0(z) = p_2(z)$. Now I assume that the recursion relation holds for $n \leq N$, and I seek to show it holds for $n = N+1$. I expand in minors along the last column to evaluate $det(zI - J_{N+1})$ to get:

$$\begin{aligned} det(zI - J_{N+1}) &= (z - \delta_{N+1})det(zI - J_N) - \gamma_{N+1}^2 det(zI - J_{N-1}) \\ &= (z - \delta_{N+1})p_N(z) - \gamma_{N+1}^2 p_{N-1}(z), \end{aligned}$$

as claimed.