Math 6620 Problem Set 3

1) Consider the function $f(x) = \frac{1}{1+x^2}$ on the interval $-5 \le x \le 5$. For each $n \ge 1$, define $h = 10/n$, $x_j = -5 + j * h$ for $j = 0, 1, 2, \ldots, n$, and let $p_n(x)$ be the polynomial of degree n which interpolates f at the nodes $x_0, x_1, \ldots, x_n$. Compute p_n for $n = 1, 2, \ldots, 20$, plot $f(x)$ and $p_n(x)$ for each n, and estimate the maximum error $|f(x) - p_n(x)|$ for $x \in (-5, 5)$. Describe what you find.

2) Consider the function $f(x) = \cos(\pi x)$ on the interval $x \in [0, 1]$.
a) Write a program to compute the cubic spline $s(x)$ for f on the interval $[0, 1]$, with the conditions $s''(0) = s''(1) = 0$ and determine the splines for $n = 5$ and $n = 10$ subintervals.
b) What is the linear system that we must solve to determine the cubic periodic spline $s(x)$, that is, the cubic spline that satisfies the conditions $s'(x_0) = s'(x_n)$ and $s''(x_0) = s''(x_n)$? You may assume that $f(x_0) = f(x_n)$.

3) Consider the function $f(x) = x^2 e^x$ on the interval $[a, b] = [0, 3]$. Compute the trapezoidal rule approximation to the integral of f over [a,b] with $n = 10$, $n = 20$, and $n = 40$. What is the error in each case? (You can find the exact integral of f using integration by parts.) Use Richardson Extrapolation with these three trapezoidal rule approximations to obtain the best approximation to the true integral that you can. What is the approximate integral and what is the error in using it?

4) Let P_j denote the set of polynomials of degree j with leading coefficient 1. Let $w(x)$ be a nonnegative and continuous weight function on the finite interval $[a, b]$, and define an inner product $(f, g) = \int_a^b f(x)g(x)w(x)dx$ for any f and g continuous on $[a, b]$. Prove the following theorem (hint: induction)
Theorem: There exist polynomials $p_j(x) \in P_j$, $j = 0, 1, 2, \ldots$, such that

$$(p_i, p_k) = 0 \quad \text{for i} \ne \text{k.}$$

These polynomials are uniquely defined by the recursions

$$p_0(x) \equiv 1$$

$$p_{i+1}(x) \equiv (x - \delta_{i+1})p_i(x) - \gamma_{i+1}^2 p_{i-1}(x)$$

for $i \ge 0$, where $p_{-1}(x) \equiv 0$ and

$$\delta_{i+1} = \frac{(xp_i, p_i)}{(p_i, p_i)}$$

for $i \ge 0$, and

$$\gamma_{i+1}^2 = \begin{cases} 0 & \text{for i} = 0 \\ \frac{(p_i, p_i)}{(p_{i-1}, p_{i-1})} & \text{for i} \ge 1. \end{cases}$$

5) Consider the interval $[a, b] = [0, 1]$ and the weight function $w(x) \equiv 1$.
a) Determine the orthogonal polynomials $p_n(x)$ of degree $n = 0, 1, 2$ for this interval and weight function, and normalized so that the coefficient of x^n in $p_n(x)$ is 1.
b) For $n = 2$, determine the best least-squares polynomial approximation of degree n to $f(x) = e^x$.

6) Suppose that $\{p_n(x)\}$ are a family of orthogonal polynomials for the weight function $w(x)$ on the interval $[a, b]$. Let $x_1, x_2, \ldots, x_n$ be the roots of $p_n(x)$.
a) Consider the matrix

$$A = \begin{pmatrix} p_0(x_1) & p_0(x_2) & \ldots & p_0(x_n) \\ p_1(x_1) & p_1(x_2) & \ldots & p_1(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ p_{n-1}(x_1) & p_{n-1}(x_2) & \ldots & p_{n-1}(x_n) \end{pmatrix}.$$

a) Show that A is nonsingular.
b) Show that there is no set of nodes $x_1, x_2, \ldots, x_n$ and coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that the quadrature rule

$$\sum_{j=1}^{n} \alpha_j f(x_j)$$

exactly gives the integral $\int_a^b f(x)w(x)dx$ for all polynomials of degree less than or equal to $2n$.

7) Consider symmetric $n \times n$ tridiagonal matrices of the form

$$J_n = \begin{pmatrix} \delta_1 & \gamma_2 & 0 & 0 & 0 & \ldots & 0 \\ \gamma_2 & \delta_2 & \gamma_3 & 0 & 0 & \ldots & 0 \\ 0 & \gamma_3 & \delta_3 & \gamma_4 & 0 & \ldots & 0 \\ 0 & 0 & \gamma_4 & \delta_4 & \gamma_5 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & \gamma_{n-1} & \delta_{n-1} & \gamma_n \\ 0 & 0 & 0 & 0 & \ldots & \gamma_n & \delta_n \end{pmatrix}.$$

Define $p_{-1}(z) \equiv 0$, and $p_0(z) \equiv 1$, and let $p_n(z) = \det(zI - J_n)$ be the characteristic polynomial of J_n. Show that, for $n \geq 1$, these characteristic polynomials satisfy the three-term recursion relation

$$p_{n+1}(z) = (z - \delta_{n+1})p_n(z) - \gamma_{n+1}^2 p_{n-1}(z).$$

These polynomials are therefore the same as the orthogonal polynomials in problem 4. So the roots of each of the orthogonal polynomials can be found by solving the eigenvalue problem for the above symmetric matrix.