

Math 6620 Spring 2009 Problem Set 2  
Solutions

1) Let  $A \in \mathbb{C}^{m \times m}$  have distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$  with corresponding right eigenvectors  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$  and left eigenvectors  $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_m$ . Show that

$$\mathbf{y}_j^* \mathbf{x}_i = \begin{cases} \neq 0, & \text{if } i = j \\ = 0, & \text{if } i \neq j. \end{cases}$$

**Solution:** Form the scalar products of  $\mathbf{y}_k^*$  with both sides of  $A\mathbf{x}_j = \lambda_j\mathbf{x}_j$  and of  $\mathbf{x}_j$  with both sides of  $\mathbf{y}_k^*A = \lambda_k\mathbf{y}_k^*$ , and subtract the second from the first to obtain  $(\lambda_j - \lambda_k)\mathbf{y}_k^*\mathbf{x}_j = 0$ . If  $j \neq k$ , then  $(\lambda_j - \lambda_k) \neq 0$ , so  $\mathbf{y}_k^*\mathbf{x}_j = 0$  for  $j \neq k$ .

Since eigenvectors that correspond to distinct eigenvalues are linearly independent,  $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$  span  $\mathbb{C}$ . Suppose  $\mathbf{y}_j^*\mathbf{x}_j = 0$  for some  $j$ . Then,  $\mathbf{y}_j^*\mathbf{x}_k = 0$  for all  $k = 1, \dots, m$ . Therefore  $\mathbf{y}_j = 0$ . But this is a contradiction, since  $\mathbf{y}_j$  is an eigenvector and therefore must be nonzero. Hence, we conclude that  $\mathbf{y}_j^*\mathbf{x}_j \neq 0$ .

2) Trefethen, page 189, problem 24.2(a,c). Here is Gerschgorin's Theorem which holds for any  $m \times m$  matrix  $A$ . Every eigenvalue of  $A$  lies in a least one of the  $m$  circular disks in the complex plane with centers  $a_{ii}$  and radii  $\sum_{j \neq i} |a_{ij}|$ . Moreover, if  $n$  of these disks form a connected domain that is disjoint from the other  $m - n$  disks, then there are precisely  $n$  eigenvalues of  $A$  within this domain.

**Solution:** a) Prove the first part of Gerschgorin's Theorem. Let  $\lambda, \mathbf{x}$  be an eigenpair of  $A$  with  $\|\mathbf{x}\|_\infty = 1$ , and suppose that  $x_k = \|\mathbf{x}\|_\infty$ . Considering the  $k^{\text{th}}$  equation of the system  $A\mathbf{x} = \lambda\mathbf{x}$  gives

$$\sum_{j=1}^m a_{kj}x_j = \lambda x_k.$$

Since  $x_k = 1$ ,

$$\sum_{j \neq k} a_{kj}x_j + a_{kk} = \lambda,$$

from which we conclude that

$$|\lambda - a_{kk}| = \left| \sum_{j \neq k} a_{kj}x_j \right| \leq \sum_{j \neq k} |a_{kj}x_j| \leq \sum_{j \neq k} |a_{kj}|.$$

The last inequality holds because  $x_j \leq 1$ .

c) For the matrix

$$A = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{bmatrix}$$

with  $|\epsilon| < 1$ , Gerschgorin's Theorem tells us that for any eigenvalue  $\lambda$  of  $A$ , at least one of the inequalities  $|\lambda - 8| \leq 1$ ,  $|\lambda - 4| \leq 1 + |\epsilon|$ , or  $|\lambda - 1| \leq |\epsilon|$ , holds. Since  $A$  is symmetric, its eigenvalues are real, and so all eigenvalues lie in the union of the intervals  $|\lambda - 8| \leq 1$ ,  $|\lambda - 4| \leq 1 + |\epsilon| < 2$ , or  $|\lambda - 1| \leq |\epsilon| < 1$ .

3) Trefethen, page 194, problem 25.1: (a) Let  $A \in \mathbb{C}^{m \times m}$  be tridiagonal and Hermitian, with all of its sub- and superdiagonal entries nonzero. Prove that the eigenvalues are distinct. (b) On the other hand, let  $A$  be upper-Hessenberg, with all of its subdiagonal entries nonzero. Give an example that shows that the eigenvalues of  $A$  are not necessarily distinct.

**Solution:** a) Let  $B = A - \lambda I$  for some complex number  $\lambda$ . We will show that  $B$  has rank at least  $m - 1$ . The matrix  $B(2 : m, 1 : m - 1)$  is upper triangular with nonzero diagonal entries so it is full rank, that is, it has rank  $m - 1$ . Hence the larger matrix  $B$  has rank at least  $m - 1$ . If  $\lambda$  is an eigenvalue of  $A$ , this implies that it has geometric multiplicity 1. On the other hand, since  $A$  is Hermitian, it has a complete set of (orthonormal) eigenvectors. Suppose that  $A$  had an eigenvalue with algebraic multiplicity  $\mu > 1$ . Since this eigenvalue cannot be defective for Hermitian  $A$ , it must have geometric multiplicity  $\mu > 1$  as well. This contradicts the result just proved, so  $A$  must have distinct eigenvalues.

b) The matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has triple eigenvalue 1.

4) Trefethen, page 200, problem 26.1: Let  $A \in \mathbb{C}^{m \times m}$  and  $\epsilon > 0$ . Show that for  $z \in \mathbb{C}$ , the following statements are equivalent:

1.  $z$  is an eigenvalue of  $A + \delta A$  for some matrix  $\delta A$  with  $\|\delta A\|_2 \leq \epsilon$ .
2. There exists  $\mathbf{u} \in \mathbb{C}^m$  with  $\|(A - zI)\mathbf{u}\|_2 \leq \epsilon$  and  $\|\mathbf{u}\|_2 = 1$ .
3.  $\sigma_m(zI - A) \leq \epsilon$  where  $\sigma_m$  denotes smallest singular value.
4.  $\|(zI - A)^{-1}\|_2 \geq \epsilon^{-1}$ , using the convention that  $\|(zI - A)^{-1}\|_2 = \infty$  for  $z$  an eigenvalue of  $A$ .

**Solution:**

(1)  $\Rightarrow$  (2): Assume  $z$  is an eigenvalue of  $A + \delta A$  for some  $\delta A$  with  $\|\delta A\|_2 \leq \epsilon$ , and let  $\mathbf{u}$  be an associated eigenvector with  $\|\mathbf{u}\|_2 = 1$ . Then,  $(A + \delta A)\mathbf{u} = z\mathbf{u}$  implies that  $(A - zI)\mathbf{u} = -\delta A\mathbf{u}$ . Therefore,  $\|(A - zI)\mathbf{u}\|_2 = \|\delta A\mathbf{u}\|_2 \leq \|\delta A\|_2 \leq \epsilon$ .

(2)  $\Rightarrow$  (3): Suppose  $\mathbf{u} \in \mathbb{C}^m$  is such that  $\|(A - zI)\mathbf{u}\|_2 \leq \epsilon$  with  $\|\mathbf{u}\|_2 = 1$ . Let  $A - zI = U\Sigma V^*$  be the SVD of  $A - zI$ . The columns  $\mathbf{v}_1, \dots, \mathbf{v}_m$  of  $V$  form an orthonormal basis for  $\mathbb{C}^m$ , so we may express  $\mathbf{u}$  as a linear combination  $\mathbf{u} = \sum_{i=1}^m a_i \mathbf{v}_i$ . Then, letting  $\mathbf{u}_1, \dots, \mathbf{u}_m$  denote the columns of  $U$  and  $\sigma_1, \dots, \sigma_m$  denote the singular values of  $A - zI$  in descending order, we have:

$$\begin{aligned} \epsilon^2 &\geq \|(A - zI)\mathbf{u}\|_2^2 = \left\| \sum_{i=1}^m a_i \sigma_i \mathbf{u}_i \right\|_2^2 = \sum_{i=1}^m (a_i \sigma_i)^2 \\ &\geq \sum_{i=1}^m (a_i \sigma_m)^2 = \sum_{i=1}^m (a_i)^2 \sigma_m^2 = \sigma_m^2 \|\mathbf{u}\|_2^2 = \sigma_m^2. \end{aligned}$$

In this chain of relations, I have used that  $\{\mathbf{u}_i\}$  are orthonormal, that  $\{\mathbf{v}_i\}$  are orthonormal, and that  $\sigma_m \leq \sigma_i$  for  $i = 1, \dots, m$ . Thus,  $\sigma_m(zI - A) \leq \epsilon$ .

(3)  $\Rightarrow$  (4): Suppose that  $\sigma_m(zI - A) \leq \epsilon$ . If  $z$  is an eigenvalue of  $A$ , then  $\|(zI - A)^{-1}\|_2 = \infty$  and (4) holds trivially. Otherwise,

$$\|(zI - A)^{-1}\|_2 = \sigma_{\max}((zI - A)^{-1}) = \frac{1}{\sigma_m(zI - A)} \geq \epsilon^{-1}.$$

(4)  $\Rightarrow$  (1): Suppose that  $\|(zI - A)^{-1}\|_2 \geq \epsilon^{-1}$ . If  $z$  is an eigenvalue of  $A$ , then (1) holds trivially by choosing  $\delta A = 0$ . Otherwise,

$$\|(zI - A)^{-1}\|_2 = \frac{1}{\sigma_m(zI - A)} \geq \epsilon^{-1}.$$

Again, let  $zI - A = U\Sigma V^*$  be the SVD of  $zI - A$ . Let  $\Sigma_{\delta A}$  be  $\sigma_m(zI - A)$  times the identity matrix and let  $\delta A = U\Sigma_{\delta A}V^*$ . It follows that  $\|\delta A\|_2 = \sigma_m(zI - A) \leq \epsilon$ . Also, we have that  $zI - (A + \delta A) = (zI - A) - \delta A$  is singular since it has SVD  $U(\Sigma - \Sigma_{\delta A})V^*$ , and the last diagonal entry of  $(\Sigma - \Sigma_{\delta A})$  is zero. Thus,  $z$  is an eigenvalue of  $A + \delta A$ .

5) Trefethen, page 201, problem 26.3. *Bauer-Fike Theorem*: Suppose that  $A \in \mathbb{C}^{m \times m}$  is diagonalizable with  $A = V\Lambda V^{-1}$  and let  $\delta A \in \mathbb{C}^{m \times m}$  be arbitrary. Then every eigenvalue of  $A + \delta A$  lies in at least one of the  $m$  circular disks in the complex plane of radius  $\kappa(V)\|\delta A\|_2$  centered at the eigenvalues of  $A$ . Here,  $\kappa(V)$  is the 2-norm condition number of  $V$ .

(a) Prove the Bauer-Fike Theorem using the equivalence of (1) and (4) in problem 26.1.

(b) Suppose that  $A$  is normal. Show that for each eigenvalue  $\tilde{\lambda}_j$  of  $A + \delta A$ , there is an eigenvalue  $\lambda_j$  of  $A$  such that  $|\tilde{\lambda}_j - \lambda_j| \leq \|\delta A\|_2$ .

**Solution:** (a) Let  $z$  be an eigenvalue of  $A + \delta A$ . If  $z$  is also an eigenvalue of  $A$ , then the result follows trivially. Otherwise  $(zI - A)^{-1}$  exists and the equivalence of (1) and (4) in problem 26.1 implies that  $\|(zI - A)^{-1}\|_2 \geq \|\delta A\|_2^{-1}$ . Thus,

$$\begin{aligned} \frac{1}{\|\delta A\|_2} &\leq \|(zI - A)^{-1}\|_2 \\ &= \|(VzIV^{-1} - V\Lambda V^{-1})^{-1}\|_2 \\ &= \|(V(zI - \Lambda)V^{-1})^{-1}\|_2 \\ &= \|(V(zI - \Lambda)^{-1}V^{-1})\|_2 \\ &\leq \|V\|_2 \|(zI - \Lambda)^{-1}\|_2 \|V^{-1}\|_2. \end{aligned}$$

Since the matrix  $(zI - \Lambda)^{-1}$  is diagonal, its 2-norm is the largest magnitude diagonal entry, so

$$\|(zI - \Lambda)^{-1}\|_2 = \frac{1}{\min_{i=1, \dots, m} |z - \lambda_i|}.$$

Combining this with the above inequalities, we find that

$$\min_{i=1, \dots, m} |z - \lambda_i| \leq \|V\|_2 \|V^{-1}\|_2 \|\delta A\|_2 = \kappa(V) \|\delta A\|_2.$$

That is,  $z$  lies in at least one of the  $m$  circular disks referred to in the Bauer-Fike theorem.

(b) Suppose that  $A$  is normal. Then,  $A$  is unitarily diagonalizable with  $A = Q\Lambda Q^*$ . By part(a), if  $\tilde{\lambda}$  is an eigenvalue of  $A + \delta A$ , then there exists an eigenvalue  $\lambda$  of  $A$  such that

$$|\tilde{\lambda} - \lambda| \leq \kappa(Q) \|\delta A\|_2 = \|\delta A\|_2,$$

since  $\kappa(Q) = 1$  for unitary  $Q$ .

6) Trefethen, page 210, problem 27.5. Inverse iteration depends on the solution of a system of equations that may be exceedingly ill-conditioned with condition number on the order of  $\epsilon^{-1}$  where  $\epsilon$  is machine precision. Show that this ill-conditioning is not a problem in inverse iteration.

**Solution:** Suppose  $A$  is a real symmetric matrix with eigenvalues  $\lambda_1, \dots, \lambda_m$  and corresponding eigenvectors  $\mathbf{q}_1, \dots, \mathbf{q}_m$ , and suppose that

$$|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_{m-1}| \gg |\lambda_m|.$$

Suppose that  $\mathbf{v}$  is a vector with nonzero components in the directions of all of  $A$ 's eigenvectors, so that

$$\mathbf{v} = \sum_{j=1}^m c_j \mathbf{q}_j,$$

where  $c_j \neq 0$  for all  $j$ . Then the exact solution  $\mathbf{w}$  to the linear system  $A\mathbf{w} = \mathbf{v}$  is given by

$$\begin{aligned} \mathbf{w} = A^{-1}\mathbf{v} &= A^{-1} \sum_{j=1}^m c_j \mathbf{q}_j \\ &= \sum_{j=1}^m c_j \frac{1}{\lambda_j} \mathbf{q}_j \\ &= c_m \frac{1}{\lambda_m} \mathbf{q}_m + \sum_{j=1}^{m-1} c_j \frac{1}{\lambda_j} \mathbf{q}_j. \end{aligned}$$

Since, by assumption  $\frac{1}{|\lambda_m|} \gg \frac{1}{|\lambda_j|}$  for  $j = 1, \dots, m-1$ ,

$$\frac{\mathbf{w}}{\|\mathbf{w}\|_2} = \mathbf{q}_m + O\left(\frac{\lambda_m}{\lambda_{m-1}}\right).$$

Now suppose that  $A\mathbf{w} = \mathbf{v}$  is solved backward stably, yielding a computed solution  $\tilde{\mathbf{w}}$ . Then  $\tilde{\mathbf{w}}$  satisfies  $(A + \delta A)\tilde{\mathbf{w}} = \mathbf{v}$ , where  $\|\delta A\|_2 = \|A\|_2 O(\epsilon)$ . That is,  $A\tilde{\mathbf{w}} = \mathbf{v} - \delta A\tilde{\mathbf{w}}$ . The vector  $\mathbf{v} - \delta A\tilde{\mathbf{w}}$  on the right-hand-side of this equation will generally have nonzero components in the directions of all of the eigenvectors  $\mathbf{q}_i$ .

$$\mathbf{v} - \delta A\tilde{\mathbf{w}} = \sum_{j=1}^m d_j \mathbf{q}_j,$$

where  $d_j \neq 0$  for all  $j$ . It follows that  $\tilde{\mathbf{w}}$  may be expressed

$$\tilde{\mathbf{w}} = d_m \frac{1}{\lambda_m} \mathbf{q}_m + \sum_{j=1}^{m-1} d_j \frac{1}{\lambda_j} \mathbf{q}_j,$$

so that

$$\frac{\tilde{\mathbf{w}}}{\|\tilde{\mathbf{w}}\|_2} = \mathbf{q}_m + O\left(\frac{\lambda_m}{\lambda_{m-1}}\right).$$

So it is possible for the coordinates  $c_j$  and  $d_j$  to be such that  $\mathbf{w}$  and  $\tilde{\mathbf{w}}$  are far away from each other, but the normalized vectors  $\mathbf{w}/\|\mathbf{w}\|_2$  and  $\tilde{\mathbf{w}}/\|\tilde{\mathbf{w}}\|_2$  are close together.

7) Trefethen, page 218, problem 28.2(a,b) The preliminary reduction to tridiagonal form would be of little use if the steps of the QR algorithm did not preserve this structure. Fortunately, they do.

(a) In the QR factorization  $A = QR$  of a symmetric tridiagonal matrix  $A$  which entries of  $R$  and  $Q$  are in general nonzero? (b) Show that the tridiagonal structure is recovered when the product  $RQ$  is formed.

**Solution:**

(a)  $Q$  is upper-Hessenberg. To show this, fix a row  $i$  of  $Q$ . To show that  $Q$  is upper-Hessenberg is to show that  $Q_{ij} = 0$  for  $j = 1, \dots, i - 2$ . Since  $A$  is tridiagonal,  $A_{ij} = 0$  if  $|i - j| \geq 2$ . Thus, since  $A = QR$ , for  $j = 1, \dots, i - 2$ , we have

$$0 = A_{ij} = \sum_{k=1}^m Q_{ik} R_{kj} = \sum_{k=1}^j Q_{ik} R_{kj},$$

since  $R$  is upper triangular. In particular, for  $j = 1$ , this becomes  $0 = A_{i1} = Q_{i1} R_{11}$  for  $i \leq 3$ . Since the diagonal elements of  $R$  are in general (i.e., for a full-ranked matrix) nonzero, it follows that  $Q_{i1} = 0$  for  $i \geq 3$ . Then, for  $j = 2$ , we have  $0 = A_{i2} = Q_{i1} R_{12} + Q_{i2} R_{22} = Q_{i2} R_{22}$  for  $i \leq 4$ . Therefore,  $Q_{i2} = 0$  for  $i \leq 4$ . It follows inductively that  $Q_{ij} = 0$  for  $j = 1, \dots, i - 2$  for arbitrary fixed  $i$ , so  $Q$  is upper-Hessenberg.

$R$  is upper triangular and zero everywhere except on its diagonal and first two superdiagonals. That is  $R_{ij} = 0$  for  $j \geq i + 3$ . To prove this, consider  $R = Q^T A$ . Fix row  $i$  of  $R$  and suppose that  $j \geq i + 3$ . Then we have,

$$R_{ij} = \sum_{k=1}^m (Q^T)_{ik} A_{kj} = \sum_{k=1}^{i+1} (Q^T)_{ik} A_{kj},$$

since  $Q^T$  is lower-Hessenberg. But, for  $j \geq i + 3$  and  $k \leq i + 1$  we have  $j - k \geq 2$ , so that  $A_{kj} = 0$  and therefore  $R_{ij} = 0$  if  $j \geq i + 3$ .

(b)  $\tilde{A} = RQ$  is tridiagonal. To show this, note that  $\tilde{A}$  is symmetric since  $\tilde{A} = RQ = Q^T A Q$  and hence  $\tilde{A}^T = (Q^T A Q)^T = Q^T A^T Q = Q^T A Q = \tilde{A}$ , since  $A$  is symmetric. Therefore to show that  $\tilde{A}$  is tridiagonal, it is enough to show that it is upper-Hessenberg. For row  $i$  of  $\tilde{A}$ , and suppose  $j \leq i - 2$ . Then,

$$\tilde{A}_{ij} = \sum_{k=1}^m R_{ik} Q_{kj} = \sum_{k=i}^{i+2} R_{ik} Q_{kj},$$

by the second claim of part (a). For  $j \leq i - 2$  and  $k \geq i$ , we have  $k \geq j + 2$ , and therefore  $Q_{kj} = 0$ . Thus,  $\tilde{A}_{ij} = 0$  for  $j = 1, \dots, i - 2$  and  $\tilde{A}$  is upper-Hessenberg and therefore tridiagonal.

8) Trefethen, page 218, problem 29.1. For part (b) just use MATLAB's built in QR factorization command `qr`.

**Solution:** Computational problem.

9) Given  $A \in \mathbf{C}^{n \times n}$ , use the Schur decomposition to show that for every  $\epsilon > 0$ , there exists a diagonalizable matrix  $B$  such that  $\|A - B\|_2 \leq \epsilon$ . This shows that the set of diagonalizable matrices is dense in  $\mathbf{C}^{n \times n}$  and that the Jordan Canonical form is not a continuous matrix decomposition.

**Solution:** Let  $A = QTQ^*$  be the Schur decomposition of  $A$ . If the diagonal elements of  $T$  are distinct, then  $A$  has distinct eigenvalues and is diagonalizable. Set  $B = A$  in this case. If  $T$  has diagonal entries which repeat, perturb them by amounts less than  $\epsilon$  so that the resulting matrix  $\tilde{T}$  has distinct diagonal elements. Let  $B = Q\tilde{T}Q^*$ . Then  $\|A - B\|_2 = \|T - \tilde{T}\|_2 < \epsilon$ , since  $T$  and  $\tilde{T}$  differ only on the diagonal and there they differ by less than  $\epsilon$ . So arbitrarily close to a defective matrix is a diagonalizable matrix and so the Jordan Canonical Form is not a continuous function of the matrix entries.

10) Suppose  $A_k \rightarrow A$  and that  $Q_k^* A_k Q_k = T_k$  is a Schur decomposition of  $A_k$ . Show that  $\{Q_k\}$  has a converging subsequence  $\{Q_{k_i}\}$  with the property that for  $Q \equiv \lim_{i \rightarrow \infty} Q_{k_i}$ , the matrix  $T \equiv Q^* A Q$  is upper triangular. This shows that the eigenvalues of a matrix are continuous functions of its entries.

**Solution:** Suppose that the above matrices are  $m \times m$ . The set of  $m \times m$  matrices  $Q$  such that  $\|Q\|_2 = 1$  is compact, so every sequence of such matrices including  $\{Q_k\}$  has a convergent subsequence, say  $\{Q_{k_i}\}$ . Let  $Q \equiv \lim_{k_i \rightarrow \infty} Q_{k_i}$  and let  $T \equiv Q^* A Q$ . I claim that  $T$  is upper triangular. Let  $T_{k_i} \equiv Q_{k_i}^* A Q_{k_i}$ . Since  $Q_{k_i} \rightarrow Q$ ,  $A_{k_i} \rightarrow A$ , and matrix multiplication is continuous, it follows that  $T_{k_i} = Q_{k_i}^* A Q_{k_i} \rightarrow Q^* A Q = T$ . Since  $T$  is the limit of upper triangular matrices, it also is upper triangular, and so  $A = Q T Q^*$  is the Schur decomposition of  $A$ .

11) For an eigenvalue  $\lambda$  of  $A$ , let  $x$  be right unit eigenvector and  $y$  be a left unit eigenvector. Then, we showed that the condition number of  $\lambda$  is  $1/s(\lambda)$  where  $s(\lambda) \equiv |y^* x|$ . For the matrix

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$$

with scalar  $a, b, c$ , show that  $s(a) = s(b) = (1 + |c/(a-b)|^2)^{-\frac{1}{2}}$ . What does this say about the condition number of the eigenvalues of this matrix if  $a \rightarrow b$ ?

**Solution:** Recall that the condition number of an eigenvalue  $\lambda$  is

$$\kappa(\lambda) = \frac{1}{\mathbf{y}^* \mathbf{x}} = \frac{1}{s(\lambda)},$$

where  $\mathbf{x}$  and  $\mathbf{y}$  are right and left unit eigenvectors corresponding to  $\lambda$ . For the matrix

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix},$$

the eigenvalues are  $\lambda = a$  and  $\lambda = b$ . For  $\lambda = a$ ,  $\mathbf{x} = [1, 0]^T$  and

$$\mathbf{y} = \frac{1}{\sqrt{\left(\frac{b-a}{c}\right)^2 + 1}} \left[ \frac{b-a}{c}, -1 \right]^T.$$

Hence

$$|\mathbf{y}^* \mathbf{x}| = \frac{1}{\sqrt{\left(\frac{b-a}{c}\right)^2 + 1}} \left| \frac{b-a}{c} \right| = \frac{1}{\sqrt{1 + \left(\frac{c}{b-a}\right)^2}}.$$

As  $b \rightarrow a$ , this goes to 0, unless  $c = 0$ . Hence,  $\kappa(\lambda) \rightarrow \infty$  as  $b \rightarrow a$  unless  $c = 0$ .

12) Suppose  $A$  is *normal*, i.e.,  $AA^* = A^*A$ . Show that if  $A$  is also triangular, it is diagonal. Use this to show that an  $n$  by  $n$  matrix is normal if and only if it has  $n$  orthonormal eigenvectors. Hint: Show that  $A$  is normal if and only if its Schur form is normal.

**Solution:** We first show that if  $A$  is normal and upper triangular, it is diagonal. The proof for a lower triangular matrix then follows by considering  $A^*$ . So suppose  $A$  is normal and upper triangular. From the equation  $A^*A = AA^*$ , we know that  $(A^*A)_{jj} = (AA^*)_{jj}$  for each  $j = 1, \dots, m$ . This implies

$$(A^*A)_{jj} = \sum_{k=1}^m (A^*)_{jk}(A)_{kj} = \sum_{k=1}^m (\bar{A})_{kj}(A)_{kj} = \sum_{k=1}^j |A_{kj}|^2.$$

The last equality holds because  $A$  is upper-triangular. Similarly,

$$(AA^*)_{jj} = \sum_{k=1}^m (A)_{jk}(A^*)_{kj} = \sum_{k=1}^m (A)_{jk}(\bar{A})_{jk} = \sum_{k=j}^m |A_{jk}|^2.$$

For  $j = 1$ , the equation  $(A^*A)_{jj} = (AA^*)_{jj}$  becomes  $|A_{11}|^2 = |A_{11}|^2 + |A_{12}|^2 + \dots + |A_{1m}|^2$  which implies that the off-diagonal entries in row 1 of  $A$  are all zero. Next, for  $j = 2$ , we have  $|A_{22}|^2 = |A_{12}|^2 + |A_{22}|^2 + \dots + |A_{2m}|^2 = |A_{22}|^2 + \dots + |A_{2m}|^2$  which implies that the off-diagonal entries of row 2 of  $A$  are also all zero. Proceeding row-by-row in this manner, we see that  $A_{ij} = 0$  for  $i < j$ , so  $A$  is lower triangular and therefore diagonal.

Next we show that  $m$ -by- $m$  matrix  $A$  is normal if and only if it has  $m$  orthonormal eigenvectors. To this end, suppose  $A$  is normal and  $A = QTQ^*$  is the Schur factorization for  $A$ . Then,

$$\begin{aligned} TT^* &= (Q^*AQ)(Q^*AQ)^* \\ &= (Q^*AQ)(Q^*A^*Q) \\ &= Q^*AA^*Q = Q^*A^*AQ \\ &= (Q^*A^*Q)(Q^*AQ) = T^*T \end{aligned}$$

so  $T$  is normal. Clearly, we can go in the other direction; so that if  $T$  is normal, we can conclude that  $A$  is normal as well. So if  $A$  is normal, then  $T$  is normal and triangular and therefore diagonal from the previous argument. So  $A = QTQ^*$  is in fact an eigenvalue decomposition of  $A$  with the orthonormal columns of  $Q$  as eigenvectors. Hence, if  $A$  is normal, it has  $m$  orthonormal eigenvectors.

Now suppose that  $A$  has  $m$  orthonormal eigenvectors  $\mathbf{q}_1, \dots, \mathbf{q}_m$  and corresponding eigenvalues  $\lambda_1, \dots, \lambda_m$ . Let the matrix  $Q$  have the vectors  $\mathbf{q}_1, \dots, \mathbf{q}_m$  as columns, and let  $\Lambda$  be the diagonal matrix with diagonal entries  $\lambda_1, \dots, \lambda_m$ . Then,  $A = Q\Lambda Q^*$ . It follows that  $AA^* = Q\Lambda Q^*(Q\Lambda Q^*)^* = Q\Lambda Q^*Q\Lambda^*Q^* = Q\Lambda\Lambda^*Q^* = Q\Lambda^*\Lambda Q^* = Q\Lambda^*Q^*Q\Lambda Q^* = A^*A$ , so  $A$  is normal.