Math 6620 Problem Set 2
Due date to be announced

1) Let $A \in \mathbb{C}^{m\times m}$ have distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ with corresponding right eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ and left eigenvectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m$. Show that

$$\mathbf{y}_j^* \mathbf{x}_i = \begin{cases} \neq 0, & \text{if } \mathrm{i} = \mathrm{j} \\ = 0, & \text{if } \mathrm{i} \neq \mathrm{j}. \end{cases}$$

2) Trefethen, page 189, problem 24.2(a,c)

3) Trefethen, page 194, problem 25.1

4) Trefethen, page 200, problem 26.1

5) Trefethen, page 201, problem 26.3

6) Trefethen, page 210, problem 27.5

7) Trefethen, page 218, problem 28.2(a,b)

8) Trefethen, page 218, problem 29.1. For part (b) just use MATLAB's builtin QR factorization command qr.

9) Given $A \in \mathbf{C}^{n\times n}$, use the Schur decomposition to show that for every $\epsilon > 0$, there exists a diagonalizable matrix B such that $\|A - B\|_2 \leq \epsilon$. This shows that the set of diagonalizable matrices is dense in $\mathbf{C}^{n\times n}$ and that the Jordan Canonical form is not a continuous matrix decomposition.

10) Suppose $A_k \to A$ and that $Q_k^* A_k Q_k = T_k$ is a Schur decomposition of A_k. Show that $\{Q_k\}$ has a converging subsequence $\{Q_{k_i}\}$ with the property that for $Q \equiv \lim_{i\to\infty} Q_{k_i}$, the matrix $T \equiv Q^* A Q$ is upper triangular. This shows that the eigenvalues of a matrix are continuous functions of its entries.

11) For an eigenvalue λ of A, let x be right unit eigenvector and y be a left unit eigenvector. Then, we showed that the condition number of λ is $1/s(\lambda)$ where $s(\lambda) \equiv |y^* x|$. For the matrix

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$$

with scalar a, b, c, show that $s(a) = s(b) = (1 + |c/(a-b)|^2)^{-\frac{1}{2}}$. What does this say about the condition number of the eigenvalues of this matrix if $a \to b$?

12) Suppose A is *normal*, i.e., $AA^* = A^*A$. Show that if A is also triangular, it is diagonal. Use this to show that an n by n matrix is normal if and only if it has n orthonormal eigenvectors. Hint: Show that A is normal if and only if its Schur form is normal.