Math 5440 Problem Set 2 – Solutions

1: (Logan, 1.2 #1) How would the derivation of the basic conservation law $u_t + \phi_x = f$ change if the tube had variable cross-sectional area $A = A(x)$ rather than a constant cross-sectional area?

Rewriting Equation (1.7) with $A(x)$ instead of $A$, we have

$$\frac{d}{dt} \int_a^b u(x,t) A(x) \, dx = A(a)\phi(a,t) - A(b)\phi(b,t) + \int_a^b f(x,t) A(x) \, dx.$$  

By the Fundamental Theorem of Calculus, we can turn the first two terms of the right-hand side into an integral. Assuming then that $u$ has continuous first partials, we can move the $\frac{d}{dt}$ inside the left-hand integral to yield

$$\int_a^b \frac{\partial}{\partial t}(u(x,t)A(x)) \, dx = -\int_a^b \frac{\partial}{\partial x}(A(x)\phi(x,t)) \, dx + \int_a^b f(x,t) A(x) \, dx.$$  

Rewriting this as one large integral, we have

$$\int_a^b Au_t + (\phi A)_x - Af \, dx = 0$$  

for any interval $[a,b]$. Therefore, since this interval is arbitrary, the integrand must be identically 0, so

$$Au_t + (\phi A)_x = Af.$$  

If we suppose that $A(x) > 0$ everywhere, then our PDE becomes

$$u_t + \phi_x + \phi \frac{A'(x)}{A(x)} = f.$$  

\[ \blacksquare \]
2. (Logan, 1.2 # 6) Solve the initial boundary value problem

\[
\begin{align*}
    u_t + cu_x &= -\lambda u, \quad x, t > 0 \\
    u(x, 0) &= 0, \quad x > 0, \quad u(0, t) = g(t), \quad t > 0.
\end{align*}
\]

In this problem you will have to treat the domains \( x > ct \) and \( x < ct \) differently; the boundary condition affects the region \( x < ct \) and the initial condition affects the region \( x > ct \).

Let \( \zeta = x - ct \) and \( \tau = t \). Then the first partials of \( u(x, t) = u(\zeta + ct, \tau) = U(\zeta, \tau) \) are

\[
    u_x = U_\zeta \zeta_x + U_\tau \tau_x = U_\zeta \quad \text{and} \quad u_t = U_\zeta \zeta_t + U_\tau \tau_t = -cU_\zeta + U_\tau.
\]

Substituting these quantities into the original PDE yields

\[
    -cU_\zeta + U_\tau + cU_\zeta = U_\tau = -\lambda U.
\]

Therefore, \( U_\tau + \lambda U = 0 \). Solving for \( U(\zeta, \tau) \) yields

\[
    U(\zeta, \tau) = h(\zeta)e^{-\lambda \tau}
\]

where \( h \) is some unknown function of \( \zeta \). Then, converting back to the original \((x, t)\) variable system, our solution to the advection-decay equation is

\[
    u(x, t) = U(\zeta, \tau) = h(\zeta)e^{-\lambda \tau} = h(x - ct)e^{-\lambda t}.
\]

We now match this solution to the initial and boundary conditions.

Suppose \( x < ct \). Then at \( x = 0, ct > 0 \) and \( t > 0 \), so we can apply the boundary condition \( u(0, t) = g(t) \). Thus,

\[
    u(0, t) = h(-ct)e^{-\lambda t} = g(t),
\]

letting \( z = -ct \), we see

\[
    h(z) = g(-z/c)e^{-\lambda z/c} \quad \text{for} \quad z < 0.
\]

Now suppose \( x > ct \). Then at \( t = 0, x > 0 \), so we can apply our initial condition \( u(x, 0) = 0 \). Then

\[
    u(x, 0) = h(x)e^0 = h(x) = 0 \quad \text{for} \quad x > 0.
\]

We’ve now determined \( h(z) \) for positive and negative \( z \); substituting back into (1) yields the solution to this IBVP as

\[
    u(x, t) = \begin{cases} 
        g(t - x/c)e^{-\lambda x/c}, & x < ct, \\
        0, & x > ct.
    \end{cases}
\]

We leave the reader to ponder what happens when \( x = ct \).
To study the absorption of nutrients by a grasshopper we model its digestive tract by a tube of length $l$ and cross-sectional area $A$. Nutrients of concentration $n = n(x,t)$ flow through the tract at speed $c$, and they are adsorbed locally at a rate proportional to $\sqrt{n}$. What is the PDE model? If the tract is empty at $t = 0$ and then nutrients are introduced at the concentration $n_0$ at the mouth ($x = 0$) for $t > 0$, formulate an initial boundary value problem for $n = n(x,t)$. Solve this PDE model and sketch a graph of the nutrient concentration exiting the tract (at $x = l$) for $t > 0$. Physically, why is the solution $n(x,t) = 0$ for $x > ct$?

Here, nutrients flow through the tract at speed $c$ and are removed from the tract at a rate proportional to $\sqrt{n}$, so $f(x,t,n) = -k\sqrt{n}$. The PDE model for this system is then

$$n_t + cn_x = -k\sqrt{n},$$

where $k > 0$ is some constant. The digestive tract is empty at $t = 0$, so $n(x,0) = 0$ for $0 \leq x \leq l$. Nutrients are introduced at the mouth ($x = 0$) at concentration $n_0$ for $t > 0$, so $n(0,t) = n_0$ for $t > 0$. This is our IBVP for $n$.

Switching to characteristic coordinates $\xi = x - ct$ and $\tau = t$ and defining $n(x,t) = n(\xi + c\tau, \tau) = N(\xi, \tau)$ we find via the chain rule that $n_x = N_\xi$ and $n_t = -cN_\xi + N_\tau$, so that our PDE is now

$$N_\tau = -k\sqrt{N}.$$  

We must be careful solving this ODE; applying some physical insight will keep us from going astray. First note that $N = 0$ is one solution to this ODE. Second, note for $N > 0$, that $N_\tau < 0$, so that $N$ is decreasing.

We can solve the ODE by separating variables and integrating with respect to $\tau$, we have

$$2\sqrt{N} = \int \frac{dN}{\sqrt{N}} = \int -k d\tau = -k\tau + g(\xi),$$

where $g(\xi)$ is some unknown function of $\xi$. Therefore, solving for $n$ we have

$$n(x,t) = N(\xi, \tau) = \frac{1}{4}(g(\xi) - k\tau)^2$$

However, when we check this solution, we find that for $k\tau > g(\xi)$, that $N$ is increasing with $\tau$ – in fact the solution is spurious in this range, corresponding to taking $\sqrt{N} < 0$. The correct solution is

$$N(\xi, \tau) \equiv \begin{cases} \frac{1}{4}(g(\xi) - k\tau)^2 & k\tau \leq g(\xi) \\ 0 & k\tau \geq g(\xi) \end{cases}$$

which, as the reader should check, is both continuous and differentiable at $k\tau = g(\xi)$.

We now apply our initial and boundary conditions. Suppose $x > ct$; at $t = 0$, $x > 0$ and we apply the initial condition $n(x,0) = 0$. Since

$$0 = n(x,0) = N(x,0) = \frac{1}{4}(g(x))^2 \quad \text{for } x > 0$$

3: (Logan, 1.2 # 8)
we conclude that \( g(x) = 0 \) for \( x > 0 \).

Suppose \( x < ct \); at \( x = 0, t > 0 \) and we apply the boundary condition \( n(0, t) = n_0 \).

Since

\[
n_0 = n(0, t) = N(-ct, t) = \frac{1}{4} (g(-ct) - kt)^2 \quad \text{for} \quad t > 0
\]

we conclude that \( g(-ct) = kt + 2\sqrt{n_0} \) for \( t > 0 \). Combining the two we find

\[
g(z) = \begin{cases} 
2\sqrt{n_0} - kt/c & z < 0 \\
0 & z > 0
\end{cases}
\]

or

\[
n(x, t) = N(\xi, \tau) \equiv \begin{cases} 
\left( \sqrt{n_0} - \frac{kx}{2c} \right)^2 & x \leq 2c\sqrt{n_0}/k \quad \text{and} \quad x < ct, \\
0 & x > 2c\sqrt{n_0}/k \quad \text{or} \quad x > ct.
\end{cases}
\]

(2)

Note that if the parameters \( n_0, k, c, \) and the length of the digestive tract \( l \) satisfy \( l \leq 2c\sqrt{n_0}/k \), then \( x < 2c\sqrt{n_0}/k \) for all \( 0 \leq x \leq l \), and the solution formula simplifies to

\[
n(x, t) = N(\xi, \tau) \equiv \begin{cases} 
\left( \sqrt{n_0} - \frac{kx}{2c} \right)^2 & x < ct, \\
0 & x > ct.
\end{cases}
\]

but if the length of the digestive track is longer than \( 2c\sqrt{n_0}/k \), we have to use the first formula (2) for the solution.

The solution for the first case in which \( l > 2c\sqrt{n_0}/k \) is graphed here:

![Graph of n(x, t) for first case](image1)

Figure 0.1: Plots of \( n(x, t) \) at \( t = 1, 2, 3, 4, 5, 6 \) for parameters \( n_0 = 10, c = 1, k = 1, \) and \( l = 10 \).

and the solution for the other case \( l \leq 2c\sqrt{n_0}/k \) is graphed here:

![Graph of n(x, t) for second case](image2)

Figure 0.2: Plots of \( n(x, t) \) at \( t = 1, 2, 3, 4, 5, 6, 7 \) for parameters \( n_0 = 10, c = 1, k = 1, \) and \( l = 5 \).
We also see that $n(l,t)$ is piecewise constant. When $l > 2c\sqrt{n_0}/k$ all the nutrients have been digested before it reaches the end of the digestive track and $n(l,t) = 0$. If $l < 2c\sqrt{n_0}/k$ the solution at $x = l$ is

$$n(l,t) \equiv \begin{cases} \left( \frac{\sqrt{n_0}}{k} - \frac{lt}{c} \right)^2 & l/c < t, \\ 0 & l/c > t, \end{cases}$$

which we graph here:

![Graph](image)

Figure 0.3: Plot of $n(l,t)$ for parameters $n_0 = 10$, $c = 1$, $k = 1$, and $l = 5$.

Basically, the solution becomes non-zero at time $t = l/c$ when the nutrients ingested at $(x,t) = (0,0)$ first reach the end of the track. In fact the solution vanishes for $x < ct$ for the same reason – the nutrients which enter the digestive track at $x = 0$ for $t > 0$ haven’t had time to reach the point $x$ yet.

\[\blacksquare\]
4: (Logan, 1.3 # 1) Heat flows longitudinally through a metal bar of length 10 centimeters, and the temperature \( u = u(x, t) \) satisfies the diffusion equation \( u_t = ku_{xx} \), where \( k = 0.02 \) square centimeters per second. Suppose the temperatures at some fixed time \( T \) at \( x = 4, 6, 8 \) cm are 58, 64 and 72 degrees, respectively. Estimate \( u_{xx}(6, T) \) using a difference approximation. Will the temperature at \( x = 6 \) increase or decrease in the next instant of time? Estimate the temperature at \( x = 6 \) at \( T + 0.5 \) seconds. (Recall from calculus that

\[
 f''(a) \approx \frac{f(a - h) - 2f(a) + f(a + h)}{h^2}
\]

where \( h \) is a small increment.)

Using the difference approximation suggested we have

\[
 u_t(6, T) \approx k \frac{u(4, T) - 2u(6, T) + u(8, T)}{2^2} = 0.01 \text{ deg/sec}
\]

The temperature at \( x = 6 \) will increase. An estimate of the temperature at \( x = 6 \) at time \( t = T + 0.5 \) is obtained using Taylor series in \( t \), and the above approximation for \( u_t(6, T) \):

\[
 u(6, T + 0.5) \approx u(6, T) + u_t(6, T)(0.5) = 64 + .01 \cdot .5 = 64.005.
\]
5: (Logan, 1.3 # 2) Let $u = u(x, t)$ satisfy the PDE model

\[ u_t = k u_{xx}, \quad 0 < x < \ell, \quad t > 0, \]

\[ u(0, t) = u(\ell, t) = 0, \quad t > 0, \]

\[ u(x, 0) = u_0(x), \quad 0 \leq x \leq \ell. \]

Show that

\[ \int_0^\ell u(x, t)^2 \, dx \leq \int_0^\ell u_0(x)^2 \, dx, \quad t \geq 0. \]

Hint: Let $E(t) = \int_0^\ell u(x, t)^2 \, dx$ and show that $E'(t) \leq 0$. What can be said about $u(x, t)$ if $u_0(x) = 0$?

From the definition of $E(t)$, we see that

\[ E'(t) = \int_0^\ell \frac{\partial}{\partial t} u(x, t)^2 \, dx = \int_0^\ell u u_t \, dx = 2k \int_0^\ell u u_{xx} \, dx. \]

The last step used the PDE $u_t = k u_{xx}$. Integrating the last expression by parts we obtain

\[ E'(t) = 2k \left\{ uu_x |_0^\ell - \int_0^\ell (u_x)^2 \, dx \right\} \]

Using the boundary conditions, the first term in braces is zero, so

\[ E'(t) = -2k \int_0^\ell (u_x)^2 \, dx. \]

Since the integrand is non-negative, so is the integral. Hence the right hand side is non-positive, $E'(t) \leq 0$, and $E(t) \leq E(0)$ which is the statement to be proved. If $u_0(x) \equiv 0$, then $E(0) = 0$, and $E(t) = 0$. This implies that $u(x, t) = 0$ for all $t$. ■
6: (Logan, 1.3 # 3) Show that the problem

\[ u_t = ku_{xx}, \quad 0 < x < \ell, \quad t > 0, \]
\[ u(0, t) = g(t), \quad u(\ell, t) = h(t), \quad t > 0, \]
\[ u(x, 0) = u_0(x), \quad 0 \leq x \leq \ell. \]

with nonhomogeneous boundary conditions can be transformed into a problem with homogeneous boundary conditions. Hint: Introduce a new dependent variable \( w \) by subtracting from \( u \) a linear function of \( x \) that satisfies the boundary conditions at any fixed \( t \). In the transformed problem for \( w \), notice that the PDE picks up a source term, so you are really trading boundary conditions for source terms.

Let \( s(x, t) = h(t) \frac{x}{\ell} + (1 - \frac{x}{\ell})g(t) \). Then, \( s(0, t) = g(t) \) and \( s(\ell, t) = h(t) \). Note that \( s_t = h'(t) \frac{x}{\ell} + (1 - \frac{x}{\ell})g'(t) \) and \( s_{xx} = 0 \). Let \( w(x, t) = u(x, t) - s(x, t) \). Then \( w_t = u_t - s_t = u_t - h'(t) \frac{x}{\ell} + (1 - \frac{x}{\ell})g'(t) \) and \( w_{xx} = u_{xx} \). So \( u_t = ku_{xx} \) if and only if \( w_t = kw_{xx} - s_t = kw_{xx} - h'(t) \frac{x}{\ell} + (1 - \frac{x}{\ell})g'(t) \). Also \( w(0, t) = u(0, t) - s(0, t) = 0 \) and \( w(\ell, t) = u(\ell, t) - s(\ell, t) = 0 \). So \( w(x, t) \) satisfies an nonhomogeneous diffusion equation, with homogeneous boundary conditions. □
Considering all cases, find the form of the steady-state solutions to the advection-diffusion equation

\[ u_t = Du_{xx} - cu_x \]

and the advection-diffusion-growth equation

\[ u_t = Du_{xx} - cu_x + ru. \]

A steady-state solution of the advection-diffusion-growth equation satisfies

\[ 0 = Du''(x) - cu'(x) + ru \]

This is a linear second order ODE with constant coefficients; we seek solutions of the form \( u(x) = \exp(ax) \), and substituting into the ODE we find that \( a \) must satisfy \( Da^2 - ca + r = 0 \), so \( a_\pm = \frac{c \pm \sqrt{c^2 - 4Dr}}{2D} \). There are three cases, depending on whether \( a_\pm \) are real and distinct, real and equal, or complex.

\[
\begin{align*}
    u(x) &= A \exp(a_+ x) + B \exp(a_- x) & \text{if } a_\pm \in \mathbb{R} \text{ and } a_+ \neq a_- \\
    &= A x \exp(ax) + B \exp(ax) & \text{if } a_\pm \in \mathbb{R} \text{ and } a_+ = a_- \\
    &= A \exp(ax) \cos(bx) + B \exp(ax) \sin(bx) & \text{if } a_\pm = a \pm ib.
\end{align*}
\]

The advection-diffusion equation is a special case of this with \( r = 0 \) from which we see that \( a_+ = c/D \) and \( a_- = 0 \). So in this case the solution has the form

\[ u(x) = A \exp(cx/D) + B. \]
8: Let erf(x) be the **error function** defined by

\[
erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds.
\]

a) Graph erf(x) for \(-10 < x < 10\). What are \(\text{erf}(0)\), \(\text{erf}(\infty)\) and \(\text{erf}(-\infty)\)?

b) Show

\[
u(x, t) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{x}{2\sqrt{t}} \right) \right]
\]

solves the heat equation \(u_t = u_{xx}\). You may use MAPLE to do the algebra.

c) Plot \(u(x, t)\) at time \(t = 0.125, 0.25, 0.5, 1, 2\) on the same axes. What is the initial temperature distribution?

d) Plot the solution surface \(u(x, t)\) for \(-6 < x < 6\) and \(0 \leq t < 2\).

e) Show \(u_x(x, t)\) also solves the heat equation. Find and plot this solution at some sample times.

See the Maple worksheet – hw2.mws.
Consider the problem:

\[ U_t + U_x = e^{-x^2} \quad t > 0, \quad -\infty < x < \infty \]

\[ U(x, 0) = 0 \]

a) Use the method of characteristics to find the solution for the problem. Remember that along a characteristic you can write \( x \) as a function of \( t \). Your answer will contain an integral or an error function.

b) Use MAPLE to plot \( U(x, t) \) at \( t = 0, 1, 2, 4, 8 \) for \( -10 < x < 10 \).

c) Plot the solution surface \( U(x, t) \) for \( 0 < t < 10 \) and \( -10 < x < 10 \).

d) Give a physical interpretation of the PDE and the solution. What happens as \( t \to \infty \)?

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a) Let \( \zeta = x - t \) and \( \tau = t \). Then the first partials of \( u(x, t) = u(\zeta + \tau, \tau) = U(\zeta, \tau) \) are

\[ u_x = U_{\zeta x} x + U_{\tau x} x = U_{\zeta} \quad \text{and} \quad u_t = U_{\zeta t} x + U_{\tau t} x = -U_{\zeta} + U_{\tau}. \]

Substituting these quantities into the original PDE yields

\[ U_{\tau} = e^{-x^2} = e^{(\zeta + \tau)^2}. \]

Solving for \( U(\zeta, \tau) \) yields

\[ U(\zeta, \tau) = \int_{0}^{\tau} e^{(\zeta + \tau')^2} d\tau' + A(\zeta) \]

where \( A \) is some unknown function of \( \zeta \). Then, converting back to the original \( (x, t) \) variable system, our solution is

\[ u(x, t) = U(\zeta, \tau) = \int_{0}^{t} e^{(x-t+\tau')^2} d\tau' + A(x-t) = \int_{x-t}^{x} e^{-s^2} ds + A(x-t) \]

where we have made the change of variables in the integral \( s = x - t + \tau' \).

We now apply the initial condition,

\[ u(x, 0) = A(x) = 0 \]

which leads to the solution

\[ u(x, t) = \int_{x-t}^{x} e^{-s^2} ds \]

b)-d) The solution is graphed in the accompanying MAPLE worksheet hw2.mws.
10: Let $u(x, t)$ be defined by

$$u(x, t) = \frac{1}{2} \int_{x-t}^{x+t} e^{-s^2} \, ds.$$ 

a) Show $u(x, t)$ solves the wave equation $u_{tt} = u_{xx}$. What is the initial displacement $(u(x, 0))$ and velocity $(u_t(x, 0))$? You may use MAPLE to do the algebra.

b) Plot $u(x, t)$ at time $t = 0, 1, 2, 4$ on the same axes.

c) Plot the solution surface $u(x, t)$ for $-6 < x < 6$ and $0 \leq t < 4$.

See the Maple worksheet hw2.mws.