

Mathematical Biology 5120-S09: Problem Set 3

Due Friday March 13.

After trying the problems yourself, I encourage you to talk to each other about how to solve them. If you cannot work a problem out, don't leave it — be sure to come and ask Karin or me for help. Be sure to show all details on each problem for full credit.

1. Suggest a set of partial differential equations to describe the following processes:
 - a) A predator-prey system in which both species move randomly in a one-dimensional setting.
 - b) A predator-prey system in which the predator moves towards higher prey densities and the prey moves towards lower predator densities.
 - c) A pair of reacting and diffusing chemicals such that species 1 activates the formation of both substances and species 2 inhibits the formation of both substances. Design your model so that it has a homogeneous (independent of location) steady state that is stable.
 - d) A population of cells that secrete a chemical substance. Assume that the cells orient and move chemotactically in gradients of this chemical and also move randomly. Further assume that the chemical diffuses and is gradually broken down at some rate.
2. What kinds of processes might be described by the following equations?
 - a) $\frac{\partial f}{\partial t} = \nabla \cdot (f\mathbf{v}) - \mu f$ f = fish density
 - b) $\frac{\partial p_i}{\partial t} = D_i \Delta p_i + r p_i (1 - p_i - \beta_{ij} p_j)$ for $(i, j = 1, 2)$. Here, $\Delta q = q_{xx} + q_{yy}$ for any function q .
3. Consider the Keller-Segel chemotaxis equations:

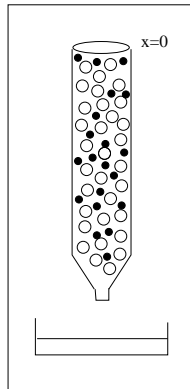
$$\begin{aligned}\frac{\partial a}{\partial t} &= -\frac{\partial}{\partial x} \left\{ -\mu \frac{\partial a}{\partial x} + \chi^a \frac{\partial c}{\partial x} \right\} \\ \frac{\partial c}{\partial t} &= D \frac{\partial^2 c}{\partial x^2} + f a - k c\end{aligned}$$

Explain how the equations would be modified to incorporate the following further assumptions:

- a) The chemotactic sensitivity increases linearly with the chemical concentration.
- b) The cells' random motions decrease as the cell density increases.
- c) The cell population increases logistically with carrying capacity proportional to the concentration of the chemical.

4. Here we consider a model of a separation column (see figure) for removing protein from a solution. Let

- $u_1(x, t)$ equal the concentration of *free* protein at distance x along the column.
- $u_2(x, t)$ equal the concentration of protein bound (reversibly) to stationary beads in the column at location x .
- B equal the total number of binding sites for protein on beads per unit distance. B is constant.
- P_0 equal the protein concentration in the stock solution that is fed into the column at $x = 0$.



The column is idealized as half-infinite, and described by locations x for $x \geq 0$. We model the operation of the column by the following equations:

$$\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial x} \left(D \frac{\partial u_1}{\partial x} - v u_1 \right) - (k_1(B - u_2)u_1 - k_2 u_2) \quad (1)$$

$$\frac{\partial u_2}{\partial t} = (k_1(B - u_2)u_1 - k_2 u_2) \quad (2)$$

- Interpret the terms in these equations and define the positive constants k_1 , k_2 , D , and v . What boundary conditions apply at $x = 0$, and what conditions on $u_1(x, t)$ and $u_2(x, t)$ should hold as $x \rightarrow \infty$?
- Determine whether the system has (bounded) nonuniform steady-state solutions. If so, find them.
- Determine whether the system has spatially-uniform steady-state solutions. If so, find them.
- Look for traveling wave solutions of the equations. Do bounded traveling wave solutions exist? If so, what restrictions are there on the wave speed S ? (Hint: Write the differential equations one gets by assuming that the unknowns $u_1(x, t) = U_1(x - St)$ and $u_2(x, t) = U_2(x - St)$. While there will initially be first and second order derivatives of U_1 and first order derivatives of U_2 , show how to manipulate the equations so that you can integrate once, and derive a pair of differential equations for U_1' and U_2' . Do phase plane analysis on this pair of equations.)