Math 5120 – Some notes on Random Walking Particles

Consider a particle which is located at $x = 0$ at time $t = 0$ and which in each timestep of length Δt takes a step of size Δx to the right with probablility 1/2 or takes a step of the same size to the left with probability 1/2. Let $x(n)$ denote the location of the particle after n steps. We derived in class an expression for the probability that $x(n) = j\Delta x$ for $j = -n, -n+1, \ldots, n-1, n$. It is

$$W(j,n) = \mathrm{Pr}\{\mathrm{x(n) = j\Delta x}\} = \frac{\mathrm{n!}}{\left(\frac{\mathrm{n+j}}{2}\right)!\left(\frac{\mathrm{n-j}}{2}\right)!}\left(\frac{1}{2}\right)^{\mathrm{n}}.$$

This is an exact formula valid for any n. Since we're primarily interested in very large n and values of j much less than n, we seek an approximation to this exact formula that may give more insight in that case.

To find this approximate formula, we make use of three approximations, the first of which is

$$\ln(m!) \approx \left(m + \frac{1}{2}\right)\ln m - m + \frac{1}{2}\ln(2\pi). \tag{1}$$

This is called Stirling's formula and the expression on the right side gives a very good approximation to the left side when m is very large. We will use this formula three times with $m = n$, $m = \frac{1}{2}(n+j)$ and $m = \frac{1}{2}(n-j)$. The other approximations are derived from Taylor series expansions and are

$$\ln\left(1 + \frac{j}{n}\right) = \frac{j}{n} - \frac{1}{2}\left(\frac{j}{n}\right)^2 + \ldots \tag{2}$$

$$\ln\left(1 - \frac{j}{n}\right) = -\frac{j}{n} - \frac{1}{2}\left(\frac{j}{n}\right)^2 - \ldots. \tag{3}$$

Now from the formula for $W(j,n)$ at the top, we see that

$$\ln W(j,n) = \ln(n!) - \ln\left\{\left(\frac{n+j}{2}\right)!\right\} - \ln\left\{\left(\frac{n-j}{2}\right)!\right\} + n\ln\frac{1}{2}. \tag{4}$$

We substitute into this formula the following approximations from Stirling's formula,

$$\ln(n!) \approx \left(n + \frac{1}{2}\right)\ln n - n + \frac{1}{2}\ln(2\pi).$$

$$\begin{aligned}\ln\left\{\left(\frac{n+j}{2}\right)!\right\} &\approx \frac{1}{2}(n+j+1)\ln\left\{\left(\frac{n+j}{2}\right)\right\} - \frac{1}{2}(n+j) + \frac{1}{2}\ln(2\pi) \\ &= \frac{1}{2}(n+j+1)\left[\ln\frac{1}{2} + \ln n + \ln\left(1 + \frac{j}{n}\right)\right] - \frac{1}{2}(n+j) + \frac{1}{2}\ln(2\pi)\end{aligned}$$

$$
\begin{aligned}
\ln\left\{\left(\frac{n+j}{2}\right)!\right\} &\approx \frac{1}{2}(n-j+1)\ln\left\{\left(\frac{n-j}{2}\right)\right\} - \frac{1}{2}(n-j) + \frac{1}{2}\ln(2\pi) \\
&= \frac{1}{2}(n+j+1)\left[\ln\frac{1}{2} + \ln n + \ln\left(1+\frac{j}{n}\right)\right] - \frac{1}{2}(n+j) + \frac{1}{2}\ln(2\pi)
\end{aligned}
$$

We obtain the approximation

$$
\begin{aligned}
\ln W(j,n) &\approx \left(n+\frac{1}{2}\right)\ln n - n + \frac{1}{2}\ln(2\pi) \\
&- \frac{1}{2}(n+j+1+n-j+1)\ln\frac{1}{2} - \frac{1}{2}(n+j+1+n-j+1)\ln n \\
&- \frac{1}{2}(n+1)\left(\ln\left(1+\frac{j}{n}\right) + \ln\left(1-\frac{j}{n}\right)\right) \\
&- \frac{1}{2}j\left(\ln\left(1+\frac{j}{n}\right) - \ln\left(1-\frac{j}{n}\right)\right) \\
&+ \frac{1}{2}(n+j+n-j) - \ln(2\pi) + n\ln\frac{1}{2}.
\end{aligned}
$$

Simplifying this expression, we find that

$$
\begin{aligned}
\ln W(j,n) &\approx \left(n+\frac{1}{2}-(n+1)\right)\ln n + (-n+n) - \frac{1}{2}\ln(2\pi) - (n+1)\ln\frac{1}{2} \\
&- \frac{1}{2}(n+1)\left(-\left(\frac{j}{n}\right)^2\right) - \frac{j}{2}\frac{2j}{n} + n\ln\frac{1}{2} \\
&= -\frac{1}{2}\ln n - \frac{1}{2}\ln(2\pi) - \ln\frac{1}{2} + \frac{1}{2}\frac{j^2}{n} + \frac{1}{2}\left(\frac{j}{n}\right)^2 - \frac{j^2}{n} \\
&\approx -\frac{1}{2}\ln n - \frac{1}{2}\ln(2\pi) - \ln\frac{1}{2} - \frac{1}{2}\frac{j^2}{n} \\
&= -\frac{1}{2}\ln n - \frac{1}{2}\ln(2\pi) + \ln 2 - \frac{1}{2}\frac{j^2}{n}
\end{aligned}
$$

In going from the third to the fourth line in this expression we dropped the $\left(\frac{j}{n}\right)^2$ term. Since $j << n$, the square of $\frac{j}{n}$ is much smaller than the other terms in the expression. So we have found that

$$\ln W(j,n) \approx -\frac{1}{2}\ln n - \frac{1}{2}\ln(2\pi) + \ln 2 - \frac{1}{2}\frac{j^2}{n}$$

and exponentiating this expression, we find that

$$W(j,n) \approx \sqrt{\frac{2}{\pi n}} e^{-\frac{1}{2}\frac{j^2}{n}}.$$