STRONG EXISTENCE AND UNIQUENESS OF THE TILT-INDEXED BUSEMANN PROCESS IN THE PLANAR CORNER GROWTH MODEL

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Abstract. We show that the Busemann process indexed by tilts in the super-differential of the limit shape exists and is unique in the strong sense in the i.i.d. planar corner growth model. This means that every probability space that supports the field of i.i.d. weights supports a copy of the process and any two realizations of the process are equal almost surely.

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1. Introduction

The *corner growth model*, or *directed last-passage percolation* (LPP), is a cornerstone model of random growth in the Kardar-Parisi-Zhang universality class which can profitably be thought of as a directed version of first-passage percolation (FPP), a prototypical example of a random (psuedo-)metric on the lattice.

FPP is constructed by assigning non-negative weights to the edges of the lattice and then defining the distance between sites to be the infimum over the weights collected along all self-avoiding paths between those sites. As usual, the optimizing paths are known as geodesics. In FPP, substantial recent attention [1, 8, 9, 19] has focused on the structure of semi-infinite geodesics, which are also known as geodesic rays.

The planar corner growth model has a similar structure with a few differences. Edge weights are replaced by vertex weights and the set of admissible paths is restricted to those which either go up or right in each step. This path restriction allows the vertex weights to take both negative and positive values. By convention, we also replace the minimum over paths with a maximum over paths. This model arises naturally in the context of tandem queueing, for example, where the passage time satisfies the same recursion as the time at which labeled customers are served at labeled service stations. Once again, we have optimizing paths, which we call geodesics by way of analogy to first-passage percolation. Similarly, significant recent attention has been focused on the structure of semi-infinite geodesics.

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In the context of metric geometry, a frequently used tool to understand the structure of geodesic rays are objects known as Busemann functions, originally introduced in [4]. In lattice growth models, a slight generalization of this notion, which we call *generalized Busemann functions*, can be viewed as the natural coupling of all ergodic and translation invariant stationary distributions of the model. See the introductions to [18, 22, 23] for a discussion of these connections and a more thorough review of the related literature.

The last decade has seen several constructions of generalized Busemann functions in various models. In both first- [8] and last-passage percolation [14, 28], as well as related models like directed polymers [21] and generalizations like random walks in random potentials [17], most of this work shows what is known as *weak existence* in the stochastic analysis literature. This means that existence statements take the form: "there exists a probability space on which generalized Busemann functions are defined."

Throughout the study of random growth models, a quantity known as the *limit shape* or *free energy* plays a central role. In general, the existence results mentioned above show that for each element of the sub-differential (super-differential in LPP) of the shape, a generalized Busemann function field exists. In planar last-passage percolation, these can be glued together by monotonicity to form a *Busemann process* indexed by the *tilts* in the sub-gradient. It is known [23] that this Busemann process encodes geometric properties of geodesics into its analytic behavior.

A line of work originating with Newman [30] shows that Busemann functions are almost sure attractors in an appropriate sense, which proves both *strong existence* (meaning that every probability space supporting the weights supports the Busemann process) and *strong uniqueness* (any two generalized Busemann functions associated to the same deterministic tilt are equal almost surely). These methods require control of the curvature and differentiability properties of the limit shape. Proving these types of estimates is a long-standing and difficult open problem outside of exactly solvable models. A collection of results in this vein implying strong existence and uniqueness in the exactly solvable Exponential LPP originally appear in [5, 7, 12].

About a decade ago, Ahlberg and Hoffman introduced a theory of "random coalescing geodesics" in FPP in [1]. One of the contributions of that paper is a proof of strong existence of the Busemann functions without unproven hypotheses on the limit shape: namely, that what we call generalized Busemann functions previously constructed in the weak sense in planar first-passage percolation previously constructed by Damron and Hanson in [8] can be pulled back to the original probability space. In [1], this was phrased in terms of the associated covariant geodesics but we will see in the sequel, the two perspectives are equivalent. They also introduce a labeling scheme that allows them to prove a similar strong uniqueness statement to the one we prove below (and much more besides). There are some technical differences between our treatment and theirs, but our methods are largely inspired by those of [1]. We do not use the geodesic labeling scheme introduced in [1], preferring to work directly with the tilt indexing.

As discussed in [22, 23] among other places, one can essentially think of fields of Busemann functions for percolation models as eternal solutions of a (discrete) stochastic partial differential equation. From that stochastic analytic point of view, whether or not strong solutions exist and if they satisfy strong uniqueness are standard topics of intrinsic interest. We refer the reader to [25, 26] for a higher-level perspective on strong existence and uniqueness of stochastic equations. Aside from this intrinsic interest, our purpose in writing this paper, and showing strong existence in particular, is to provide the right setting for future work. Working on an extended probability space is unnatural and introduces technical issues. For example, weight modification arguments that control infinite geodesics generated by Busemann functions become delicate because the extra noise from the extension is not independent of the weight field. This extended space is also in general only stationary, rather than ergodic, under shifts. This complicates the application of standard results in the area. See, for example, Remark A.5 in [23], which discusses working around this technical issue.

Strong uniqueness is of interest for related reasons. Without control on regularity of the limit shape, the only uniqueness results that existed concerned the finite-dimensional marginal [6, 11, 31] distributions of generalized Busemann functions. We include [11] here because the proof of Theorem 5.6 in that work can be made general under the same hypotheses as [6], though it is not phrased as such. These results were not sufficiently strong to show that any two such processes would have to be equal, leaving open the possibility of multiple Busemann processes existing. Pre-existing distributional uniqueness results also require extra hypotheses: applying the results of [6, 11] would require the weights to be unbounded from above, while applying [31] would require the weights to be bounded from below.

Our results, like those of [1], are limited to the planar setting. The reason why both strong existence and uniqueness are accessible in a planar setting but remain open in non-planar settings is geodesic coalescence and a path ordering coming from the nearest-neighbor paths and planarity. In both first- and last-passage percolation, this coalescence comes from the Licea-Newman argument [27].

We rely on a seminal result of Martin [29] concerning the curvature of the limit shape of the i.i.d. corner growth model near the boundary. That result requires i.i.d. weights, while [1] are able to work somewhat more generally in their more abstract Condition A2. We have made no attempt to prove Martin's result in a setting comparable to that condition in order to generalize beyond the i.i.d. setting.

As a consequence of strong existence, we also obtain a significant extension of the previously-known ergodicity properties of the Busemann process in the i.i.d. corner growth model. Prior to this work, it was known that Busemann functions which correspond to extremal tilt vectors in the subdifferential were separately ergodic under the e_1 and e_2 shifts [6]. Because the Busemann process can be realized as a function of an i.i.d. field, this can now be upgraded to strong mixing under all non-trivial lattice shifts.

1.1. **Outline.** In Section 2, we state the assumptions of the model, define terms, recall basic facts from the previous literature, and then state our main results. The proofs of the two main results concerning strong existence and uniqueness appear in Section 3. Section 4 then shows an equivalence between two ways of setting up the problems we consider. Appendix A collects some technical lemmas. Finally, Appendix B proves that every non-trivial geodesic generates a finite Busemann function.

1.2. **Notation and conventions.** $\mathbb{Z}$ are the integers, $\mathbb{Z}_+$ the nonnegative and $\mathbb{Z}_-$ the nonpositive integers. $\mathbb{N} = \{1, 2, 3, \dots\}$. $\mathbb{Q}$ are the rational numbers, and $\mathbb{R}$ are the real numbers. $\mathbb{Z}_{>a}$ are integers $> a$. Similarly for $\mathbb{Z}_{<a}$, $\mathbb{Z}_{\geqslant a}$, and $\mathbb{Z}_{\leqslant a}$. $[\![a, b]\!] = [a, b] \cap \mathbb{Z}$.

When a symbol is needed, $\mathcal{L}$ denotes the Lebesgue measure on $[0, 1]$. We write integrals with respect to this measure using the standard notation $\int_0^1 f(s)\,ds$.

The end of a non-proof structured environment ends with a $\triangle$ to help delineate these from the rest of the text.

2. Setting and main results

2.1. **Probability space.** $(\widehat{\Omega}, \widehat{\mathfrak{S}}, \widehat{\mathbb{P}})$ denotes a generic Polish probability space equipped with a group of continuous bijections $\widehat{T} = \{\widehat{T}_x : x \in \mathbb{Z}^2\}$ from $\widehat{\Omega}$ onto itself. In particular, $\widehat{T}_0$ is the identity map and $\widehat{T}_x \widehat{T}_y = \widehat{T}_{x+y}$ for all $x, y \in \mathbb{Z}^2$. Let $\widehat{\mathbb{E}}$ denote expectation relative to $\widehat{\mathbb{P}}$. Suppose

$$\widehat{\mathbb{P}} \text{ is invariant under } \widehat{T}_x \text{ for each } x \in \mathbb{Z}^2.$$

Let $\widehat{\mathcal{I}}$ be the σ-algebra of events that are invariant under the group of shifts $\widehat{T}$. A generic element of $\widehat{\Omega}$ is denoted by $\widehat{\omega}$. As usual, the $\widehat{\omega}$ can be dropped from the arguments of random variables.

We are given random variables $\omega_x : \widehat\Omega \to \mathbb{R}$, $x \in \mathbb{Z}^2$, which satisfy the shift-covariance property

$$\omega_x(\widehat T_z\widehat\omega) = \omega_{x+z}(\widehat\omega) \tag{2.1}$$

almost surely under $\widehat{\mathbb{P}}$. We assume also that

$$\{\omega_x : x \in \mathbb{Z}^2\} \text{ are i.i.d. under } \widehat{\mathbb{P}} \text{ and satisfy } \widehat{\mathbb{E}}[|\omega_0|^{2+\epsilon}] < \infty \text{ for some } \epsilon > 0. \tag{2.2}$$

To dispense with trivialities we assume

$$\widehat{\mathbb{E}}[\omega_0^2] > \widehat{\mathbb{E}}[\omega_0]^2. \tag{2.3}$$

Let $\mathfrak{S}$ denote the σ-algebra on $\widehat\Omega$ generated by $\omega = \{\omega_x : x \in \mathbb{Z}^2\}$. For $x \in \mathbb{Z}^2$ let $\mathfrak{S}_x^+ \subset \mathfrak{S}$ denote the σ-algebra on $\widehat\Omega$ generated by $\{\omega_y : y - x \in \mathbb{Z}_+^2\}$.

Ω denotes the product space $\mathbb{R}^{\mathbb{Z}^2}$ and $\mathcal{F} = \mathcal{B}(\mathbb{R}^{\mathbb{Z}^2})$ its Borel σ-algebra. We slightly abuse notation and use the symbol $\omega = (\omega_x)_{x\in\mathbb{Z}^2}$ to denote both the $\widehat\Omega \to \Omega$ mapping $\widehat\omega \mapsto \omega(\widehat\omega) = (\omega_x(\widehat\omega))_{x\in\mathbb{Z}^2}$ and a generic element of Ω. The shifts $T = \{T_x : x \in \mathbb{Z}^2\}$ on Ω are defined by

$$(T_z\omega)_x = \omega_{x+z}. \tag{2.4}$$

The shifts on the two spaces are related via the following identity, valid for $x, z \in \mathbb{Z}^2$ and $\widehat{\mathbb{P}}$-almost all $\widehat\omega \in \widehat\Omega$:

$$(T_z[\omega(\widehat\omega)])_x \overset{(2.4)}{=} [\omega(\widehat\omega)]_{z+x} \overset{(\text{def.})}{=} \omega_{z+x}(\widehat\omega) \overset{(2.1)}{=} \omega_x(\widehat T_z\widehat\omega). \tag{2.5}$$

For $x \in \mathbb{Z}^2$ let $\mathcal{F}_x^+ \subset \mathcal{F}$ denote the σ-algebra on Ω generated by $\{\omega_y : y - x \in \mathbb{Z}_+^2\}$. We denote by $\mathbb{P}(\cdot) = \widehat{\mathbb{P}}\{\widehat\omega : \omega(\widehat\omega) \in \cdot\}$ the probability measure induced by pushing forward $\widehat{\mathbb{P}}$ by the map $\widehat\omega \mapsto \omega(\widehat\omega)$. By (2.2), $\mathbb{P}$ is an i.i.d. product measure on Ω.

2.2. **Path spaces and order relations.** A path $\pi_{m:n} = (\pi_i)_{i=m}^n$ of points in $\mathbb{Z}^2$ is called *up-right* if it only takes steps in $\{e_1, e_2\}$, meaning $\pi_{i+1} - \pi_i \in \{e_1, e_2\}$ for all integers $i \in [\![m, n-1]\!]$. This definition extends naturally to semi-infinite and bi-infinite up-right paths. Paths are indexed by antidiagonal levels, that is, $\pi_i \cdot (e_1 + e_2) = i$ for all i in the relevant range.

The symbol o denotes the index of the point of origin of a path: γ_o is the first point of γ, regardless of the actual index. When $u \leqslant v$ are points on a path γ, $\gamma_{u:v}$ is the segment of γ from u to v, including the endpoints u and v. Then $\gamma_{\gamma_m:\gamma_n}$ is abbreviated by $\gamma_{m:n}$. If m is an index and $u \in \gamma$, then mixtures $\gamma_{m:u}$ and $\gamma_{u:m}$ are also entirely unambiguous.

For $\ell \in \mathbb{Z}$ and $u \in \mathbb{Z}^2$ with $u \cdot (e_1 + e_2) = \ell$, $\mathbb{X}_u$ denotes the space of semi-infinite up-right paths on $\mathbb{Z}^2$ that start at u. This path space is compact in the product-discrete topology, which can be metrized by $d(\gamma, \pi) = \sum_{i=\ell}^\infty 2^{-(i-\ell+1)}\mathbb{1}_{\{\gamma_i \neq \pi_i\}}$. $\mathbb{X} = \bigcup_{u\in\mathbb{Z}^2} \mathbb{X}_u$ is the space of all up-right lattice paths.

We use $\preceq$ to denote southeast type partial order relations on various spaces. Our convention is then that $b \succeq a$ means $a \preceq b$, $a \precneqq b$ means $a \preceq b$ but $a \neq b$, and $b \succneqq a$ means $a \precneqq b$.

For $h, h' \in \mathbb{R}^2$, $h \preceq h'$ means $h \cdot e_1 \leqslant h' \cdot e_1$ and $h \cdot e_2 \geqslant h' \cdot e_2$. In particular, the simplex $[e_2, e_1] = \{(t, 1-t) \in \mathbb{R}^2 : 0 \leqslant t \leqslant 1\}$ of direction vectors is ordered so that $\zeta \preceq \eta$ if and only if $\zeta \cdot e_1 \leqslant \eta \cdot e_1$.

An asymptotic version of southeast ordering is defined for semi-infinite up-right paths $\pi, \gamma \in \mathbb{X}$ as follows: $\gamma_{k:\infty} \preceq_a \pi_{\ell:\infty}$ means that $\pi_{\ell:\infty}$ is *eventually* weakly to the right of $\gamma_{k:\infty}$. Equivalently, there exists an integer $m \geqslant k \vee \ell$ such that $\gamma_n \preceq \pi_n$ for all integers $n \geqslant m$. If there exists an integer $m \geqslant k \vee \ell$ such that $\gamma_n = \pi_n$ for all $n \geqslant m$, then these paths *coalesce*, abbreviated by $\gamma_{k:\infty} \curlywedge \pi_{\ell:\infty}$. Equivalently, $\gamma_{k:\infty} \preceq_a \pi_{\ell:\infty}$ and $\gamma_{k:\infty} \succeq_a \pi_{\ell:\infty}$.

For $B, B' \in \mathbb{R}^{\mathbb{Z}^2}$, $B \preceq B'$ means $B(x, x+e_1) \geqslant B'(x, x+e_1)$ and $B(x, x+e_2) \leqslant B'(x, x+e_2)$ for all $x \in \mathbb{Z}^2$.

These partial order relations turn out to be total order relations and will happen to be all consistent with each other when we use them on the spaces we study (the super-differential of the shape function, semi-infinite geodesic trees, and recovering cocycles).

2.3. **Last-passage percolation.** Given weights $\omega = (\omega_x)_{x\in\mathbb{Z}^2} \in \mathbb{R}^{\mathbb{Z}^2}$ and two distinct points $u \leqslant v$ (coordinatewise) in $\mathbb{Z}^2$, let

$$L_{u,v} = \max\Big\{ \sum_{x\in\pi\setminus\{v\}} \omega_x : \pi \text{ is up-right from } u \text{ to } v \Big\}.$$

A maximizing path in the above is called a *point-to-point geodesic* (from u to v). The *rightmost geodesic* π from u to v is the unique geodesic between the two points that is to the right of any other geodesic from u to v: if γ is another geodesic, then $\gamma \preceq \pi$.

A *semi-infinite geodesic*, starting at $u \in \mathbb{Z}^2$ with $m = u\cdot(e_1+e_2)$, is a path π with $\pi_m = u$, $\pi_{i+1} - \pi_i \in \{e_1, e_2\}$ for all integers $i \geqslant m$, and such that $\pi_{k:\ell}$ is a geodesic from π_k to π_ℓ, for any pair of integers $\ell > k \geqslant m$. The semi-infinite geodesic is said to be *locally-rightmost* if each finite segment is the rightmost geodesic between its endpoints.

When π and γ are both locally-rightmost geodesics that start at the same point $\pi_o = \gamma_o$, $\pi \preceq_a \gamma$ is equivalent to $\pi \preceq \gamma$.

For $u \in \mathbb{Z}^2$ and $m \in \mathbb{Z}$ with $m \geqslant k = u\cdot(e_1+e_2)$ let $\mathcal{G}^\omega_{u,m}$ denote the set of rightmost geodesics that start at u and end at some $x \in u + \mathbb{Z}^2_+$ with $m = x\cdot(e_1+e_2) \geqslant k$. The set of semi-infinite up-right paths $\pi_{k:\infty}$ that start at $\pi_k = u$ and satisfy $\pi_{k:m} \in \mathcal{G}^\omega_{u,m}$ for all $m \in \mathbb{Z}_{\geqslant k}$ is denoted by $\mathcal{G}^\omega_u$. This is the set of all locally-rightmost semi-infinite geodesics started at u. A discussion of measurability of $\mathcal{G}^\omega_u$ appears in Appendix A. It is immediate from the definition that

$$u + \mathcal{G}^{T_u\omega}_0 = \mathcal{G}^\omega_u \quad \text{for all } u \in \mathbb{Z}^2 \text{ and } \omega \in \Omega. \tag{2.6}$$

The (deterministic) uniqueness of finite rightmost geodesics implies that $\mathcal{G}^\omega_u$ is a tree. Therefore, we will hereafter refer to a locally-rightmost semi-infinite geodesic as a *geodesic ray*. There are two *trivial* geodesic rays in $\mathcal{G}^\omega_u$ given by $u + \mathbb{Z}_+e_1$ and $u + \mathbb{Z}_+e_2$. That these are always geodesic rays follows from the path structure.

The uniqueness of rightmost point-to-point geodesics implies that $\preceq$ is a total order on $\mathcal{G}^\omega_u$. Precisely, the following three facts hold for each pair of geodesics π and γ in $\mathcal{G}^\omega_u$:

$$\pi \preceq \gamma \preceq \pi \text{ is equivalent to } \pi = \gamma. \tag{2.7}$$

$$\text{If } \pi \precnsim \gamma \text{ then } \pi \text{ and } \gamma \text{ separate at some point and never intersect again.} \tag{2.8}$$

$$\text{Exactly one of } \pi \precnsim \gamma,\ \pi \succnsim \gamma \text{ and } \pi = \gamma \text{ holds.} \tag{2.9}$$

2.4. **Limit shape.** By the shape theorem [29], there exists a *shape function* $\mathrm{g} : \mathbb{R}^2_+ \to \mathbb{R}$ such that with $\mathbb{P}$-probability one

$$\lim_{n\to\infty} \max_{x\in\mathbb{Z}^2_+ : |x|_1 = n} \frac{|L_{0,x} - \mathrm{g}(x)|}{n} = 0. \tag{2.10}$$

g is symmetric, concave, and positively homogeneous of degree one. Homogeneity implies g is determined by its restriction to $\mathcal{U} = [e_2, e_1]$.

The super-differential of g at $\xi \in \mathbb{R}^2_+$ is

$$\partial\mathrm{g}(\xi) = \{h \in \mathbb{R}^2 : \mathrm{g}(\zeta) - \mathrm{g}(\xi) \leqslant h\cdot(\zeta - \xi) \text{ for all } \zeta \in \mathbb{R}^2_+\}. \tag{2.11}$$

By homogeneity, $\partial\mathrm{g}(\xi) = \partial\mathrm{g}(c\xi)$ for any $c > 0$. Thus $\partial\mathrm{g}(\bullet)$ is also determined by points on $\mathcal{U}$. Concavity implies the existence of one-sided derivatives at relative interior points $\xi \in \mathrm{ri}\,\mathcal{U}$:

$$\nabla\mathrm{g}(\xi\pm)\cdot e_1 = \lim_{\varepsilon\searrow 0} \frac{\mathrm{g}(\xi \pm \varepsilon e_1) - \mathrm{g}(\xi)}{\pm\varepsilon} \quad \textit{and} \quad \nabla\mathrm{g}(\xi\pm)\cdot e_2 = \lim_{\varepsilon\searrow 0} \frac{\mathrm{g}(\xi \mp \varepsilon e_2) - \mathrm{g}(\xi)}{\mp\varepsilon}.$$

By [20, Lemma 4.7(c)] differentiability of g at $\xi \in \operatorname{ri}\mathcal{U}$ is the same as $\nabla g(\xi+) = \nabla g(\xi-)$. More generally, these values are the extreme points of the convex set $\partial g(\xi)$.

In addition to the shape theorem mentioned above, a contribution of Martin [29] shows a certain universal behavior of the limit shape in planar last-passage percolation with i.i.d. weights near the coordinate axis boundaries. In particular, this result implies that the limit shape must have infinitely many faces and that $\partial g(e_1) = \partial g(e_2) = \varnothing$. These facts are often critically important in non-triviality arguments and this is the primary reason why we must assume the weights are i.i.d. in (2.2).

Remark 2.1. The result of [29] mentioned above has a slightly weaker moment assumption than in (2.2). The existence result in [21] recorded below as Theorem 2.7 also relies on a variational characterization of the limit shape from [14], which requires this stronger moment hypothesis. △

An important index set for Busemann functions is the total superdifferential of the shape function, denoted by

$$\partial g(\mathcal{U}) = \{h \in \mathbb{R}^2 : \text{there exists } \xi \in \mathcal{U} \text{ with } h \in \partial g(\xi)\}. \tag{2.12}$$

In the sequel, we will call elements of this set *tilts*.

2.5. Recovering cocycles and Busemann functions.

Definition 2.2. *A function $A : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ is a* cocycle *if*

$$A(x, y) + A(y, z) = A(y, z) \quad \textit{for all } x, y, z \in \mathbb{Z}^2. \tag{2.13}$$

△

Definition 2.3. *Given real weights $\omega = (\omega_x)_{x\in\mathbb{Z}^2} \in \mathbb{R}^{\mathbb{Z}^2}$, a function $A : \mathbb{Z}^2 \to \mathbb{R}$ is said to* recover *the weights ω if it satisfies the following* recovery property*:*

$$A(x, x + e_1) \wedge A(x, x + e_2) = \omega_x \quad \textit{for all } x \in \mathbb{Z}^2. \tag{2.14}$$

△

Given a recovering cocycle A, a semi-infinite up-right path π is called an A-geodesic in weights ω if it satisfies

$$A(u, v) = L_{u,v}(\omega). \tag{2.15}$$

for all $u \leqslant v$ on π. Such a path is always a geodesic in weights ω because for any other up-right path $(x_i)_{i=m}^n$ from u to v,

$$\sum_{i=m}^{n-1} \omega_{x_i} \leqslant \sum_{i=m}^{n-1} A_\pi(x_i, x_{i+1}) = A_\pi(u, v) = L_{u,v}(\omega).$$

By Theorem B.1 in Appendix B, under (2.2), there exists an event $\Omega_0 \in \mathfrak{S}$ of full $\mathbb{P}$-probability on which, for every non-trivial geodesic ray π, the limits

$$A_\pi(\omega, x, y) = \lim_{n\to\infty} (L_{x,\pi_n}(\omega) - L_{y,\pi_n}(\omega)) \tag{2.16}$$

define a recovering cocycle. This is the definition originally introduced by Busemann [4] in metric geometry. Thus, A_π is called the *Busemann function* generated by π.

The definition of A_π implies that (2.15) holds for any $v \geqslant u$ on π. Therefore, π is always an A_π-geodesic.

The fact that $L_{x+z,y+z}(\omega) = L_{x,y}(T_z\omega)$ gives that if π is a semi-infinite geodesic in the weights $T_z\omega$ then $z + \pi = (z + u : u \in \pi)$ is a semi-infinite geodesic in the weights ω and

$$A_{z+\pi}(\omega, x + z, y + z) = A_\pi(T_z\omega, x, y). \tag{2.17}$$

Lemma B.3 states that, $\mathbb{P}$-almost surely, for any nontrivial $\gamma \preceq \pi$ in $\mathcal{G}^\omega$, $A_\gamma \preceq A_\pi$. Consequently, if $\gamma \curlywedge \pi$, then $A_\gamma = A_\pi$.

2.6. **Generalized Busemann functions and the Busemann process.** The last observation motivates restricting attention to objects which are shift covariant.

Definition 2.4. *A measurable function $\widehat{B} : \widehat{\Omega} \times \mathbb{Z}^2 \to \mathbb{R}$ is* shift-covariant *if for $\widehat{\mathbb{P}}$-almost every $\widehat{\omega}$,*

$$\widehat{B}(\widehat{\omega}, x+z, y+z) = \widehat{B}(\widehat{T}_z\widehat{\omega}, x, y) \quad \text{for all } x, y, z \in \mathbb{Z}^2. \tag{2.18}$$

It is said to be $L^1(\widehat{\Omega}, \widehat{\mathfrak{S}}, \widehat{\mathbb{P}})$ if

$$\widehat{\mathbb{E}}[|\widehat{B}(x,y)|] < \infty \quad \text{for all } x, y \in \mathbb{Z}^2. \tag{2.19}$$

△

The main objects that we consider in this work are shift covariant, recovering, L^1 cocycles, which we call generalized Busemann functions.

Definition 2.5. *A shift-covariant $L^1(\widehat{\Omega}, \widehat{\mathfrak{S}}, \widehat{\mathbb{P}})$ recovering cocycle is an $L^1(\widehat{\Omega}, \widehat{\mathfrak{S}}, \widehat{\mathbb{P}})$ shift-covariant measurable function $\widehat{B} : \widehat{\Omega} \times \mathbb{Z}^2 \to \mathbb{R}$ that is $\widehat{\mathbb{P}}$-almost surely a recovering cocycle. The space of shift-covariant $L^1(\widehat{\Omega}, \widehat{\mathfrak{S}}, \widehat{\mathbb{P}})$ recovering cocycles is denoted by $\widehat{\mathcal{K}}$. Such objects are called generalized Busemann functions.* △

For $\widehat{B} \in \widehat{\mathcal{K}}$ define the random 2-vector $\mathbf{h}(\widehat{B}) = \mathbf{h}(\widehat{B}, \widehat{\omega}) \in \mathbb{R}^2$ via

$$\mathbf{h}(\widehat{B}) \cdot e_i = -\widehat{\mathbb{E}}[\widehat{B}(0, e_i) \,|\, \widehat{\mathcal{I}}], \quad i \in \{1, 2\}. \tag{2.20}$$

By [21, Theorem 4.4] (see [20, Theorem B.3] for the details), for $\widehat{\mathbb{P}}$-a.e. $\widehat{\omega}$,

$$\lim_{n\to\infty} n^{-1} \max_{|x|_1 \leqslant n} |\widehat{B}(\widehat{\omega}, 0, x) + \mathbf{h}(\widehat{B}, \widehat{\omega}) \cdot x| = 0. \tag{2.21}$$

We have the following lemma connecting generalized Busemann functions to the superdifferential of the shape function. Recall the set $\partial \mathrm{g}(\mathcal{U})$ defined in (2.12).

Lemma 2.6. [21, Lemma 4.5] *A generalized Busemann function $\widehat{B} \in \widehat{\mathcal{K}}$ has the following properties:*

(a) *$-\mathbf{h}(\widehat{B})$ takes values in $\partial \mathrm{g}(\mathcal{U})$, $\widehat{\mathbb{P}}$-almost surely.*

(b) *If $-\widehat{\mathbb{E}}[\mathbf{h}(\widehat{B})] \in \partial \mathrm{g}(\xi)$ for some $\xi \in \mathcal{U}$, then $-\mathbf{h}(\widehat{B}) \in \partial \mathrm{g}(\xi)$ $\widehat{\mathbb{P}}$-almost surely.*

(c) *If $-\widehat{\mathbb{E}}[\mathbf{h}(\widehat{B})] \in \{\nabla \mathrm{g}(\xi+), \nabla \mathrm{g}(\xi-)\}$ for some $\xi \in \mathcal{U}$, then $\mathbf{h}(\widehat{B}) = \widehat{\mathbb{E}}[\mathbf{h}(\widehat{B})]$ $\widehat{\mathbb{P}}$-almost surely.* △

As mentioned in the introduction, under some additional hypotheses on the weights, it is known from the results of [6, 31] that for each $h \in -\partial \mathrm{g}(\mathcal{U})$, there is at most one distribution of a generalized Busemann function with the property that $\widehat{\mathbb{P}}\{\mathbf{h}(\widehat{B}) = h\} = 1$.

Existence of stationary queueing fixed points for the tandem queueing model connected to the general i.i.d. weight corner growth model was originally established by Mairesse and Prabhakar [28] under the assumption that the weights are bounded from below with > 2 moments, but phrased in queueing language. These were used to generate generalized Busemann functions for the corner growth model in [16]. Connections to geodesics were explored in [15]. [21] subsequently removed the boundedness below requirement for existence. By monotonicity, these constructions also build a *Busemann process* on the extended space. This is a covariant, recovering cocycle-valued stochastic process indexed by $-\partial \mathrm{g}(\mathcal{U}) \times \{+, -\}$ as described by the next theorem.

Theorem 2.7. [21, Theorem 4.7] *There exists a probability space $(\widehat{\Omega}, \widehat{\mathfrak{S}}, \widehat{\mathbb{P}})$, equipped with an additive group of continuous bijections $\widehat{T} = \{\widehat{T}_x : x \in \mathbb{Z}^2\}$ and satisfying the hypotheses of Section 2.1, on which there exists a stochastic process*

$$(\widehat{B}^{h\square}(x,y) : x, y \in \mathbb{Z}^2, h \in -\partial \mathrm{g}(\mathcal{U}), \square \in \{+, -\}) \tag{2.22}$$

with the following properties:

(a) (*No $\pm$ distinction at fixed h*) *For each $h \in -\partial\mathrm{g}(\mathcal{U})$,*

$$\widehat{\mathbb{P}}(\widehat{B}^{h-}(x,y) = \widehat{B}^{h+}(x,y)) = 1.$$

When $\widehat{B}^{h-}(\widehat{\omega},x,y) = \widehat{B}^{h+}(\widehat{\omega},x,y)$, call the common value $\widehat{B}^{h}(\widehat{\omega},x,y)$.

(b) (*Generalized Busemann function*) *For each $h \in -\partial\mathrm{g}(\mathcal{U}), \widehat{B}^h \in \widehat{\mathcal{K}}$.*

(c) (*Mean $-h$*) *For each $h \in -\partial\mathrm{g}(\mathcal{U})$, $\widehat{\mathbb{E}}[\widehat{B}(0,e_i)] = -h \cdot e_i$.*

(d) (*Monotonicity*) *For $h, h' \in -\partial\mathrm{g}(\mathcal{U})$ with $h \cdot e_1 \leqslant h' \cdot e_1$, all $x \in \mathbb{Z}^2$, and $\widehat{\mathbb{P}}$ almost every $\widehat{\omega}$*

$$\widehat{B}^{h-}(x,x+e_1) \geqslant \widehat{B}^{h+}(x,x+e_1) \geqslant \widehat{B}^{h'-}(x,x+e_1) \geqslant \widehat{B}^{h'-}(x,x+e_1)$$

and

$$\widehat{B}^{h-}(x,x+e_2) \leqslant \widehat{B}^{h+}(x,x+e_2) \leqslant \widehat{B}^{h'-}(x,x+e_2) \leqslant \widehat{B}^{h'-}(x,x+fe_2).$$

(e) (*Left-/right-continuity*) *For $\widehat{\mathbb{P}}$ almost all $\widehat{\omega}$, for all $h \in -\partial\mathrm{g}(\mathcal{U})$,*

$$\widehat{B}^{h-}(x,y) = \lim_{\substack{-\partial\mathrm{g}(\mathcal{U})\ni h'\to h\\ h'\cdot e_1\nearrow h\cdot e_1}} \widehat{B}^{h\pm}(x,y) \qquad \text{and } \widehat{B}^{h-}(x,y) = \lim_{\substack{-\partial\mathrm{g}(\mathcal{U})\ni h'\to h\\ h'\cdot e_1\searrow h\cdot e_1}} \widehat{B}^{h\pm}(x,y).$$

(f) (*Backward independence*) *For any $I \subseteq \mathbb{Z}^2$, the random variables $\{\omega_x, \widehat{B}^{h\pm}(x,y) : y \geqslant x, x \in I, h \in -\partial\mathrm{g}(\mathcal{U})\}$ are independent from the set of weights behind I, $\{\omega_x : x$ is not $\geqslant z$, for all $z \in I\}$.* △

Remark 2.8. The Busemann process is also often indexed by directions. This indexing corresponds to restricting the process to the subset of $\partial\mathrm{g}(\mathcal{U})$ given by the extreme points of each super-differential interval, $-h \in \{\nabla\mathrm{g}(\xi\square), : \xi \in \mathcal{U}, \square \in \{+,-\}\}$. In principle, it is possible that tilt-indexing gives a richer process if there exist directions of non-differentiability. If the shape is differentiable, as is widely believed to be true in the setting of this work, then the two are equivalent. △

2.7. **Shift-covariant systems of geodesic rays.**

Definition 2.9. *A* random geodesic *out of $u \in \mathbb{Z}^2$ is a measurable mapping $\widehat{\pi} : \widehat{\Omega} \to \mathbb{X}_u$ such that*

$$\widehat{\mathbb{P}}\{\widehat{\omega} : \widehat{\pi}(\widehat{\omega}) \in \mathcal{G}_u^{\omega(\widehat{\omega})}\} = 1.$$ △

Definition 2.10. *A* system of random geodesics *is a family of random geodesics $\{\widehat{\pi}^u(\widehat{\omega}) : u \in \mathbb{Z}^2\}$ such that for each $u \in \mathbb{Z}^2$, $\widehat{\pi}^u$ is a random geodesic out of u.*

The system is coalescing *if*

$$\widehat{\mathbb{P}}\big\{\forall u, v \in \mathbb{Z}^2 : \widehat{\pi}^u(\widehat{\omega}) \curlywedge \widehat{\pi}^v(\widehat{\omega})\big\} = 1.$$

The system is said to be shift-covariant if, $\widehat{\mathbb{P}}$-almost surely,

$$\widehat{\pi}^u(\widehat{\omega}) = u + \widehat{\pi}^0(\widehat{T}_u\widehat{\omega}) \quad \text{for } u \in \mathbb{Z}^2.$$

$\widehat{\Pi}_c$ denotes the set of shift-covariant coalescing systems of random geodesics. △

Remark 2.11. [1] refers to what we call a *shift-covariant system of coalescing geodesics* as *random coalescing geodesics*. We use this slightly different terminology because in last-passage percolation, it has been proven that there exist random systems of coalescing geodesics which are not shift-covariant. For example, the system of rightmost geodesics in the exponential last-passage percolation going in the direction of the competition interface rooted at the origin. See [23, Theorem 3.11]. A similar statement can be expected to hold in first-passage percolation as well. △

Any shift-covariant system of random geodesics can be generated by the member emanating from 0. Conversely, every random geodesic $\widehat{\pi}$ out of 0 generates a shift-covariant system of random geodesics $\widehat{\pi}^u$, $u \in \mathbb{Z}^2$, defined by

$$\widehat{\pi}^u(\widehat{\omega}) = u + \widehat{\pi}(\widehat{T}_u\widehat{\omega}) \in \mathcal{G}_u^{\omega(\widehat{\omega})} \quad \text{for } u \in \mathbb{Z}^2. \tag{2.23}$$

With this notation, $\widehat{\pi}^0 = \widehat{\pi}$. We will abbreviate $\{\widehat{\pi}^u : u \in \mathbb{Z}^2\}$ by writing $\widehat{\pi}^\bullet$.

For $\widehat{B} \in \widehat{\mathcal{K}}$ and $x \in \mathbb{Z}^2$ let $\widehat{S}_x^{\widehat{B}} : \widehat{\Omega} \to \{e_1, e_2\}$ be the $\widehat{\mathfrak{S}}$-measurable random variable defined by

$$\widehat{S}_x^{\widehat{B}}(\widehat{\omega}) = \begin{cases} e_1 & \text{if } \widehat{B}(\widehat{\omega}, x, x+e_1) \leqslant \widehat{B}(\widehat{\omega}, x, x+e_2), \\ e_2 & \text{if } \widehat{B}(\widehat{\omega}, x, x+e_1) > \widehat{B}(\widehat{\omega}, x, x+e_2). \end{cases}$$

Think of $\widehat{S}^{\widehat{B}} = \{\widehat{S}_x^{\widehat{B}} : x \in \mathbb{Z}^2\}$ as placing arrows at the lattice sites so that x points to $x + \widehat{S}_x^{\widehat{B}}(\widehat{\omega})$.

For $u \in \mathbb{Z}^2$ let $\phi^{\widehat{B},u}(\widehat{\omega})$ denote the path that starts at u and follows the arrows given by $\widehat{S}^{\widehat{B}}(\widehat{\omega})$. For $\widehat{\mathbb{P}}$-almost every $\widehat{\omega}$, these paths satisfy (2.15) for $A = \widehat{B}(\widehat{\omega})$. Therefore, these are semi-infinite geodesics in the LPP model with weights $\omega(\widehat{\omega})$. We know from [15, Lemma 4.1] that these are also, $\widehat{\mathbb{P}}$-almost surely, locally-rightmost semi-infinite geodesics, that is, $\phi^{\widehat{B},u}(\widehat{\omega}) \in \mathcal{G}_u^{\omega(\widehat{\omega})}$. Therefore, we refer to $\phi^{\widehat{B},u}(\widehat{\omega})$ as the $\widehat{B}$-geodesic (ray) out of u. $\widehat{S}^{\widehat{B}}$ is a measurable way to encode all the (locally-rightmost) semi-infinite $\widehat{B}$-geodesics.

The shift-covariance of $\widehat{B}$ gives, $\widehat{\mathbb{P}}$-almost surely, $\widehat{S}_x^{\widehat{B}}(\widehat{T}_z\widehat{\omega}) = \widehat{S}_{x+z}^{\widehat{B}}(\widehat{\omega})$ for all $x, z \in \mathbb{Z}^2$, and hence

$$\phi^{\widehat{B},u+z}(\widehat{\omega}) = z + \phi^{\widehat{B},u}(\widehat{T}_z\widehat{\omega}). \tag{2.24}$$

Lemma A.4 in the appendix says that

$$\widehat{\mathbb{P}}\big\{\widehat{\omega} : \phi^{\widehat{B},u}(\widehat{\omega}) \notin \{u + \mathbb{Z}_+e_1, u + \mathbb{Z}_+e_2\}\big\} = 1. \tag{2.25}$$

Definition 2.12. *Given a shift-covariant $L^1(\widehat{\Omega}, \widehat{\mathfrak{S}}, \widehat{\mathbb{P}})$ recovering cocycle $\widehat{B}$ we say that it* has coalescing geodesic rays *if $\phi^{\widehat{B},u}(\widehat{\omega}) \curlywedge \phi^{\widehat{B},v}(\widehat{\omega})$ for all $u, v \in \mathbb{Z}^2$ and $\widehat{\mathbb{P}}$-almost all $\widehat{\omega}$. Denote the space of shift-covariant $L^1(\widehat{\Omega}, \widehat{\mathfrak{S}}, \widehat{\mathbb{P}})$ recovering cocycles with coalescing geodesic rays by $\widehat{\mathcal{K}}_c$.* △

We introduce analogous notation on the canonical space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.13. *Let $\mathcal{K}$ be the space of T-covariant $L^1(\Omega, \mathcal{F}, \mathbb{P})$ recovering cocycles. Let $\mathcal{K}_c \subset \mathcal{K}$ be the subspace of cocycles that have coalescing geodesics. A cocycle $B \in \mathcal{K}$ is said to be forward measurable, if for any $u \in \mathbb{Z}^2$ and any $x, y \in u + \mathbb{Z}_+^2$, $B(x, y)$ is $\mathcal{F}_u^+$-measurable. Let $\mathcal{K}^+ \subset \mathcal{K}$ be the subspace of cocycles that are forward measurable and let $\mathcal{K}_c^+ = \mathcal{K}_c \cap \mathcal{K}^+$.* △

2.8. **Main results.** Our first result is strong existence of Busemann functions.

Theorem 2.14. *Let $\widehat{B} \in \widehat{\mathcal{K}}_c$ and $h = \widehat{\mathbb{E}}[\mathbf{h}(\widehat{B})]$. Assume that $\widehat{\mathbb{P}}\{\mathbf{h}(\widehat{B}) = h\} = 1$. Then there exists a forward measurable cocycle $B \in \mathcal{K}_c^+$ such that $\widehat{B}(\widehat{\omega}) = B(\omega(\widehat{\omega}))$, $\widehat{\mathbb{P}}$-almost surely.* △

The second result is strong uniqueness.

Theorem 2.15. *Let $B_1, B_2 \in \mathcal{K}_c$. Assume $\mathbf{h}(B_1) = \mathbf{h}(B_2)$. Then $B_1 = B_2$, $\mathbb{P}$-almost surely.* △

With those in mind, we are now ready to construct the tilt-indexed Busemann process on the canonical space Ω. Let

$$\mathcal{H} = \{\mathbf{h}(B) : B \in \mathcal{K}_c\}. \tag{2.26}$$

denote the collection of tilts associated to covariant recovering cocycles with coalescing geodesics which are defined on Ω. Because $\mathbb{P}$ is ergodic, $\mathbf{h}(B)$ is non-random for each $B \in \mathcal{K}_c$. Recall from Lemma 2.6 that $-\widehat{\mathbb{E}}[\mathbf{h}(\widehat{B})] \in \{\nabla \mathrm{g}(\xi+), \nabla \mathrm{g}(\xi-)\}$ for some $\xi \in \mathcal{U}$ is also sufficient for a non-random tilt: $\mathbf{h}(\widehat{B}) = \widehat{\mathbb{E}}[\mathbf{h}(\widehat{B})]$ $\widehat{\mathbb{P}}$-almost surely. Applying Theorem 2.14 to the cocycles from Theorem 2.7 and recalling part (i) of Lemma 2.6, we see that

$$\{-\nabla \mathrm{g}(\xi\square) : \xi \in \mathcal{U}, \square \in \{+,-\}\} \subset \mathcal{H} \subset -\partial \mathrm{g}(\mathcal{U}) \tag{2.27}$$

Remark 2.16. If g is differentiable, as is widely expected under assumption (2.2), both inclusions in (2.27) are equalities. As mentioned in Remark 2.8, if directions of non-differentiability exist, then the associated super-differential is a line segment. If an interior tilt exists which is associated to an ergodic cocycle (i.e. an element of $\widehat{\mathfrak{S}}$ with $\mathbf{h}(\widehat{B})$ deterministic), then the first inclusion is strict. The second inclusion is an equality if and only if every tilt is associated to an ergodic cocycle, i.e., if there are no *gaps* in $\mathcal{H}$. Brito and Hoffman [3] give an example of an ergodic FPP model where there are only four semi-infinite geodesics on the entire lattice, which provides an example where *gaps* in the associated $\mathcal{H}$ exist. △

The index set of the tilt-indexed Busemann process on Ω will be $\mathcal{H} \times \{-,+\}$. The process is obtained by taking left and right limits of an appropriate countable set of elements in $\mathcal{K}_c$.

Theorem 2.17. *There exists a stochastic process*

$$(B^{h\square}(x,y) : x,y \in \mathbb{Z}^2,\ h \in \mathcal{H},\ \square \in \{+,-\}) \tag{2.28}$$

on $(\Omega, \mathcal{F}, \mathbb{P})$ *with the following properties:*

(a) (*No* $\pm$ *distinction at fixed* h) *For each* $h \in \mathcal{H}$,

$$\mathbb{P}(B^{h-}(x,y) = B^{h+}(x,y)) = 1.$$

When $B^{h-}(\omega,x,y) = B^{h+}(\omega,x,y)$, *call the common value* $B^h(\omega,x,y)$.

(b) (*Forward measurable Busemann function*) *For each* $h \in \mathcal{H}$, $B^h \in \mathcal{K}_c^+$.

(c) (*Mean* $-h$) *For each* $h \in \mathcal{H}$, $\mathbb{E}[B^h(0,e_i)] = -h \cdot e_i$.

(d) (*Monotonicity*) *For* $h,h' \in \mathcal{H}$ *with* $h\cdot e_1 \leqslant h' \cdot e_1$, *all* $x \in \mathbb{Z}^2$, *and* $\mathbb{P}$ *almost every* ω

$$B^{h-}(x,x+e_1) \geqslant B^{h+}(x,x+e_1) \geqslant B^{h'-}(x,x+e_1) \geqslant B^{h'-}(x,x+e_1)$$

and

$$B^{h-}(x,x+e_2) \leqslant B^{h+}(x,x+e_2) \leqslant B^{h'-}(x,x+e_2) \leqslant B^{h'-}(x,x+e_2).$$

(e) (*Left-/right-continuity*) *For* $\mathbb{P}$ *almost all* ω, *for all* $h \in \mathcal{H}$,

$$B^{h-}(x,y) = \lim_{\substack{\mathcal{H}\ni h'\to h \\ h'\cdot e_1 \nearrow h\cdot e_1}} B^{h\pm}(x,y) \ \text{ and }\ B^{h-}(x,y) = \lim_{\substack{\mathcal{H}\ni h'\to h \\ h'\cdot e_1 \searrow h\cdot e_1}} B^{h\pm}(x,y).$$

Moreover, this process is unique in the sense that any two processes satisfying the above conditions are equal almost surely. △

Remark 2.18. If one instead wishes to work with the (potentially) smaller direction-indexed Busemann process coming from restricting to $\{-\nabla \mathrm{g}(\xi\pm) : \xi \in \mathcal{U}\} \subset \mathcal{H}$, strong uniqueness of the process still holds. △

Remark 2.19. The shift-covariance implicitly contained in part (b) of Theorem 2.17 is inherited by the full process. In particular, we have that

$$(B^{\bullet}(x,y) : x,y \in \mathbb{Z}^2) \circ T_z = (B^{\bullet}(z+x,z+y) : x,y \in \mathbb{Z}^2)$$

$\mathbb{P}$ almost surely. △

As a consequence of the above observation and the fact that $\mathbb{P}$ is an i.i.d. measure on $\Omega = \mathbb{R}^{\mathbb{Z}^2}$, we have strong mixing of the process.

Corollary 2.20. *Call* $B^{\bullet} = (B^{\bullet}(x,y) : x,y \in \mathbb{Z}^2)$ *and let* $T \in \{T_z : z \neq 0\}$. *Then* $B^{\bullet}$ *is strongly mixing under* T. *Explicitly, this means that for all events* $A \in \mathcal{F}$ *and Borel* C,

$$\lim_{n\to\infty} \mathbb{P}(A, B^{\bullet} \circ T^n \in C) = \mathbb{P}(A)\mathbb{P}(B^{\bullet} \in C). \qquad △$$

3. Strong existence and uniqueness of generalized Busemann functions

The main aim of the first part of this section is to start with a given generalized Busemann function on $\widehat{\Omega}$ with coalescing geodesics, $\widehat{B} \in \widehat{\mathcal{K}}_c$, construct an $\mathfrak{S}$-measurable version of its coalescing geodesics, and subsequently use this to obtain an $\mathfrak{S}$-measurable version of $\widehat{B}$ itself. While one could try to rely on general measure-theoretic techniques (e.g., sections) to produce a system of coalescing geodesics—similar to the approach used in [21, Theorem 3.2] to establish the $\mathbb{P}$-a.s. existence of directed geodesics after proving their $\widehat{\mathbb{P}}$-a.s. existence—the challenge lies in maintaining shift-covariance across all $T_x\omega$.

The basic idea, following a similar construction in [1], is to apply a variant of the classical inverse CDF sampling method to the conditional distribution on the path space of a $\widehat{B}$-geodesic ray rooted at u, given $\mathfrak{S}$. Because the path space is totally ordered, we can define quantile functions and this method works essentially the same way for real random variables. The outcome is a process of random geodesics which is shift-covariant by construction. We then show that under appropriate hypotheses on $\widehat{B}$, this process is essentially constant and defines a family of coalescing geodesics, which then generate a Busemann function that is equal to $\widehat{B}$ almost surely. Strong uniqueness comes from showing that there is a total ordering on such objects indexed by the tilt vector and so in particular any two generalized Busemann functions with the same deterministic tilt vector must be equal. Extending these properties to the full process is essentially immediate from monotonicity.

We begin with left- and right-isolated geodesics. These play a central role in the argument.

3.1. **Isolated geodesics.** Since $\mathcal{G}_u^\omega$ is totally ordered and compact, it has the greatest lower bound and least upper bound properties. For $u \in \mathbb{Z}^2$, let $m = u \cdot (e_1 + e_2)$ and define these collections of semi-infinite geodesics rooted at u:

$$\begin{aligned}
\mathrm{LI}_u^\omega &= \bigcup_{\substack{\sigma_{m:n} \text{ up-right, } n \in \mathbb{Z}_{\geqslant m} \\ \exists \pi \in \mathcal{G}_u^\omega : \pi_{m:n} = \sigma_{m:n}}} \big\{ \inf\{\pi \in \mathcal{G}_u^\omega : \pi_{m:n} = \sigma_{m:n}\} \big\} \quad \text{and} \\
\mathrm{RI}_u^\omega &= \bigcup_{\substack{\sigma_{m:n} \text{ up-right,} n \in \mathbb{Z}_{\geqslant m} \\ \exists \pi \in \mathcal{G}_u^\omega : \pi_{m:n} = \sigma_{m:n}}} \big\{ \sup\{\pi \in \mathcal{G}_u^\omega : \pi_{m:n} = \sigma_{m:n}\} \big\}.
\end{aligned} \tag{3.1}$$

Note that the conditions in the unions above and the geodesic rays appearing in (3.1) (e.g. $\sup\{\pi \in \mathcal{G}_u^\omega : \pi_{m:n} = \sigma_{m:n}\}$ for some finite up-right path $\sigma_{m:n}$ with $\sigma_m = u$) are $\mathfrak{S}_u^+$-measurable, because they can be constructed inductively from the arrows $S_{u,x}^i(\omega)$ described at the beginning of Appendix A.

A non-trivial geodesic ray $\lambda \in \mathcal{G}_u^\omega$ is *left-isolated* if it is not a limit of members of $\mathcal{G}_u^\omega$ from the left, equivalently, $\lambda \gneq \sup\{\gamma \in \mathcal{G}_u^\omega : \gamma \lneq \lambda\}$. Analogously, a non-trivial $\rho \in \mathcal{G}_u^\omega$ is *right-isolated* if $\rho \lneq \inf\{\gamma \in \mathcal{G}_u^\omega : \gamma \gneq \rho\}$. A non-trivial ray is right-isolated if and only if it is in RI_u^ω and it is left-isolated if and only if it is in LI_u^ω. To see this, take a non-trivial left-isolated ray λ and consider the first index n at which λ differs from the supremum of rays strictly less than it. Then λ is the infimum of all geodesic rays which contain the segment $\lambda_{u:n}$. The right-isolated case is similar. Note also that RI_u^ω is right-dense in $\mathcal{G}_u^\omega$ and LI_u^ω is left-dense in $\mathcal{G}_u^\omega$.

Let

$$\mathcal{G}^\omega = \bigcup_{u \in \mathbb{Z}^2} \mathcal{G}_u^\omega$$

be the collection of all geodesic rays from all starting points.

On the union $\mathcal{G}^\omega$, $\pi \preceq_a \gamma$ and $\gamma \preceq_a \pi$ happen together if and only if $\pi \curlywedge \gamma$. In this situation, after their first meeting the two remain together, again by virtue of the uniqueness of rightmost point-to-point geodesics.

3.2. **Strong existence.** Let ν^ω denote the conditional distribution on $(\widehat\Omega, \widehat{\mathfrak S})$ given $\mathfrak S$. Recall the countable right-dense set of $\mathfrak S_u^+$-measurable right-isolated geodesics $\mathrm{RI}_u^\omega \subset \mathcal G_u^\omega$, defined in (3.1), and note that by definition $\mathrm{RI}_u^{T_z\omega} = \mathrm{RI}_{u+z}^\omega$ for all $u, z \in \mathbb Z^2$.

For $u \in \mathbb Z^2$ and $s \in \mathbb Q \cap [0,1]$, define an $\mathfrak S$-measurable geodesic ray $\boldsymbol\pi^{\widehat B,u,s} \in \mathcal G_u^\omega$ via

$$\boldsymbol\pi^{\widehat B,u,s}(\widehat\omega) = \inf\bigl\{\rho \in \mathrm{RI}_u^{\omega(\widehat\omega)} : \nu^{\omega(\widehat\omega)}(\phi^{\widehat B,u} \preceq \rho) \geqslant s\bigr\}. \tag{3.2}$$

The set in (3.2) is not empty because it contains the trivial geodesic $u+\mathbb Z_+e_1$. We took the infimum in (3.2) over the countable set $\mathrm{RI}_u^{\omega(\widehat\omega)}$ to ensure measurability. We verify that this expression is measurable in Corollary A.2 below.

Extend this definition to $s \in [0,1]$ by setting

$$\boldsymbol\pi^{\widehat B,u,s}(\widehat\omega) = \sup\bigl\{\boldsymbol\pi^{\widehat B,u,r}(\widehat\omega) : r \in \mathbb Q \cap [0,1],\, r < s\bigr\}. \tag{3.3}$$

In Lemma 3.1(a), we show that (3.2) and (3.3) agree on rational s. In Lemma 3.1(c), we show that the infimum can be taken over the uncountable tree $\mathcal G_u^{\omega(\widehat\omega)}$ for any $s \in [0,1]$.

Denote by $\boldsymbol\pi^{\widehat B,u,\bullet}$ the process $\bigl(\boldsymbol\pi^{\widehat B,u,s} : s \in [0,1]\bigr)$, defined through (3.3). Since $\mathcal G_u^\omega$ is closed, we have, for $\widehat{\mathbb P}$-almost every $\widehat\omega$, $\boldsymbol\pi^{\widehat B,u,s}(\widehat\omega) \in \mathcal G_u^{\omega(\widehat\omega)}$ for all $s \in [0,1]$. In the statement of the next result, $\mathrm D_{\mathrm{LCRL}}([0,1], \mathbb X_u)$ and $\mathrm D_{\mathrm{LCRL}}([0,1], \mathcal G_u^\omega)$ are Skorokhod spaces of left-continuous paths with right limits (see [10, Section 3.5] for a definition of the Skorokhod topology) taking values in the compact metric spaces $\mathbb X_u$ and $\mathcal G_u^\omega$, respectively.

In what follows, the phrase "for $\widehat{\mathbb P}$-almost every $\widehat\omega$" means the existence of an $\widehat{\mathfrak S}$-measurable event of full $\widehat{\mathbb P}$-measure such that the stated property holds for each $\widehat\omega$ in this event. By taking intersections with all shifts by $\{\widehat T_x : x \in \mathbb Z^2\}$, this event can without loss of generality be assumed to be shift invariant.

Lemma 3.1. *For each $\widehat B \in \widehat{\mathcal K}$, the process $\boldsymbol\pi^{\widehat B,u,\bullet}$ satisfies the following properties:*

(a) *For $\widehat{\mathbb P}$-almost every $\widehat\omega$, the definition in (3.3) agrees with (3.2) for rational s, so $\boldsymbol\pi^{\widehat B,u,\bullet}(\widehat\omega)$ is well-defined.*

(b) *$\widehat\omega \mapsto \boldsymbol\pi^{\widehat B,u,\bullet}(\widehat\omega)$ is an $\mathfrak S$-measurable, $\mathrm D_{\mathrm{LCRL}}([0,1], \mathbb X_u)$-valued random variable which almost surely takes values in $\mathrm D_{\mathrm{LCRL}}([0,1], \mathcal G_u^{\omega(\widehat\omega)})$.*

(c) *For $\widehat{\mathbb P}$-almost every $\widehat\omega$ and all $s \in [0,1]$,*

$$\boldsymbol\pi^{\widehat B,u,s}(\widehat\omega) = \inf\bigl\{\gamma \in \mathcal G_u^{\omega(\widehat\omega)} : \nu^{\omega(\widehat\omega)}(\phi^{\widehat B,u} \preceq \gamma) \geqslant s\bigr\}. \tag{3.4}$$

Consequently, $\boldsymbol\pi^{\widehat B,u,r}(\widehat\omega) \preceq \boldsymbol\pi^{\widehat B,u,s}(\widehat\omega)$ for $r \leqslant s$ in $[0,1]$.

(d) *For $\widehat{\mathbb P}$-almost every $\widehat\omega$ and all $\gamma \in \mathcal G_u^{\omega(\widehat\omega)}$,*

$$\bigl\{s \in [0,1] : \boldsymbol\pi^{\widehat B,u,s}(\widehat\omega) \preceq \gamma\bigr\} = \bigl[0, \nu^{\omega(\widehat\omega)}(\phi^{\widehat B,u} \preceq \gamma)\bigr]. \tag{3.5}$$

(e) *For $\widehat{\mathbb P}$-almost every $\widehat\omega$, all $u, z \in \mathbb Z^2$, and all $s \in [0,1]$,*

$$\boldsymbol\pi^{\widehat B,u+z,s}(\widehat\omega) = z + \boldsymbol\pi^{\widehat B,u,s}(\widehat T_z\widehat\omega). \tag{3.6}$$

△

Proof. First note that if $r < s$ and both are rational, then using definition (3.2), we have $\boldsymbol\pi^{\widehat B,u,r} \preceq \boldsymbol\pi^{\widehat B,u,s}$. Monotonicity and the fact that $\mathcal G_u^\omega$ has the least upper bound and greatest lower bound properties imply the existence of left and right limits of these paths, which lie in $\mathcal G_u^\omega \subset \mathbb X_u$.

We check that (3.4) holds for rational s, with the definition in (3.2). For such s, we have

$$\boldsymbol\pi^{\widehat B,u,s} = \inf\bigl\{\rho \in \mathrm{RI}_u^\omega : \nu^\omega(\phi^{\widehat B,u} \preceq \rho) \geqslant s\bigr\} \succeq \inf\bigl\{\gamma \in \mathcal G_u^\omega : \nu^\omega(\phi^{\widehat B,u} \preceq \gamma) \geqslant s\bigr\} = \widetilde{\boldsymbol\pi}^s,$$

because the set in the infimum on the left is a subset of the one on the right. Continuity of probability implies that

$$\nu^{\omega}(\phi^{\widehat{B},u} \preceq \widetilde{\boldsymbol{\pi}}^{s}) \geqslant s. \tag{3.7}$$

Either $\widetilde{\boldsymbol{\pi}}^{s} \in \mathrm{RI}^{\omega}_{u}$, in which case we have $\widetilde{\boldsymbol{\pi}}^{s} = \boldsymbol{\pi}^{\widehat{B},u,s}$, or there exists a sequence $\rho^{n} \in \mathrm{RI}^{\omega}_{u}$ with $\rho^{n} \searrow \widetilde{\boldsymbol{\pi}}^{s}$. But then we must have $\nu^{\omega}(\phi^{\widehat{B},u} \preceq \rho^{n}) \geqslant s$ by monotonicity, which implies that $\boldsymbol{\pi}^{\widehat{B},u,s} \preceq \inf\{\rho^{n} : n \in \mathbb{N}\} = \widetilde{\boldsymbol{\pi}}^{s}$ and again $\widetilde{\boldsymbol{\pi}}^{s} = \boldsymbol{\pi}^{\widehat{B},u,s}$.

Next, observe that if s is any number in $[0,1]$ for which (3.4) holds, then for all $\gamma \in \mathcal{G}^{\omega}_{u}$

$$\boldsymbol{\pi}^{\widehat{B},u,s} \preceq \gamma \Longleftrightarrow s \leqslant \nu^{\omega}(\phi^{\widehat{B},u} \preceq \gamma). \tag{3.8}$$

$\Leftarrow$ comes from (3.4) and $\Rightarrow$ comes from (3.7) and that $\widetilde{\boldsymbol{\pi}}^{s} = \boldsymbol{\pi}^{\widehat{B},u,s}_{0:\infty}$. In particular, since we showed above that (3.4) holds for all rational $s \in [0,1]$, we now know that (3.8) holds for all such s.

With this observation in mind, we check that the process $\boldsymbol{\pi}^{\widehat{B},u,\bullet}$ is well-defined, i.e., that with the definition in (3.2), we have left-continuity over the rational $s \in [0,1]$. For $s \in [0,1]$, call

$$\overline{\boldsymbol{\pi}}^{s} = \sup\bigl\{\boldsymbol{\pi}^{\widehat{B},u,r} : r < s, r \in \mathbb{Q} \cap [0,1]\bigr\}.$$

The monotonicity observed at the beginning of the proof implies that if $s \in [0,1]$ is rational, then $\overline{\boldsymbol{\pi}}^{s} \preceq \boldsymbol{\pi}^{\widehat{B},u,s}$. On the other hand, from $\nu^{\omega}(\phi^{\widehat{B},u} \preceq \overline{\boldsymbol{\pi}}^{s}) \geqslant \nu^{\omega}(\phi^{\widehat{B},u} \preceq \boldsymbol{\pi}^{\widehat{B},u,r}) \geqslant r$ for all rational r with $r < s$, we see that $\nu^{\omega}(\phi^{\widehat{B},u} \preceq \overline{\boldsymbol{\pi}}^{s}) \geqslant s$, which implies $\boldsymbol{\pi}^{\widehat{B},u,s} \preceq \overline{\boldsymbol{\pi}}^{s}$ by (3.8). This is left-continuity on the rationals and part (a) follows.

The definition (3.3) is left-continuous with right limits by construction and monotonicity on the rationals. $\mathfrak{S}$-measurability then follows from the fact that the path $\boldsymbol{\pi}^{\widehat{B},u,\bullet}$ is determined by the values of $\boldsymbol{\pi}^{\widehat{B},u,s}$ defined according to (3.2) for $s \in \mathbb{Q} \cap [0,1]$, the $\mathfrak{S}^{+}_{u}$-measurability of the paths in RI^{ω}_{u}, and the $\mathfrak{S}$-measurability of ν^{ω}. Part (b) is proved.

Next, we show that (3.4) holds also for irrational s. For s irrational, we denote the infimum in (3.4) by $\widetilde{\boldsymbol{\pi}}^{s}$, similarly to what was done above. Arguing exactly as above, by continuity of probability, we have $\nu^{\omega}(\phi^{\widehat{B},u} \preceq \widetilde{\boldsymbol{\pi}}^{s}) \geqslant s$. Since we already proved (3.4) for rationals, we get that for all r rational with $r < s$, $\boldsymbol{\pi}^{\widehat{B},u,r} \preceq \widetilde{\boldsymbol{\pi}}^{s}$. Then by (3.3), $\boldsymbol{\pi}^{\widehat{B},u,s} \preceq \widetilde{\boldsymbol{\pi}}^{s}$.

On the other hand, we already showed that $\nu^{\omega}(\phi^{\widehat{B},u} \preceq \boldsymbol{\pi}^{\widehat{B},u,r}) \geqslant r$ holds for all rational $r \in [0,1]$. Hence, $\nu^{\omega}(\phi^{\widehat{B},u} \preceq \boldsymbol{\pi}^{\widehat{B},u,s}) \geqslant \nu^{\omega}(\phi^{\widehat{B},u} \preceq \boldsymbol{\pi}^{\widehat{B},u,r}) \geqslant r$ for all rational $r < s$, which implies that $\nu^{\omega}(\phi^{\widehat{B},u} \preceq \boldsymbol{\pi}^{\widehat{B},u,s}) \geqslant s$ and so the reverse inequality $\boldsymbol{\pi}^{\widehat{B},u,s} \succeq \widetilde{\boldsymbol{\pi}}^{s}$ also holds. Part (c) is proved, which in turn implies that (3.8) holds for all $s \in [0,1]$ and proves part (d).

We verify (e) using (3.4) for each s. First, we note that by definition, $\gamma \in \mathcal{G}^{\omega}_{u}$ if and only if $z + \gamma \in \mathcal{G}^{T_{-z}\omega}_{u+z}$, where the addition is understood as the translation $(z+\gamma)_i = z + \gamma_i$. Similarly, for $\gamma \in \mathbb{X}_{u+z}$, the shift covariance of $\widehat{B}$ in (2.18) implies that $\widehat{T}_{-z}\{\widehat{\omega} : \phi^{\widehat{B},u+z}(\widehat{\omega}) \preceq \gamma\} = \{\widehat{\omega} : \phi^{\widehat{B},u}(\widehat{\omega}) \preceq \gamma - z\}$. Changing variables by $\gamma' = \gamma - z$, we deduce part (e):

$$\begin{aligned}\boldsymbol{\pi}^{\widehat{B},u+z,s}(\widehat{T}_{-z}\widehat{\omega}) &= \inf\bigl\{\gamma \in \mathcal{G}^{\omega(\widehat{T}_{-z}\widehat{\omega})}_{u+z} : \nu^{\omega(\widehat{T}_{-z}\widehat{\omega})}(\phi^{\widehat{B},u+z} \preceq \gamma) \geqslant s\bigr\} \\ &= z + \inf\bigl\{\gamma' \in \mathcal{G}^{\omega(\widehat{\omega})}_{u} : \nu^{\omega(\widehat{\omega})}(\phi^{\widehat{B},u} \preceq \gamma') \geqslant s\bigr\} = z + \boldsymbol{\pi}^{\widehat{B},u,s}(\widehat{\omega}). \qquad \square\end{aligned}$$

Note that there is no dependence in $\boldsymbol{\pi}^{\widehat{B},u,s}$ on $\widehat{\omega}$ through the superscript $\widehat{B}$. The $\widehat{B}$ in the superscript is just to remind us that if we use a different cocycle we get a different path. Due to the $\mathfrak{S}$-measurability in Lemma 3.1(b), instead of $\boldsymbol{\pi}^{\widehat{B},u,s}_{0:\infty}(\widehat{\omega})$ we write $\boldsymbol{\pi}^{\widehat{B},u,s}(\omega(\widehat{\omega}))$ and frequently simplify it further to $\boldsymbol{\pi}^{\widehat{B},u,s}(\omega)$ because now these paths can be regarded as functions of either $\widehat{\omega} \in \widehat{\Omega}$ or $\omega \in \Omega$.

Example 3.2. Suppose that the conditional distribution $\nu^\omega\{\widehat{x}^{u,\widehat{B}}_{0:\infty} \in \cdot\}$ is supported on finitely many distinct geodesics $\pi^1 \precnsim \pi^2 \precnsim \cdots \precnsim \pi^m$ with $a_k = \nu^\omega\{\phi^{\widehat{B},u} = \pi^k\}$, $b_0 = 0$, and $b_k = a_1 + \cdots + a_k$, for $k \in \{1, \dots, m\}$. Then

$$\nu^\omega\{\phi^{\widehat{B},u} \preceq \gamma\} = \begin{cases} 0, & \text{if } \gamma \precnsim \pi^1, \\ b_k, & \text{if } \pi^k \preceq \gamma \precnsim \pi^{k+1} \text{ for } 1 \leqslant k < m, \text{ and} \\ 1, & \text{if } \gamma \succeq \pi^m_{0:\infty}. \end{cases}$$

Thus, for $s \in [0, 1]$,

$$\boldsymbol{\pi}^{\widehat{B},u,s} = \inf\big\{\gamma \in \mathcal{G}^\omega_u : \nu^\omega(\phi^{\widehat{B},u} \preceq \gamma) \geqslant s\big\} = \begin{cases} u + \mathbb{Z}_+ e_2 & \text{if } s = 0 \text{ and} \\ \pi^k & \text{if } b_{k-1} < s \leqslant b_k \text{ for } k \in \{1, \dots, m\}. \end{cases} \qquad \triangle$$

Recall that $\mathcal{L}$ is the Lebesgue measure on $[0, 1]$, but we write integrals with respect to this measure using the standard notation $\int_0^1 f(s)\, ds$.

Lemma 3.3. *Fix $\widehat{B} \in \widehat{\mathcal{K}}$ and $u \in \mathbb{Z}^2$ and define $\boldsymbol{\pi}^{\widehat{B},u,\bullet}$ as in Lemma 3.1. Equip $\Omega \times \mathbb{X}_u$ with the product Borel σ-algebra. The distribution on $\Omega \times \mathbb{X}_u$ of $(\omega(\widehat{\omega}), \phi^{\widehat{B},u}(\widehat{\omega}))$ under $\widehat{\mathbb{P}}(d\widehat{\omega})$ is the same as the distribution of $(\omega, \boldsymbol{\pi}^{\widehat{B},u,s}(\omega))$ under $\mathbb{P}(d\omega) \otimes \mathcal{L}(ds)$.* $\triangle$

Proof. To prove the lemma, it suffices to show that for all $A \in \mathcal{B}(\mathbb{X}_u)$ and $V \in \mathcal{F}$,

$$\widehat{\mathbb{P}}(\omega \in V,\, \phi^{\widehat{B},u} \in A) = \int_0^1 \mathbb{P}(\boldsymbol{\pi}^{\widehat{B},u,s} \in A, V)\, ds.$$

For this to hold, it is sufficient to consider A of the form $A = \{\gamma \in \mathbb{X}_u : \gamma \preceq \pi\}$ for fixed $\pi \in \mathbb{X}_u$ because the collection of sets of this form is closed under intersections and generates $\mathcal{B}(\mathbb{X}_u)$. See Lemma A.3.

Fix $\pi \in \mathbb{X}_u$. We then have

$$\widehat{\mathbb{P}}(\omega \in V,\, \phi^{\widehat{B},u} \preceq \pi) = \widehat{\mathbb{E}}\big[\nu^\omega(\phi^{\widehat{B},u} \preceq \pi)\mathbb{1}_V(\omega)\big] = \mathbb{E}\big[\nu^\omega(\phi^{\widehat{B},u} \preceq \pi)\mathbb{1}_V(\omega)\big].$$

Call $\gamma^\omega = \sup\{\lambda \in \mathrm{LI}^\omega_u : \lambda \preceq \pi\} \in \mathcal{G}^\omega_u$. Then γ^ω is $\mathfrak{S}^+_u$-measurable and the set inside the supremum is non-empty (as it always contains $u + \mathbb{Z}_+ e_2$). Moreover, for any $\rho \in \mathcal{G}^\omega_u$, $\rho \preceq \gamma^\omega$ if and only if $\rho \preceq \pi$.

Working on the $\mathbb{P}$-almost sure event where $\nu^\omega(\phi^{\widehat{B},u} \in \mathcal{G}^\omega_u) = 1$, we may write

$$\nu^\omega(\phi^{\widehat{B},u} \preceq \pi) = \nu^\omega(\phi^{\widehat{B},u} \preceq \gamma^\omega).$$

By Lemma 3.1(d) and the Fubini-Tonelli theorem, we have

$$\mathbb{E}\big[\nu^\omega(\phi^{\widehat{B},u} \preceq \pi)\mathbb{1}_V(\omega)\big] = \mathbb{E}\big[\nu^\omega(\phi^{\widehat{B},u} \preceq \gamma^\omega)\mathbb{1}_V(\omega)\big] = \mathbb{E}\Big[\int_0^1 \mathbb{1}_V(\omega)\mathbb{1}\{\boldsymbol{\pi}^{\widehat{B},u,s} \preceq \gamma^\omega\}\, ds\Big]$$
$$= \int_0^1 \mathbb{P}(\boldsymbol{\pi}^{\widehat{B},u,s} \preceq \gamma^\omega, V)\, ds.$$

For each s, restricting to the $\mathbb{P}$-almost sure event on which $\boldsymbol{\pi}^{\widehat{B},u,s} \in \mathcal{G}^\omega_u$, this last expression is equal to

$$\int_0^1 \mathbb{P}(\boldsymbol{\pi}^{\widehat{B},u,s} \preceq \pi, V)\, ds.$$

The result now follows. $\square$

Next we narrow the assumptions to include coalescence of $\widehat{B}$-geodesics. In the statement of the next result, for $u, v \in \mathbb{Z}^2$, we say that $(\pi, \gamma) \in \mathcal{G}^\omega_u \times \mathcal{G}^\omega_v$ is a *coalescing pair* if $\pi \curlywedge \gamma$.

Lemma 3.4. *Fix $\widehat{B} \in \widehat{\mathcal{K}}_c$. Then with $\boldsymbol{\pi}^{\widehat{B},u,\bullet}$ and $\boldsymbol{\pi}^{\widehat{B},v,\bullet}$ defined as in Lemma 3.1, for $\mathbb{P}$-almost every ω, for any $u, v \in \mathbb{Z}^2$, and for all coalescing pairs $(\pi, \gamma) \in \mathcal{G}_u^\omega \times \mathcal{G}_v^\omega$,*

$$\big\{s \in [0,1] : \boldsymbol{\pi}^{\widehat{B},u,s}(\omega) \preceq \pi\big\} = \big\{s \in [0,1] : \boldsymbol{\pi}^{\widehat{B},v,s}(\omega) \preceq \gamma\big\}. \tag{3.9}$$

△

Proof. By $\widehat{B} \in \widehat{\mathcal{K}}_c$ and Fubini's theorem, the event $\widehat{\Omega}_1 = \{\widehat{\omega} : \nu^\omega(\phi^{\widehat{B},u} \curlywedge \phi^{\widehat{B},v}) = 1\}$ has $\widehat{\mathbb{P}}$-probability one. Let $\widehat{\Omega}_2$ be the intersection of $\widehat{\Omega}_1$ with the full $\widehat{\mathbb{P}}$-probability event in Lemma 3.1(d). Take $\omega \in \widehat{\Omega}_2$. Then for any $u, v \in \mathbb{Z}^2$, $(\pi, \gamma) \in \mathcal{G}_u^\omega$ such that $\pi \curlywedge \gamma$, we have

$$\nu^\omega(\phi^{\widehat{B},v} \preceq \gamma) - \nu^\omega(\phi^{\widehat{B},u} \preceq \pi) \leqslant \nu^\omega(\phi^{\widehat{B},u} \gnsim \pi,\, \phi^{\widehat{B},v} \preceq \gamma) \leqslant 1 - \nu^\omega(\phi^{\widehat{B},u} \curlywedge \phi^{\widehat{B},v}) = 0$$

and

$$\nu^\omega(\phi^{\widehat{B},u} \preceq \pi) - \nu^\omega(\phi^{\widehat{B},v} \preceq \gamma) \leqslant \nu^\omega(\phi^{\widehat{B},u} \preceq \pi,\, \phi^{\widehat{B},v} \gnsim \gamma) \leqslant 1 - \nu^\omega(\phi^{\widehat{B},u} \curlywedge \phi^{\widehat{B},v}) = 0.$$

Thus, $\nu^\omega(\phi^{\widehat{B},u} \preceq \pi) = \nu^\omega(\phi^{\widehat{B},v} \preceq \gamma)$. Lemma 3.1(d) then gives (3.9) for all $\widehat{\omega} \in \widehat{\Omega}_2$ and coalescing $\pi \in \mathcal{G}_u^\omega$ and $\gamma \in \mathcal{G}_v^\omega$. □

Lemma 3.5. *Fix $\widehat{B} \in \widehat{\mathcal{K}}_c$. For all $u, v \in \mathbb{Z}^2$, with $\boldsymbol{\pi}^{\widehat{B},u,\bullet}$ and $\boldsymbol{\pi}^{\widehat{B},v,\bullet}$ defined as in Lemma 3.1,*

$$\int_0^1 \mathbb{P}(\boldsymbol{\pi}^{\widehat{B},u,s} \curlywedge \boldsymbol{\pi}^{\widehat{B},v,s})\, ds = 1.$$

△

Proof. Fix $u, v \in \mathbb{Z}^2$. Using Lemma 3.3,

$$\begin{aligned} 1 = \widehat{\mathbb{P}}\{\phi^{\widehat{B},u} \curlywedge \phi^{\widehat{B},v}\} &\leqslant \widehat{\mathbb{P}}\big\{\widehat{\omega} : \exists \pi \in \mathcal{G}_v^{\omega(\widehat{\omega})} \text{ s.t. } \pi \curlywedge \phi^{\widehat{B},u}\big\} \\ &= \int_0^1 \mathbb{P}\big\{\omega : \exists \pi \in \mathcal{G}_v^\omega \text{ s.t. } \pi \curlywedge \boldsymbol{\pi}^{\widehat{B},u,s}\big\}\, ds. \end{aligned}$$

By symmetry, switching the roles of u and v,

$$1 = \int_0^1 \mathbb{P}\big\{\omega : \exists \pi \in \mathcal{G}_u^\omega \text{ s.t. } \pi \curlywedge \boldsymbol{\pi}^{\widehat{B},v,s}\big\}\, ds.$$

Thus there exists a measurable set $D_0 \subset \Omega \times (0,1]$ such that $\mathbb{P} \otimes \mathcal{L}(D_0) = 1$ and $\forall (s, \omega) \in D_0$:

$$\exists \gamma \in \mathcal{G}_v^\omega \text{ s.t. } \gamma \curlywedge \boldsymbol{\pi}^{\widehat{B},u,s} \quad \text{and} \quad \exists \gamma' \in \mathcal{G}_u^\omega \text{ s.t. } \gamma' \curlywedge \boldsymbol{\pi}^{\widehat{B},v,s}. \tag{3.10}$$

By Lemma 3.4, there exists an event $\Omega_0 \in \mathfrak{S}$ with $\mathbb{P}(\Omega_0) = 1$ and on which (3.9) holds for all coalescing pairs $(\pi, \gamma) \in \mathcal{G}_u^\omega \times \mathcal{G}_v^\omega$. We claim that for $(s, \omega) \in D_0 \cap (\Omega_0 \times (0,1])$, $\boldsymbol{\pi}^{\widehat{B},u,s}(\omega) \curlywedge \boldsymbol{\pi}^{\widehat{B},v,s}(\omega)$.

By (3.10) there is a coalescing pair $(\boldsymbol{\pi}^{\widehat{B},u,s}, \gamma) \in \mathcal{G}_u^\omega \times \mathcal{G}_v^\omega$. Then by (3.9), $\boldsymbol{\pi}^{\widehat{B},v,s} \preceq \gamma$. Thus $\boldsymbol{\pi}^{\widehat{B},v,s} \preceq_a \boldsymbol{\pi}^{\widehat{B},u,s}$. A symmetric argument gives $\boldsymbol{\pi}^{\widehat{B},u,s} \preceq_a \boldsymbol{\pi}^{\widehat{B},v,s}$. Hence, the two paths coalesce. □

Because geodesics proceed up-right in directed last-passage percolation, it is natural to expect that a shift-covariant family of coalescing geodesics will be measurable with respect to the weights ahead of the root. The next result records this fact if the weights are i.i.d. We phrase this result on Ω, but an analogous statement holds on $\widehat{\Omega}$ with $\mathcal{F}_u^+$ replaced by $\mathfrak{S}_u^+$ if the geodesics are $\mathfrak{S}$ measurable.

Lemma 3.6. *Let $\{\pi^u : u \in \mathbb{Z}^2\}$ be a shift-covariant system of coalescing geodesics on Ω. Then for each $u \in \mathbb{Z}^2$, π^u is $\mathcal{F}_u^+$-measurable up to sets of measure zero.* △

Proof. Recall that we assumed the weights are i.i.d. in (2.2). Let $\overline{\mathbb{P}}(d\omega, d\widetilde{\omega})$ be the probability measure on Ω^2 that couples two copies of $\mathbb{P}$ as follows: $\widetilde{\omega}_x = \omega_x$ if $x \in \mathbb{Z}_+^2$ and $(\widetilde{\omega}_x : x \notin \mathbb{Z}_+^2)$ is independent of $(\omega_x : x \notin \mathbb{Z}_+^2)$ with the same distribution.

For every $(\omega,\widetilde\omega)$, $\mathcal{G}_0^\omega=\mathcal{G}_0^{\widetilde\omega}$ and therefore $\pi^0(\omega)\preceq\pi^0(\widetilde\omega)$ or $\pi^0(\omega)\succeq\pi^0(\widetilde\omega)$. We claim that both hold almost surely. To derive a contradiction, assume that $\overline{\mathbb{P}}\{\pi^0(\omega)\neq\pi^0(\widetilde\omega)\}>0$. By symmetry

$$\overline{\mathbb{P}}\{\pi^0(\omega)\precneqq\pi^0(\widetilde\omega)\}=\overline{\mathbb{P}}\{\pi^0(\omega)\succneqq\pi^0(\widetilde\omega)\}>0.$$

Denote the two events in the above display by A and A'. By the ergodic theorem, for $\overline{\mathbb{P}}$-almost every $(\omega,\widetilde\omega)$, $T_{ke_1}(\omega,\widetilde\omega)\in A$ occurs for infinitely many integers k and $T_{\ell e_1}(\omega,\widetilde\omega)\in A'$ occurs for infinitely many integers ℓ. Thus, for $\overline{\mathbb{P}}$-almost every $(\omega,\widetilde\omega)$, there exist integers $k<\ell$ such that $T_{ke_1}(\omega,\widetilde\omega)\in A$ and $T_{\ell e_1}(\omega,\widetilde\omega)\in A'$. This implies that

$$\pi^{ke_1}(\omega)\precneqq\pi^{ke_1}(\widetilde\omega)\curlywedge\pi^{\ell e_1}(\widetilde\omega)\precneqq\pi^{\ell e_1}(\omega).$$

These inequalities prevent the coalescence $\pi^{ke_1}(\omega)\curlywedge\pi^{\ell e_1}(\omega)$, thereby contradicting the assumption that $\{\pi^u:u\in\mathbb{Z}^2\}$ is a coalescing system of geodesics. Thus $\pi^0(\omega)=\pi^0(\widetilde\omega)$ almost surely. It follows from a standard measure theoretic fact (see e.g. Lemma A.2 in [25]) that this implies that there exists a $\sigma(\omega_x:x\in\mathbb{Z}^2_+)$ measurable function $F:\Omega\to\mathbb{X}_0$ so that $\pi^0(\omega)=F(\omega)$ $\mathbb{P}$-almost surely. ☐

Recall the definition (2.16). For $s\in(0,1]$ define

$$A^s(\omega,x,y)=A_{\boldsymbol{\pi}^{\widehat B,0,s}(\omega)}(\omega,x,y). \tag{3.11}$$

Lemma 3.7. *Fix $\widehat B\in\widehat{\mathcal{K}}$. Suppose $s\in(0,1]$ satisfies*

$$\mathbb{P}\big(\forall u,v\in\mathbb{Z}^2:\boldsymbol{\pi}^{\widehat B,u,s}\curlywedge\boldsymbol{\pi}^{\widehat B,v,s}\big)=1. \tag{3.12}$$

Then A^s, defined by (3.11), is a shift-covariant recovering cocycle. For any $u\in\mathbb{Z}^2$ and any $x,y\in u+\mathbb{Z}^2_+$, $A^s(x,y)$ is $\mathcal{F}_u^+$-measurable, up to sets of $\mathbb{P}$ measure zero. △

Proof. That A^s is a recovering cocycle comes from Theorem B.1. The shift-covariance (3.6) and the coalescence (3.12) imply that $\{\boldsymbol{\pi}^{\widehat B,u,s}:u\in\mathbb{Z}^2\}$ is a shift-covariant system of coalescing geodesics. By Lemma 3.6, $\boldsymbol{\pi}^{\widehat B,u,s}$ is $\mathfrak{G}_u^+$-measurable. Take $u\in\mathbb{Z}^2$ and $x,y\in u+\mathbb{Z}^2_+$. Then $L_{x,\boldsymbol{\pi}_n^{\widehat B,u,s}}$ and $L_{y,\boldsymbol{\pi}_n^{\widehat B,u,s}}$ are both $\mathfrak{G}_u^+$-measurable (for $n\geqslant u\cdot(e_1+e_2)$) and, consequently, so is $A_{\boldsymbol{\pi}^{\widehat B,u,s}(\omega)}(\omega,x,y)$. Lemma B.3 and the coalescence $\boldsymbol{\pi}^{\widehat B,u,s}\curlywedge\boldsymbol{\pi}^{\widehat B,0,s}$ implies

$$A^s(\omega,x,y)=A_{\boldsymbol{\pi}^{\widehat B,u,s}(\omega)}(\omega,x,y), \tag{3.13}$$

for all $x,y,u\in\mathbb{Z}^2$. Thus, we see that $A^s(\omega,x,y)$ is $\mathcal{F}_u^+$-measurable, for all $x,y\in u+\mathbb{Z}^2_+$.

(3.13) also implies the shift-covariance:

$$\begin{aligned}A^s(\omega,x+z,y+z)&=A_{\boldsymbol{\pi}^{\widehat B,z,s}(\omega)}(\omega,x+z,y+z)=A_{z+\boldsymbol{\pi}^{\widehat B,0,s}(T_z\omega)}(\omega,x+z,y+z)\\&=A_{\boldsymbol{\pi}^{\widehat B,0,s}(T_z\omega)}(T_z\omega,x,y)=A^s(T_z\omega,x,y),\end{aligned}$$

where the first equality used (3.13) with $u=z$, the second equality used the shift-covariance (3.6), and the third equality used the shift-covariance (2.17). ☐

Lemma 3.8. *Take $\widehat B\in\widehat{\mathcal{K}}_c$. Then for $\widehat{\mathbb{P}}$-almost every $\widehat\omega$ and all $u,x,y\in\mathbb{Z}^2$*

$$A_{\phi^{\widehat B,u}(\widehat\omega)}(\omega(\widehat\omega),x,y)=\widehat B(\widehat\omega,x,y). \tag{3.14}$$ △

Proof. Since $\phi^{\widehat B,u}(\widehat\omega)$, $\phi^{\widehat B,x}(\widehat\omega)$, and $\phi^{\widehat B,y}(\widehat\omega)$ all coalesce, there exists a $z\in\mathbb{Z}^2$ that is on all three geodesics. The recovery property gives that $\widehat B(\widehat\omega,x,z)=L_{x,z}(\omega(\widehat\omega))=A_{\phi^{\widehat B,x}(\widehat\omega)}(\omega(\widehat\omega),x,z)$ and the coalescence of the geodesics gives $A_{\phi^{\widehat B,x}(\widehat\omega)}(\omega(\widehat\omega),x,z)=A_{\phi^{\widehat B,u}(\omega(\widehat\omega))}(\omega(\widehat\omega),x,z)$. Similarly, we have $\widehat B(\widehat\omega,y,z)=A_{\phi^{\widehat B,y}(\widehat\omega)}(\omega(\widehat\omega),y,z)=A_{\phi^{\widehat B,u}(\widehat\omega)}(\omega(\widehat\omega),y,z)$. (3.14) now follows from the cocycle property. ☐

Lemma 3.9. *Fix $\widehat{B} \in \widehat{\mathcal{K}}_c$. The distribution of $(\omega(\widehat{\omega}), \widehat{B}(\widehat{\omega}))$ under $\widehat{\mathbb{P}}(d\widehat{\omega})$ is the same as the distribution of $(\omega, A^s(\omega))$ under $\mathbb{P}(d\omega) \otimes \mathcal{L}(ds)$.* △

Proof. Apply Lemma 3.3 to the measurable mapping $(\omega, \gamma) \mapsto (\omega, A_\gamma(\omega))$ to get that the distribution of $(\omega, A_{\boldsymbol{\pi}^{\widehat{B},u,s}}(\omega))$ under $\mathbb{P}(d\omega) \otimes \mathcal{L}(ds)$ is the same as that of $(\omega(\widehat{\omega}), A_{\phi^{\widehat{B},u}(\widehat{\omega})}(\omega(\widehat{\omega})))$ under $\widehat{\mathbb{P}}(d\widehat{\omega})$. The claim then follows from (3.13) and (3.14). □

Given a shift-covariant $L^1(\widehat{\Omega}, \widehat{\mathfrak{S}}, \widehat{\mathbb{P}})$ recovering cocycle $\widehat{B}$, recall the random vector $\mathbf{h}(\widehat{B}, \widehat{\omega})$ defined in (2.20). Note that if $B \in \mathcal{K}$, from the assumption that $\mathbb{P}$ is i.i.d., $\mathbf{h}(B)$ is a deterministic vector, i.e. $\mathbf{h}(B) = \mathbb{E}[\mathbf{h}(B)]$, $\mathbb{P}$-almost surely.

Lemma 3.10. *Fix $\widehat{B} \in \widehat{\mathcal{K}}_c$. Let $h = \widehat{\mathbb{E}}[\mathbf{h}(\widehat{B})]$. Assume that $\widehat{\mathbb{P}}\{\widehat{\omega} : \mathbf{h}(\widehat{B}, \widehat{\omega}) = h\} = 1$. Then for every $s \in (0,1]$, $A^s \in \mathcal{K}_c^+$ and $\mathbf{h}(A^s) = h$, $\mathbb{P}$-almost surely. Furthermore, $\mathbb{P}\{\forall s, t \in (0,1] : A^s = A^t\} = 1$.* △

Proof. By Lemmas 3.5 and 3.7 there exists a Borel set $D_1 \subset (0,1]$ such that $\mathcal{L}(D_1) = 1$ and for each $s \in D_1$, A^s is a forward measurable shift-covariant recovering cocycle with coalescing geodesics.

The assumption $\mathbf{h}(\widehat{B}, \widehat{\omega}) = h$, identity (3.14), and the cocycle shape theorem (2.21) give for $\widehat{\mathbb{P}}$-almost every $\widehat{\omega}$

$$\varlimsup_{|x|_1 \to \infty} \frac{|A_{\phi^{\widehat{B},0}(\widehat{\omega})}(\omega(\widehat{\omega}), 0, x) + h \cdot x|}{|x|_1} = 0.$$

Then Lemma 3.3 says that there exists a Borel set $D_2 \subset (0,1]$ with $\mathcal{L}(D_2) = 1$ and such that for each $s \in D_2$,

$$\mathbb{P}\Big\{ \varlimsup_{|x|_1 \to \infty} \frac{|A_{\boldsymbol{\pi}^{\widehat{B},0,s}(\omega)}(\omega, 0, x) + h \cdot x|}{|x|_1} = 0 \Big\} = 1.$$

Thus, for each $s \in D_1 \cap D_2$, A^s is a forward-measurable shift-covariant recovering cocycle (on Ω) that satisfies

$$\varlimsup_{|x|_1 \to \infty} \frac{|A^s(0,x) + h \cdot x|}{|x|_1} = 0, \quad \mathbb{P}\text{-almost surely.}$$

Then $n^{-1}A^s(0, ne_1) \to h \cdot e_1$ and $n^{-1}A^s(0, ne_2) \to h \cdot e_2$. The shift-covariance, the cocycle property, the inequalities $A^s(\omega, x, x + e_i) \geqslant \omega_x \in L^1(\widehat{\mathbb{P}})$, and Birkhoff's ergodic theorem give integrability and $\mathbf{h}(A^s) = h$, so in particular $A^s \in \mathcal{K}_c^+$.

So far, we proved that for $\mathcal{L}$-almost every $s \in (0,1]$, $A^s \in \mathcal{K}_c^+$ and $\mathbf{h}(A^s) = h$. This implies that there exists a countable dense set $D_3 \subset (0,1]$ such that for every $s \in D_3$, $A^s \in \mathcal{K}_c^+$ and $\mathbf{h}(A^s) = h$.

The monotonicity of $\boldsymbol{\pi}^{\widehat{B},0,s}$ in s implies the monotonicity of A^s by Lemma B.3. This monotonicity and the equal expectations imply that $\mathbb{P}$-almost surely, for any $s, t \in D_3$, $A^s = A^t$. Using the monotonicity one more time extends this to all $s, t \in (0,1]$. □

From Lemmas 3.9 and 3.10 we can now establish the strong existence claimed in Theorem 2.14.

Proof of Theorem 2.14. We set $B(x,y) = \int_0^1 A^s(x,y)ds$ so that B is a Borel-measurable random field on Ω. By Lemma 3.10, $\mathbb{P}$-almost surely, $A^s = B$ for all $s \in (0,1]$ and so $B \in \mathcal{K}_c^+$. Lemma 3.9 now implies that the joint distribution of $(\omega(\widehat{\omega}), \widehat{B}(\widehat{\omega}))$ under $\widehat{\mathbb{P}}$ is the same as that of $(\omega, B(\omega))$ under $\mathbb{P}$. It follows that $\widehat{\mathbb{P}}$-almost surely, $\widehat{B}(\widehat{\omega}) = B(\omega(\widehat{\omega}))$. This last claim is essentially Lemma 2.2 in [25], but we include the proof. We can uniquely (up to sets of $\mathbb{P}$-measure zero) factorize the joint distribution of $(\omega(\widehat{\omega}), \widehat{B}(\widehat{\omega}))$ under $\widehat{\mathbb{P}}$ as the distribution $\mathbb{P}(d\omega)$ of ω together with a transition kernel $\eta(db|\omega)$ that represents the conditional distribution of $\widehat{B}$ given ω. Do the same on the other side of the equality in distribution to see that $\eta(db|\omega) = \delta_{B(\omega)}(db)$ $\mathbb{P}$-almost surely. Thus $\widehat{\mathbb{P}}\{\widehat{\omega} : \widehat{B}(\widehat{\omega}) = B(\omega(\widehat{\omega}))\} = \widehat{\mathbb{E}}[\widehat{\mathbb{P}}(\widehat{B} = B | \mathfrak{S})] = 1$. □

3.3. **Strong uniqueness.** Recall the order relation on cocycles: for $B, B' \in \mathbb{R}^{\mathbb{Z}^2}$, $B \preceq B'$ means $B(x, x+e_1) \geqslant B'(x, x+e_1)$ and $B(x, x+e_2) \leqslant B'(x, x+e_2)$ for all $x \in \mathbb{Z}^2$. Extend this order relation to an order relation $\preceq_{\text{as}}$ on $\mathcal{K}$ by defining

$$B_1 \preceq_{\text{as}} B_2 \quad \text{if } \mathbb{P}\{B_1 \preceq B_2\} = 1.$$

Clearly, if $B_1 \preceq_{\text{as}} B_2$ and $B_2 \preceq_{\text{as}} B_1$, then $B_1 = B_2$ as random variables, i.e. with $\mathbb{P}$-probability one. The next lemma shows that $\preceq_{\text{as}}$ is a total order on $\mathcal{K}_c$.

Lemma 3.11. *Let $B_1, B_2 \in \mathcal{K}_c$. Then either $\mathbb{P}(B_1 \preceq B_2) = 1$ or $\mathbb{P}(B_2 \preceq B_1) = 1$.* △

Proof. On the full $\mathbb{P}$-probability event where $\phi^{B_1,u} \curlywedge \phi^{B_1,v}$ and $\phi^{B_2,u} \curlywedge \phi^{B_2,v}$ for all $u, v \in \mathbb{Z}^2$, we have that $\phi^{B_1,0} \preceq \phi^{B_2,0}$ implies $\phi^{B_1,u} \preceq \phi^{B_2,u}$ for all $u \in \mathbb{Z}^2$. Thus the event $\{\phi^{B_1,0} \preceq \phi^{B_2,0}\}$ is shift-invariant and thereby has $\mathbb{P}$-probability of 0 or 1 by the ergodicity of $\mathbb{P}$. The same holds for the event $\{\phi^{B_1,0} \succeq \phi^{B_2,0}\}$.

Since the $\preceq$ is a total order on $\mathcal{G}_0^\omega$, we have that $\mathbb{P}$-almost surely, either $\phi^{B_1,0} \preceq \phi^{B_2,0}$ or $\phi^{B_1,0} \succeq \phi^{B_2,0}$. Since we just showed that these two events are trivial, we get that either $\phi^{B_1,0} \preceq \phi^{B_2,0}$, $\mathbb{P}$-almost surely, or $\phi^{B_1,0} \succeq \phi^{B_2,0}$, $\mathbb{P}$-almost surely. Lemmas 3.8 and B.4 imply then that either $B_1 \preceq B_2$, $\mathbb{P}$-almost surely, or $B_1 \succeq B_2$, $\mathbb{P}$-almost surely. □

Strong uniqueness follows.

Proof of Theorem 2.15. By the total order in Lemma 3.11, equality $\mathbf{h}(B_1) = \mathbf{h}(B_2)$ of the means is sufficient for almost sure equality. □

The following is an immediate corollary of Theorem 2.14.

Corollary 3.12. $\mathcal{K}_c = \mathcal{K}^+ = \mathcal{K}_c^+$. △

Proof. By an adaptation of the Licea-Newman [27] coalescence argument given in Theorem A.1 in [13], $\mathcal{K}^+ \subset \mathcal{K}_c$. Forward measurability gives the finite energy condition used in the coalescence proof. The previous inclusion then gives $\mathcal{K}^+ \subset \mathcal{K}_c^+$, so we have $\mathcal{K}_c^+ = \mathcal{K}^+$. Theorem 2.14 implies that also $\mathcal{K}_c \subset \mathcal{K}_c^+$ and hence $\mathcal{K}_c = \mathcal{K}^+ = \mathcal{K}_c^+$. □

Recall Theorem 2.7 and the consequence that $\mathcal{H}$ contains $\{-\nabla g(\xi\square) : \xi \in \mathcal{U}, \square \in \{+,-\}\}$. By Theorem 2.15, for each $h \in \mathcal{H}$ there exists a unique $B^h \in \mathcal{K}_c^+$ such that $\mathbf{h}(B^h) = h$. The following is a direct consequence of Lemma 3.11 and the fact that if $B_1 \preceq B_2$ in $\mathcal{K}$, then $\mathbf{h}(B_1) \preceq \mathbf{h}(B_2)$.

Lemma 3.13. *For any $h, h' \in \mathcal{H}$, either we have $h \preceq h'$ and $B^h \preceq B^{h'}$ or we have $h' \preceq h$ and $B^{h'} \preceq B^h$. In particular, $\preceq$ is a total order on $\mathcal{H}$ and $\mathcal{H} \ni h \mapsto B^h$ is nondecreasing.* △

Let $\mathcal{H}^0$ be a countable dense subset of $\mathcal{H}$. Using the monotonicity in Lemma 3.13 and the cocycle property (2.13) that B^h, $h \in \mathcal{H}^0$, satisfy, define the process

$$\overline{B}^{h-}(x,y) = \lim_{\mathcal{H}^0 \ni h' \nearrow h} B^{h'}(x,y) \quad \text{and} \quad \overline{B}^{h+}(x,y) = \lim_{\mathcal{H}^0 \ni h' \searrow h} B^{h'}(x,y),$$

for $x, y \in \mathbb{Z}^2$ and $h \in \mathcal{H}$. Then for $\mathbb{P}$-almost every ω, for any $h \in \mathcal{H}$ and $\square \in \{-,+\}$, $\overline{B}^{h\square}$ is a recovering cocycle.

The following lemma says that for a fixed $h \in \mathcal{H}$, the above definitions recover B^h.

Lemma 3.14. *Fix $h \in \mathcal{H}$. Then $\mathbb{P}$-almost surely, for any $x, y \in \mathbb{Z}^2$, $\overline{B}^{h-}(x,y) = \overline{B}^{h+}(x,y) = B^h(x,y)$. In particular, this holds for $\mathbb{P}$-almost every ω, simultaneously for all $h \in \mathcal{H}^0$.* △

Proof. We have that $\overline{B}^{h-} \in \mathcal{K}$, $\overline{B}^{h-} \preceq B^h$, and by monotone convergence, $\mathbf{h}(\overline{B}^{h-}) = \mathbf{h}(B^h)$. This implies that $\overline{B}^{h-} = B^h$, $\mathbb{P}$-almost surely. The case of $\overline{B}^{h+}$ is similar. □

Proof of Theorem 2.17. In view of the above lemma, we will drop the overline from $\overline{B}^{h\square}$ and just write $B^{h\square}$. Furthermore, when $B^{h-} = B^{h+}$, we drop the sign distinction and write B^h. In particular, for each $h \in \mathcal{H}$, $\mathbb{P}$-almost surely, $B^{h-} = B^{h+} = B^h$. The claimed monotonicity follows from Lemma 3.13. Using dominated convergence, this implies that the mean $-h$ condition holds. The cocycle, recovery, and covariance properties are closed under limits. By almost sure left- and right- continuity, uniqueness for fixed h implies uniqueness of the process. $\square$

We close this section with the observation that Theorem 2.14 and Corollary 3.12 imply that

$$\mathcal{H} = \{\mathbf{h}(B) : B \in \mathcal{K}_c\} = \{\mathbf{h}(B) : B \in \mathcal{K}^+\} = \big\{h \in \mathbb{R}^2 : \widehat{B} \in \widehat{\mathcal{K}}_c,\ \widehat{\mathbb{P}}(\mathbf{h}(\widehat{B}) = h) = 1\big\}. \tag{3.15}$$

4. Shift-covariant coalescing systems of random geodesics and cocycles

In [1], the authors start from a shift-covariant coalescing system of geodesics and use these to construct Busemann functions. We instead started with a field of generalized Busemann functions and then used those to build a system of coalescing geodesics. We show in this section that these two approaches are equivalent.

We say that a shift-covariant system of geodesics is *non-crossing* if

$$\widehat{\mathbb{P}}\big\{\forall u, v \in \mathbb{Z}^2 : \widehat{\pi}^u(\widehat{\omega}) \curlywedge \widehat{\pi}^v(\widehat{\omega}) \text{ or } \widehat{\pi}^u(\widehat{\omega}) \cap \widehat{\pi}^v(\widehat{\omega}) = \varnothing\big\} = 1.$$

The recovering cocycle $A_{\widehat{\pi}^u}$ generated by the random geodesic $\widehat{\pi}^u$ is a function on $\widehat{\Omega}$ defined in terms of (2.16) by

$$A_{\widehat{\pi}^u}(\widehat{\omega}, x, y) = A_{\widehat{\pi}^u(\widehat{\omega})}(\omega(\widehat{\omega}), x, y). \tag{4.1}$$

Recall the space of shift-covariant systems of coalescing geodesics, $\widehat{\mathbb{\Pi}}_c$.

Lemma 4.1. *If $\widehat{\pi}^\bullet \in \widehat{\mathbb{\Pi}}_c$, then $A_{\widehat{\pi}^0}$ is shift-covariant.* $\triangle$

Proof. If $\widehat{\pi}^\bullet \in \widehat{\mathbb{\Pi}}_c$, then using (2.17) in the second equality and the coalescence $\widehat{\pi}^0 \curlywedge \widehat{\pi}^z$ with Lemma B.3 in the last equality,

$$\begin{aligned} A_{\widehat{\pi}^0}(\widehat{T}_z\widehat{\omega}, x, y) &= A_{\widehat{\pi}^0(\widehat{T}_z\widehat{\omega})}(T_z\omega(\widehat{\omega}), x, y) = A_{z+\widehat{\pi}^0(\widehat{T}_z\widehat{\omega})}(\omega(\widehat{\omega}), x+z, y+z) \\ &= A_{\widehat{\pi}^z(\widehat{\omega})}(\omega(\widehat{\omega}), x+z, y+z) = A_{\widehat{\pi}^z(\widehat{\omega})}(\omega(\widehat{\omega}), x+z, y+z) \\ &= A_{\widehat{\pi}^0(\widehat{\omega})}(\omega(\widehat{\omega}), x+z, y+z). \end{aligned}$$ $\square$

Let $\widehat{\mathcal{K}}_{\widehat{\mathbb{\Pi}}_c}$ denote the set of shift-covariant $L^1(\widehat{\mathfrak{S}}, \widehat{\mathbb{P}})$ recovering cocycles generated by systems of shift-covariant coalescing geodesics.

Lemma 4.2. *Let $\widehat{\pi}^\bullet$ and $\widehat{\gamma}^\bullet$ be two covariant non-crossing systems of geodesics such that $\widehat{\mathbb{P}}\{\widehat{\pi}^0 \preceq \widehat{\gamma}^0\} = 1$. Assume that either $\widehat{\pi}^\bullet \in \widehat{\mathbb{\Pi}}_c$ or $\widehat{\gamma}^\bullet \in \widehat{\mathbb{\Pi}}_c$. Assume also that there exists a cocycle $\widehat{B} \in \widehat{\mathcal{K}}$ such that both $\widehat{\pi}^u$ and $\widehat{\gamma}^u$ are $\widehat{B}$-geodesics, for all $u \in \mathbb{Z}^2$. That is,*

$$\widehat{\mathbb{P}}\big\{\widehat{\omega} : \forall u \in \mathbb{Z}^2, \forall n \geqslant u \cdot (e_1 + e_2) : \widehat{B}(\widehat{\omega}, \widehat{\pi}^u_n(\widehat{\omega}), \widehat{\pi}^u_{n+1}(\widehat{\omega})) = \omega_{\widehat{\pi}^u_n(\widehat{\omega})}(\widehat{\omega})\big\} = 1 \tag{4.2}$$

and

$$\widehat{\mathbb{P}}\big\{\widehat{\omega} : \forall u \in \mathbb{Z}^2, \forall n \geqslant u \cdot (e_1 + e_2) : \widehat{B}(\widehat{\omega}, \widehat{\gamma}^u_n(\widehat{\omega}), \widehat{\gamma}^u_{n+1}(\widehat{\omega})) = \omega_{\widehat{\gamma}^u_n(\widehat{\omega})}(\widehat{\omega})\big\} = 1. \tag{4.3}$$

Then $\widehat{\mathbb{P}}\{\forall u \in \mathbb{Z}^2 : \widehat{\pi}^u = \widehat{\gamma}^u\} = 1$. $\triangle$

Proof. The two cases are proved similarly. We work out the case $\widehat{\pi}^\bullet \in \widehat{\mathbb{\Pi}}_c$. By (2.23), applied to both $\widehat{\pi}^u$ and $\widehat{\gamma}^u$, it is enough to prove $\widehat{\mathbb{P}}\{\widehat{\pi}^0 = \widehat{\gamma}^0\} = 1$. We show that $\mathcal{E}_\infty = \{\widehat{\omega} : \widehat{\pi}^0(\widehat{\omega}) \precneqq \widehat{\gamma}^0(\widehat{\omega})\}$ is a zero $\widehat{\mathbb{P}}$-probability event. To arrive at a contradiction, suppose $\widehat{\mathbb{P}}(\mathcal{E}_\infty) > 0$.

The events $\mathcal{E}_m = \{\widehat{\pi}^0_{0:m} \neq \widehat{\gamma}^0_{0:m}\}$ increase up to $\mathcal{E}_\infty$ as $m \nearrow \infty$. Hence we can pick $m \in \mathbb{N}$ so that $\widehat{\mathbb{P}}(\mathcal{E}_m) > 0$. By Poincaré recurrence, $\mathbb{P}$-almost surely $\mathcal{E}_m \subset \bigcup_{\ell \geqslant N} \widehat{T}_{-\ell e_1}\mathcal{E}_m$ for each $N \in \mathbb{N}$ [24, Section 1.3]. Then

$$\begin{aligned}\widehat{\mathbb{P}}\Big(\bigcup_{k\geqslant 1}\bigcup_{\ell\geqslant k+m}\widehat{T}_{-ke_1}\mathcal{E}_m \cap \widehat{T}_{-\ell e_1}\mathcal{E}_m\Big) &= \widehat{\mathbb{P}}\Big(\bigcup_{k\geqslant 1}\widehat{T}_{-ke_1}\Big[\mathcal{E}_m \cap \bigcup_{j\geqslant m}\widehat{T}_{-je_1}\mathcal{E}_m\Big]\Big)\\ &= \widehat{\mathbb{P}}\Big(\bigcup_{k\geqslant 1}\widehat{T}_{-ke_1}\mathcal{E}_m\Big) > 0.\end{aligned} \tag{4.4}$$

Let $\widehat{\Omega}_0$ be the full $\widehat{\mathbb{P}}$-probability event on which the geodesics $\{\widehat{\pi}^u(\widehat{\omega})\}_{u\in\mathbb{Z}^2}$ coalesce, $\{\widehat{\gamma}^u(\widehat{\omega})\}_{u\in\mathbb{Z}^2}$ are non-crossing, $\widehat{\pi}^0(\widehat{\omega}) \preceq \widehat{\gamma}^0(\widehat{\omega})$, and both events in (4.2) and (4.3) hold. The proof is concluded by showing that

$$\widehat{\mathbb{P}}\big(\widehat{\Omega}_0 \cap \widehat{T}_{-ke_1}\mathcal{E}_m \cap \widehat{T}_{-\ell e_1}\mathcal{E}_m\big) = 0 \quad \forall\, \ell \geqslant k+m,\ k \geqslant 1. \tag{4.5}$$

Since $\widehat{\mathbb{P}}(\widehat{\Omega}_0) = 1$, this contradicts (4.4), which in turn forces $\widehat{\mathbb{P}}(\mathcal{E}_\infty) = 0$.

Let $\widehat{\omega} \in \widehat{\Omega}_0 \cap \widehat{T}_{-ke_1}\mathcal{E}_m \cap \widehat{T}_{-\ell e_1}\mathcal{E}_m$ with $\ell \geqslant k+m > m$. Then the non-crossing of the geodesics $\widehat{\gamma}^0(\widehat{\omega})$, $\widehat{\gamma}^{ke_1}(\widehat{\omega})$, and $\widehat{\gamma}^{\ell e_1}(\widehat{\omega})$ implies $\widehat{\gamma}^0(\widehat{\omega}) \preceq \widehat{\gamma}^{ke_1}(\widehat{\omega}) \preceq \widehat{\gamma}^{\ell e_1}(\widehat{\omega})$. This, $\widehat{\pi}^0(\widehat{\omega}) \precnsim \widehat{\gamma}^0(\widehat{\omega})$, and the coalescence of $\{\widehat{\pi}^u(\widehat{\omega})\}_{u\in\mathbb{Z}^2}$, together imply

$$\widehat{\pi}^u(\widehat{\omega}) \precnsim \widehat{\gamma}^v(\widehat{\omega}) \quad \text{for} \quad u,v \in \{ke_1, \ell e_1\}. \tag{4.6}$$

Next, $\widehat{\omega} \in \widehat{T}_{-ke_1}\mathcal{E}_m \cap \widehat{T}_{-\ell e_1}\mathcal{E}_m$ says that $\widehat{\pi}^u(\widehat{\omega})$ and $\widehat{\gamma}^u(\widehat{\omega})$ separate in the first m steps, for both $u \in \{ke_1, \ell e_1\}$. Once separated, rightmost geodesics from u cannot meet again, and so

$$\widehat{\pi}^u_{o+m:\infty}(\widehat{\omega}) \cap \widehat{\gamma}^u_{o+m:\infty}(\widehat{\omega}) = \varnothing \quad \text{for both} \quad u \in \{ke_1, \ell e_1\}. \tag{4.7}$$

We draw the conclusions from the observations above. Since $\widehat{\pi}^{\ell e_1}(\widehat{\omega})$ starts strictly to the right of $\widehat{\gamma}^{ke_1}(\widehat{\omega})$ but by (4.6) ends up strictly to its left, $\widehat{\gamma}^{ke_1}(\widehat{\omega})$ and $\widehat{\pi}^{\ell e_1}(\widehat{\omega})$ must intersect. Denote their first intersection point by $z = \widehat{\gamma}^{ke_1}_n(\widehat{\omega}) = \widehat{\pi}^{\ell e_1}_n(\widehat{\omega})$. Since $z \in \widehat{\pi}^{\ell e_1}(\widehat{\omega})$ we have $z \geqslant \ell e_1$. This implies $n - k = (z - ke_1)\cdot(e_1+e_2) \geqslant \ell - k \geqslant m$. Since $z \in \widehat{\gamma}^{ke_1}(\widehat{\omega})$, (4.7) implies $z \notin \widehat{\pi}^{ke_1}(\widehat{\omega})$.

Let x denote the coalescence point of $\widehat{\pi}^{ke_1}(\widehat{\omega})$ and $\widehat{\pi}^{\ell e_1}(\widehat{\omega})$. Since $z \notin \widehat{\pi}^{ke_1}(\widehat{\omega})$, z must lie on $\widehat{\pi}^{\ell e_1}(\widehat{\omega})$ before x. Thus in $\mathbb{Z}^2$ ordering

$$ke_1 \leqslant x \quad \text{and} \quad \ell e_1 \leqslant z \leqslant x.$$

Since $\widehat{\omega}$ is in the event in (4.2) and since x is on $\widehat{\pi}^{ke_1}(\widehat{\omega})$ and $z \leqslant x$ are both on $\widehat{\pi}^{\ell e_1}(\widehat{\omega})$, we have $L_{ke_1,x}(\omega(\widehat{\omega})) = \widehat{B}(\widehat{\omega}, ke_1, x)$ and $L_{z,x}(\omega(\widehat{\omega})) = \widehat{B}(\widehat{\omega}, z, x)$. Similarly, since z is on $\widehat{\gamma}^{ke_1}(\widehat{\omega})$ and $\widehat{\omega}$ is in the event in (4.3) we have $L_{ke_1,z}(\omega(\widehat{\omega})) = \widehat{B}(\widehat{\omega}, ke_1, z)$. By the cocycle property,

$$\begin{aligned}&L_{ke_1,x}(\omega(\widehat{\omega})) - L_{ke_1,z}(\omega(\widehat{\omega})) - L_{z,x}(\omega(\widehat{\omega}))\\ &\qquad = \widehat{B}(\widehat{\omega}, ke_1, x) - \widehat{B}(\widehat{\omega}, ke_1, z) - \widehat{B}(\widehat{\omega}, z, x) = 0.\end{aligned}$$

This implies that z is on some geodesic from ke_1 to x. But since $z \notin \widehat{\pi}^{ke_1}(\widehat{\omega})$, z lies strictly to the right of $\widehat{\pi}^{ke_1}(\widehat{\omega})$ because $z \in \widehat{\pi}^{\ell e_1}(\widehat{\omega})$. We have a contradiction because by definition $\widehat{\pi}^{ke_1}(\widehat{\omega})$ gives the rightmost geodesic from ke_1 to x. This contradiction verifies (4.5). ☐

Lemma 4.3.

(a) *Let $\widehat{\pi}^\bullet \in \widehat{\Pi}_c$ and let $\widehat{B} = A_{\widehat{\pi}^0}$. Then $\widehat{\mathbb{P}}\{\forall u \in \mathbb{Z}^2 : \phi^{\widehat{B},u} = \widehat{\pi}^u\} = 1$.*

(b) *We have the equality of the cocycle spaces $\widehat{\mathcal{K}}_{\widehat{\Pi}_c} = \widehat{\mathcal{K}}_c$.* △

Proof. Part (a). The system $\{\phi^{\widehat{B},u}\}$ generated by $\widehat{B}$ is noncrossing because if $\phi^{\widehat{B},u}$ and $\phi^{\widehat{B},v}$ ever intersect, their subsequent steps are determined by $\widehat{B}$ and hence identical. Furthermore, $\phi^{\widehat{B},0}$ is by definition the geodesic of $\widehat{B}$ that takes e_1 steps when there are ties $\widehat{B}(x, x+e_1) = \widehat{B}(x, x+e_2)$, and hence $\widehat{\pi}^0 \preceq \phi^{\widehat{B},0}$ by Lemma B.5. Both (4.2) and (4.3) are satisfied. The claim $\phi^{\widehat{B},u} = \widehat{\pi}^u$ follows from Lemma 4.2.

Part (b). The assumption $\widehat{\pi}^\bullet \in \widehat{\Pi}_c$ and the conclusion $\phi^{\widehat{B},u} = \widehat{\pi}^u$, for all $u \in \mathbb{Z}^2$, imply in particular that the geodesics $\{\phi^{\widehat{B},u}\}_{u\in\mathbb{Z}^2}$ form almost surely a coalescing family. This implies $\widehat{\mathcal{K}}_{\widehat{\Pi}_c} \subset \widehat{\mathcal{K}}_c$.

It remains to show the opposite inclusion $\widehat{\mathcal{K}}_c \subset \widehat{\mathcal{K}}_{\widehat{\Pi}_c}$. Let $\widehat{B} \in \widehat{\mathcal{K}}_c$. This means that the geodesics $\phi^{\widehat{B},u}$ coalesce, that is, $\phi^{\widehat{B},\bullet} \in \widehat{\Pi}_c$. By Lemma B.4, $\widehat{B}$ equals the cocycle $A_{\phi^{\widehat{B},0}}$ generated by $\phi^{\widehat{B},0}$, which says that $\widehat{B} \in \widehat{\mathcal{K}}_{\widehat{\Pi}_c}$. □

Appendix A. Technical lemmas

A.1. **Measurability of the geodesic tree.** We begin with measurability of $\mathcal{G}^\omega_u$. For $u, x \in \mathbb{Z}^2$ with $x \geqslant u$ and $i \in \{1, 2\}$ define the $\mathfrak{S}^+_u$-measurable random variables

$$S^i_{u,x}(\omega) = \begin{cases} 1 & \text{if } \forall m \geqslant x \cdot (e_1 + e_2) + 1 \ \exists \sigma \in \mathcal{G}^\omega_{u,m} : x, x + e_i \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Think of $S^i_{u,x} = 1$ as opening the edge $(x, x + e_i)$. Note that if $S^i_{u,x} = 1$ then $S^j_{u,x+e_i} = 1$ for some (or both) $j \in \{1, 2\}$. Starting at u and following open edges gives a geodesic ray and, conversely, the edges of any geodesic ray started at u are all open. Hence $S_u = \{S^i_{u,x} : x \in u + \mathbb{Z}^2_+, i \in \{1, 2\}\}$ gives a $\mathfrak{S}^+_u$-measurable way to encode the random tree $\mathcal{G}^\omega_u$. It is not hard to see that $\mathcal{G}^\omega_u$ is also a closed set in the product-discrete topology on paths. Since the space of paths rooted at u is compact in this topology, this implies that $\mathcal{G}^\omega_u$ is compact.

A.2. **Measurability on $\mathbb{X}_u$.** Without loss of generality, we consider $u = 0$ for notational simplicity. We begin with some preliminary observations about the path space $\mathbb{X}_0$. Recall that for $\gamma, \pi \in \mathbb{X}_0$, the metric distance between γ and π is $d(\gamma, \pi) = \sum_{i=0}^\infty 2^{-(i+1)} \mathbb{1}_{\{\gamma_i \neq \pi_i\}}$.

We prove measurability of the expression in (3.2).

Lemma A.1. *If ν^ω is a regular conditional distribution on $(\widehat{\Omega}, \widehat{\mathfrak{S}})$ given $\mathfrak{S}$, then the function*

$$(\widehat{\omega}, \rho) \in \widehat{\Omega} \times \mathbb{X}_0 \mapsto \nu^{\omega(\widehat{\omega})}(\phi^{\widehat{B},u} \preceq \rho) \in [0, 1]$$

is jointly $(\mathfrak{S}, \mathcal{B}(\mathbb{X}_0))$-measurable. △

Proof. Denote by $F(\widehat{\omega}, \rho)$ the function in the statement. It suffices to show that F is the limit of jointly measurable functions. We begin with the observation that for each $n \in \mathbb{N}$ and $\widehat{\omega} \in \widehat{\Omega}$

$$\rho \mapsto F_n(\widehat{\omega}, \rho) = \nu^{\omega(\widehat{\omega})}(\phi^{\widehat{B},u}_{0:n} \preceq \rho_{0:n})$$

is continuous. To see this, note that if $\rho^k \to \rho$, then for all sufficiently large k, $\rho^k_{0:n} = \rho_{0:n}$. $\mathbb{X}_0$ is separable, being compact, so it follows from $\mathfrak{S}$ measurability of $\nu^{\omega(\widehat{\omega})}$ that F_n is $(\mathfrak{S}, \mathcal{B}(\mathbb{X}_0))$-jointly measurable. See, e.g., Lemma 4.51 in [2].

We have that

$$\{\phi^{\widehat{B},u} \preceq \rho\} = \bigcap_n \{\phi^{\widehat{B},u}_{0:n} \preceq \rho_{0:n}\}.$$

From continuity of measure, $F(\widehat{\omega}, \rho) = \lim_n F_n(\widehat{\omega}, \rho)$. Therefore F is measurable. □

Corollary A.2. *For $s \in [0, 1]$, the $\mathbb{X}_u$-valued random variable $\boldsymbol{\pi}^{\widehat{B},u,s}$ in (3.2) is $\mathfrak{S}$-measurable.* △

Proof. Again, we take u to be the origin for simplicity. Fix a deterministic finite admissible path $\sigma_{0:n}$ rooted at the origin. Then the path defined by $\rho_{\sigma_{0:n}}(\omega(\widehat\omega)) = \sup\{\pi \in \mathcal{G}_0^{\omega(\widehat\omega)} : \pi_{0:n} = \sigma_{0:n}\}$ if such a π exists and $\rho_{\sigma_{0:n}}(\omega(\widehat\omega)) = \mathbb{Z}_+ e_1$ otherwise is measurable by the results of the previous sub-section. Moreover, this collection enumerates RI_0^ω. It then follows from the previous result that

$$\widehat\omega \mapsto \nu^{\omega(\widehat\omega)}(\phi^{\widehat B,0} \preceq \rho_{\sigma_{0:n}}(\omega(\widehat\omega))) \in [0,1]$$

is $\mathfrak{S}$-measurable. Then for each $s \in (0,1)$ and each finite admissible path $\sigma_{0:n}$, the event

$$\{\nu^{\omega(\widehat\omega)}(\phi^{\widehat B,0} \preceq \rho_{\sigma_{0:n}}(\omega(\widehat\omega))) \geqslant s\}$$

is $\mathfrak{S}$-measurable. Fix some finite path $\pi_{0:m}$. The event

$$\{\widehat\omega : \boldsymbol{\pi}^{\widehat B,u,s}_{0:m}(\widehat\omega) = \pi_{0:m}\} \tag{A.1}$$

is measurable because it is equal to the intersection of two events: (i) there exists a path $\sigma_{0:n}$ which is on $\mathcal{G}_0^{\omega(\widehat\omega)}$, $n \geqslant m$, where $\pi_{0:m} \subseteq \sigma_{0:n}$ for which $\nu^{\omega(\widehat\omega)}(\phi^{\widehat B,u} \preceq \rho_{\sigma_{0:n}}(\omega(\widehat\omega))) \geqslant s$,

$$\bigcup_{n\geqslant m} \bigcup_{\substack{\sigma_{0:n} \text{ up-right:}\\ \sigma_{0:m}=\pi_{0:m}}} \{\sigma_{0:n} \text{ is on } \mathcal{G}_0^{\omega(\widehat\omega)}, \nu^{\omega(\widehat\omega)}(\phi^{\widehat B,u} \preceq \rho_{\sigma_{0:n}}(\omega(\widehat\omega))) \geqslant s\};$$

and (ii) for any $n \geqslant m$ and any finite path $\sigma_{0:n}$ which lies on the tree $\mathcal{G}_0^{\omega(\widehat\omega)}$ with the property that $\sigma_{0:m} \precnsim \pi_{0:m}$, we have $\nu^{\omega(\widehat\omega)}(\phi^{\widehat B,u} \preceq \rho_{\sigma_{0:n}}(\omega(\widehat\omega))) < s$:

$$\bigcap_{n\geqslant m} \bigcap_{\substack{\sigma_{0:n} \text{ up-right:}\\ \sigma_{0:m}\precnsim\pi_{0:m}}} \big(\{\sigma_{0:n} \text{ is on } \mathcal{G}_0^{\omega(\widehat\omega)}, \nu^{\omega(\widehat\omega)}(\phi^{\widehat B,0} \preceq \rho_{\sigma_{0:n}}(\omega(\widehat\omega))) < s\} \cup \{\sigma_{0:n} \text{ is not on } \mathcal{G}_0^{\omega(\widehat\omega)}\}\big).$$

Events of the type (A.1) generate $\mathcal{B}(\mathbb{X}_0)$, so the claim follows. $\square$

We also have the following Lemma concerning generation of $\mathcal{B}(\mathbb{X}_0)$.

Lemma A.3. *The family of events $\{\rho_{0:\infty} \in \mathbb{X}_0 : \rho_{0:\infty} \preceq \pi_{0:\infty}\}$, $\pi_{0:\infty} \in \mathbb{X}_0$ generates $\mathcal{B}(\mathbb{X}_0)$.* $\triangle$

Proof. Cylinder events of the form $\{\rho_{0:\infty} \in \mathbb{X}_0 : \rho_{0:n} = \gamma_{0:n}\}$ are intersections of events of the form $\{\rho_{0:\infty} \in \mathbb{X}_u : \rho_n = x\}$, $x \in \mathbb{Z}^2_+$. For a given $x \in \mathbb{Z}^2_+$, let $\gamma^x_{0:\infty}$ be the up-right path that starts with $x \cdot e_1$ e_1-steps, then takes $x \cdot e_2$ e_2-steps, getting to x, then from there only takes e_1 steps. Then, for $x \neq ne_2$, we have

$$\{\rho_{0:\infty} \in \mathbb{X}_0 : \rho_n = x\} = \{\rho_{0:\infty} \in \mathbb{X}_0 : \rho_{0:\infty} \preceq \gamma^x_{0:\infty}\} \backslash \{\rho_{0:\infty} \in \mathbb{X}_0 : \rho_{0:\infty} \preceq \gamma^{x+e_2-e_1}_{0:\infty}\}.$$

For $x = ne_2$ we have

$$\{\rho_{0:\infty} \in \mathbb{X}_0 : \rho_n = x\} = \{\rho_{0:\infty} \in \mathbb{X}_0 : \rho_{0:\infty} \preceq \gamma^x_{0:\infty}\}. \qquad \square$$

A.3. Non-existence of trivial Busemann geodesics.

Lemma A.4. *Assume (2.3), (2.2), and $\widehat{\mathbb{E}}[\omega_0^2] < \infty$. Let $\widehat B \in \widehat{\mathcal{K}}$. Then (2.25) holds.* $\triangle$

Proof. By the shift invariance of $\widehat{\mathbb{P}}$ and shift covariance of $\widehat B$, it is enough to consider $u = 0$. On the event $\phi^{\widehat B,0} = \mathbb{Z}_+ e_2$ we have, by the cocycle and the recovery properties,

$$\begin{aligned}\sum_{k=0}^{n-1} \omega_{ke_2} + \widehat B(ne_2, e_1 + ne_2) &= \sum_{k=0}^{n-1} \widehat B(ke_2, (k+1)e_2) + \widehat B(ne_2, e_1 + ne_2)\\ &= \widehat B(0, e_1) + \sum_{k=0}^{n-1} \widehat B(e_1 + ke_2, e_1 + (k+1)e_2) \geqslant \widehat B(0, e_1) + \sum_{k=0}^{n-1} \omega_{e_1+ke_2}\end{aligned}$$

from which

$$\frac{\widehat{B}(ne_2, e_1 + ne_2)}{\sqrt{n}} \geqslant \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (\omega_{e_1+ke_2} - \omega_{ke_2}) + \frac{\widehat{B}(0, e_1)}{\sqrt{n}}.$$

Now the left-hand side goes to 0 in probability and hence almost surely along some subsequence of any given subsequence, while the limsup of the right-hand side is infinite almost surely. □

Appendix B. Busemann functions generated by geodesics

First we prove that in almost every environment ω, each nontrivial semi-infinite geodesic generates a recovering cocycle, then we explore some basic properties of these cocycles.

Theorem B.1. *Assume weights are i.i.d. with $p > 2$ moments. Then there exists an event Ω_0 of full probability on which the following holds. For each $\omega \in \Omega_0$ and every semi-infinite geodesic $\pi_{k:\infty}$ in the environment ω such that $\pi_n \cdot e_i \to \infty$ for both $i \in \{1, 2\}$, there exists a finite Busemann function*

$$A(\omega, x, y) = \lim_{n\to\infty} [L_{x,\pi_n}(\omega) - L_{y,\pi_n}(\omega)] \quad \forall x, y \in \mathbb{Z}^2 \tag{B.1}$$

that recovers the weights ω:

$$\omega_x = A(\omega, x, x + e_1) \wedge A(\omega, x, x + e_2). \qquad \triangle$$

Proof. Recovery at x follows once the limits in (B.1) are proved for x and $y \in \{x + e_1, x + e_2\}$: for large enough n,

$$\begin{aligned} &L_{x,\pi_n} = \omega_x + L_{x+e_1,\pi_n} \vee L_{x+e_2,\pi_n} \\ &\quad\Longrightarrow\quad \omega_x = (L_{x,\pi_n} - L_{x+e_1,\pi_n}) \wedge (L_{x,\pi_n} - L_{x+e_2,\pi_n}) \xrightarrow[n\to\infty]{} A(x, x + e_1) \wedge A(x, x + e_2). \end{aligned}$$

We begin with a purely deterministic lemma that gives the limit (B.1) in a northeast quadrant.

Lemma B.2. *Consider a fixed weight configuration $\omega \in \mathbb{R}^{\mathbb{Z}^2}$ and $k \in \mathbb{Z}$. Suppose $\pi_{k:\infty}$ is a semi-infinite geodesic such that $\pi_n \cdot e_i \to \infty$ for $i \in \{1, 2\}$. Then for all $x \in \mathbb{Z}^2$ the monotone nondecreasing limit*

$$A(x, \pi_k) = \lim_{n\to\infty} (L_{x,\pi_n} - L_{\pi_k,\pi_n}) \tag{B.2}$$

exists in $(-\infty, \infty]$. On the quadrant $\pi_k + \mathbb{Z}_+^2$ we have a finite Busemann function

$$A(x, y) = \lim_{n\to\infty} (L_{x,\pi_n} - L_{y,\pi_n}). \tag{B.3}$$

$\triangle$

Proof. Given x, let N be any index such that $\pi_N \geqslant x$. Then for $n \geqslant N$,

$$L_{x,\pi_{n+1}} - L_{\pi_N,\pi_{n+1}} \geqslant (L_{x,\pi_n} + L_{\pi_n,\pi_{n+1}}) - (L_{\pi_N,\pi_n} + L_{\pi_n,\pi_{n+1}}) = L_{x,\pi_n} - L_{\pi_N,\pi_n}.$$

Thus this monotone nondecreasing limit exists:

$$\lim_{n\to\infty} (L_{x,\pi_n} - L_{\pi_N,\pi_n}) \in [L_{x,\pi_N}, \infty]. \tag{B.4}$$

Furthermore, since $L_{\pi_k,\pi_n} = L_{\pi_k,\pi_N} + L_{\pi_N,\pi_n}$ for $k \leqslant N \leqslant n$, we have this monotone nondecreasing limit:

$$A(x, \pi_k) = \lim_{n\to\infty} (L_{x,\pi_n} - L_{\pi_k,\pi_n}) \in [L_{x,\pi_N} - L_{\pi_k,\pi_N}, \infty]. \tag{B.5}$$

Now let $x \in \pi_k + \mathbb{Z}_+^2$. For n such that $\pi_k \leqslant x \leqslant \pi_n$,

$$L_{\pi_k,\pi_n} \geqslant L_{\pi_k,x} + L_{x,\pi_n} \implies L_{x,\pi_n} - L_{\pi_k,\pi_n} \leqslant -L_{\pi_k,x}.$$

Thus for any $x \geqslant \pi_k$ we have the finite limit

$$A(x, \pi_k) = \lim_{n\to\infty} (L_{x,\pi_n} - L_{\pi_k,\pi_n}) \in [L_{x,\pi_N} - L_{\pi_k,\pi_N}, -L_{\pi_k,x}] \qquad \forall N \text{ such that } \pi_N \geqslant x. \tag{B.6}$$

This defines a finite Busemann function

$$A(x,y) = A(x,\pi_k) - A(y,\pi_k) = \lim_{n\to\infty}(A_{x,\pi_n} - A_{y,\pi_n}) \tag{B.7}$$

in the quadrant $\pi_k + \mathbb{Z}_+^2$. □

Returning to the proof of Theorem B.1, it remains to verify that when weights are i.i.d. with $p > 2$ moments, we can construct a full-probability event Ω_0 on which $A(x,\pi_k)$ in (B.2) is finite for each $x \in \mathbb{Z}^2$ and for every semi-infinite geodesic $\pi_{k:\infty}$ such that $\pi_n \cdot e_i \to \infty$ for both $i \in \{1,2\}$. This can be achieved by combining known properties of geodesics and Busemann functions in the corner growth model. Namely, there exists a full-probability event Ω_0 on which the following properties hold.

(a) Each semi-infinite geodesic $\pi_{k:\infty}$ is directed into some $\mathcal{U}_\xi$ for some $\xi \in [e_2, e_1]$, meaning that, as $n \to \infty$, all the limit points of π_n/n lie in $\mathcal{U}_\xi$ (Theorem 2.1(i) in [15]).

(b) The only geodesics directed towards e_i are the trivial ones of the form $x + \mathbb{Z}_+ e_i$ (Lemma 5.1 in [18]).

(c) For any sequence $\{u_n\} \subset \mathbb{Z}^2$ such that, as $n \to \infty$, $u_n \cdot e_i \to \infty$ for both $i \in \{1,2\}$ and the set of limit points of $\{u_n/n\}$ is bounded away from $\{e_2, e_1\}$,

$$\varlimsup_{n\to\infty} |L_{x,u_n} - L_{y,u_n}| < \infty \quad \mathbb{P}\text{-almost surely} \quad \forall x,y \in \mathbb{Z}^2. \tag{B.8}$$

This comes from the zero-temperature version of Theorem 4.14 in [21], or by taking the intersection of the full probability events in Theorem 6.1 of [16] over a countable dense collection of exposed points and maximal linear segments in $\,]e_2, e_1[\,$.

Since we know from Lemma B.2 that $A(x,\pi_k) > -\infty$ for all x and $A(x,\pi_k) < \infty$ for x in a northeast quadrant, it is enough to prove the following statement on the event Ω_0:

$$\text{if } A(x,\pi_k) < \infty, \text{ then } A(x-e_1,\pi_k) < \infty \text{ and } A(x-e_2,\pi_k) < \infty. \tag{B.9}$$

We prove the case $A(x-e_1,\pi_k) < \infty$, the other one being entirely analogous.

We can now assume that for some $\xi \in]e_2, e_1[\,$, as $n \to \infty$, all the limit points of π_n/n lie in $\mathcal{U}_\xi$. $\mathcal{U}_\xi$ is a compact segment (possibly a singleton) contained in the open segment $\,]e_2, e_1[\,$. By the curvature of the shape function close to the extreme direction e_2 implied by Theorem 2.4 of [29], we can pick a direction $\zeta \in]e_2, \xi[\,$ so that the segment $\mathcal{U}_\zeta$ lies strictly to the northwest of the segment $\mathcal{U}_\xi$. As in equation (2.12) in [23] or in Section 4 of [15], the Busemann function $B^{\zeta+}$ defines the Busemann geodesic $\phi^{\zeta+,\pi_k}$ started at vertex π_k, which takes the horizontal step $\phi^{\zeta+,\pi_k}_{n+1} = \phi^{\zeta+,\pi_k}_n + e_1$ whenever there is a tie $B^{\zeta+}(\phi^{\zeta+,\pi_k}_n, \phi^{\zeta+,\pi_k}_n + e_1) = B^{\zeta+}(\phi^{\zeta+,\pi_k}_n, \phi^{\zeta+,\pi_k}_n + e_2)$ in the Busemann increments.

Recall that $\phi^{\zeta+,\pi_k}$ is indexed so that $\phi^{\zeta+,\pi_k}_n \cdot (e_1+e_2) = \pi_n \cdot (e_1+e_2)$ for all $n \geqslant k$. By Theorem 4.3 in [15], $\phi^{\zeta+,\pi_k}$ is directed into the segment $\mathcal{U}_{\zeta+}$. Thus the two geodesics must separate eventually, and so for large enough n, $\phi^{\zeta+,\pi_k}_n \precnsim \pi_n$. The monotonicity of planar LPP increments implies that

$$L_{x-e_1,\pi_n} - L_{x,\pi_n} \leqslant L_{x-e_1,\phi^{\zeta+,\pi_k}_n} - L_{x,\phi^{\zeta+,\pi_k}_n} \tag{B.10}$$

for any $x \in \mathbb{Z}^2$ such that the LPP values are defined. This so-called "path crossing trick" can be found for example in Lemma B.3 below. Now on Ω_0 we have this upper bound, assuming $A(x,\pi_k) < \infty$:

$$\begin{aligned} A(x-e_1,\pi_k) &= \lim_{n\to\infty}[L_{x-e_1,\pi_n} - L_{\pi_k,\pi_n}] = \lim_{n\to\infty}[L_{x-e_1,\pi_n} - L_{x,\pi_n} + L_{x,\pi_n} - L_{\pi_k,\pi_n}] \\ &\overset{(B.10)}{\leqslant} \varlimsup_{n\to\infty} \big|L_{x-e_1,\phi^{\zeta+,\pi_k}_n} - L_{x,\phi_n\zeta+,\pi_k}\big| + A(x,\pi_k) \overset{(B.8)}{<} \infty. \end{aligned}$$

This completes the proof of Theorem B.1. □

When ω lies in the event Ω_0 constructed in Theorem B.1 and γ is a nontrivial semi-infinite geodesic in the environment ω, let $A^\gamma(\omega)$ denote the recovering cocycle constructed in Theorem B.1. Nontrivial geodesics are those whose limiting directions do not include e_2 or e_1. Equivalently, a nontrivial geodesic is directed into some compact segment $\mathcal{U}_\xi$ inside the open segment $]e_2, e_1[$.

Lemma B.3. *Let $\gamma \preceq \pi$ be two nontrivial semi-infinite geodesics in an environment $\omega \in \Omega_0$. Then, $A_\gamma \preceq A_\pi$. Consequently, if $\gamma \curlywedge \pi$, then $A_\gamma = A_\pi$.* △

Proof. Take $x \in \mathbb{Z}^2$. Take n large enough so that $x + e_1 \leqslant \gamma_n$, $x + e_1 \leqslant \pi_n$, and $\gamma_n \precnsim \pi_n$. Since x is to the right of σ^{x+e_1,γ_n} and π_n is to its left, σ^{x+e_1,γ_n} must intersect σ^{x,π_n}. Let z denote the first intersection point. Then

$$L_{x,z} + L_{z,\gamma_n} \leqslant L_{x,\gamma_n} \quad \text{and} \quad L_{x+e_1,z} + L_{z,\pi_n} \leqslant L_{x+e_1,\pi_n}.$$

Add the two inequalities, use $L_{x,z} + L_{z,\pi_n} = L_{x,\pi_n}$ and $L_{x+e_1,z} + L_{z,\gamma_n} = L_{x+e_1,\gamma_n}$, and rearrange to get

$$L_{x,\pi_n} - L_{x+e_1,\pi_n} \leqslant L_{x,\gamma_n} + L_{x+e_1,\gamma_n}.$$

Take $n \to \infty$ to get $A_\pi(x, x + e_1) \leqslant A_\gamma(x, x + e_1)$. The inequality $A_\pi(x, x + e_2) \geqslant A_\gamma(x, x + e_2)$ is proved similarly. □

Given a recovering cocycle B in an environment $\omega \in \Omega$, a B-*geodesic* is an up-right path π, finite or infinite, whose steps obey minimal B-increments: $B(\pi_i, \pi_{i+1}) = B(\pi_i, \pi_i + e_1) \wedge B(\pi_i, \pi_i + e_2)$. Such a path is a geodesic. The e_1 tiebreaker geodesic $\phi^{B,u,+}$ is the semi-infinite geodesic that starts at vertex u, follows minimal increments of B, and takes an e_1 step at a tie. It is the rightmost geodesic between any two its vertices [15, Lemma 4.1]. Analogously, $\phi^{B,u,-}$ is the semi-infinite B-geodesic from u that takes an e_2 step at a tie.

Lemma B.4. *Let B be a recovering cocycle in an environment $\omega \in \Omega_0$. Suppose there exists a coalescing family $\{\pi^u\}_{u \in \mathbb{Z}^2}$ of semi-infinite B-geodesics from all initial vertices $u \in \mathbb{Z}^2$. Then $A_{\pi^u}(\omega) = B$ for every geodesic π^u from this family.* △

Proof. Given x and u, let π^u_N be the point where π^x and π^u first coalesce. Then for $n \geqslant N$, since we can follow B-geodesics,

$$L_{x,\pi^u_n} - L_{u,\pi^u_n} = B(x, \pi^u_n) - B(u, \pi^u_n) = B(x, u).$$

Letting $n \to \infty$ gives $A_{\pi^u}(x, u) = B(x, u)$. This and the cocycle property give $A_{\pi^u} = B$. □

In particular, Lemma B.4 implies that if a recovering cocycle B generates a coalescing family of cocycle geodesics, then any one of these geodesics is enough to identify B.

Lemma B.5. *Let $\pi_{k:\infty}$ be a nontrivial semi-infinite geodesic in a fixed environment $\omega \in \Omega_0$ and $A_\pi(\omega)$ the recovering cocycle constructed in Theorem B.1. The geodesic π is an A_π-geodesic. It lies between $\phi^{\pi_k,A_\pi,-}$ and $\phi^{\pi_k,A_\pi,+}$, the two geodesics generated by A_π, which resolve ties by taking e_2 and e_1 steps, respectively:*

$$\phi^{\pi_k,A_\pi,-} \preceq \rho \preceq \phi^{\pi_k,A_\pi,+}. \tag{B.11}$$
△

Proof. Since $\pi_{m:n}$ is a geodesic between π_m and π_n and goes through π_{m+1}, and then by recovery,

$$\begin{aligned} A_\pi(\pi_m, \pi_{m+1}) &= \lim_{n\to\infty} [L_{\pi_m,\pi_n} - L_{\pi_{m+1},\pi_n}] = \lim_{n\to\infty} [\omega_{\pi_m} + L_{\pi_{m+1},\pi_n} - L_{\pi_{m+1},\pi_n}] \\ &= \omega_{\pi_m} = A_\pi(\pi_m, \pi_m + e_1) \wedge A^\pi(\pi_m, \pi_m + e_2). \end{aligned}$$

This implies also (B.11). □

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