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An almost sure invariance principle for random walks in a space-time random environment

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Abstract. We consider a discrete time random walk in a space-time i.i.d. random environment. We use a martingale approach to show that the walk is diffusive in almost every fixed environment. We improve on existing results by proving an invariance principle and considering environments with an L^2 averaged drift. We also state an a.s. invariance principle for random walks in general random environments whose hypothesis requires a subdiffusive bound on the variance of the quenched mean, under an ergodic invariant measure for the environment chain.

1. Introduction

Random walk in a random environment is one of the basic models of the field of disordered systems of particles. In this model, an environment is a collection of transition probabilities $\omega = (\pi_{xy})_{x,y\in\mathbb{Z}^d} \in \mathcal{P}^{\mathbb{Z}^d}$ where $\mathcal{P} = \{(p_z)_{z\in\mathbb{Z}^d} \in [0,1]^{\mathbb{Z}^d} : \sum_z p_z = 1\}$. Let us denote by $\Omega = \mathcal{P}^{\mathbb{Z}^d}$ the space of all such transition probabilities. The space Ω is equipped with the canonical product σ-field $\mathfrak{S}$ and with the natural shift $\pi_{xy}(T_z\omega) = \pi_{x+z,y+z}(\omega)$, for $z \in \mathbb{Z}^d$. On the space of environments $(\Omega, \mathfrak{S})$, we are given a certain T-invariant probability measure $\mathbb{P}$ with $(\Omega, \mathfrak{S}, (T_z)_{z\in\mathbb{Z}^d}, \mathbb{P})$ ergodic. We will say that the environment is i.i.d. when $\mathbb{P}$ is a product measure in the sense that the random probability vectors $(\pi_{x,y})_{y\in\mathbb{Z}^d}$ are i.i.d. over distinct sites x. We will denote by $\mathbb{E}$ the expectation under $\mathbb{P}$. Let us now describe the process.

First, the environment ω is chosen from the distribution $\mathbb{P}$. Once this is done it remains fixed for all times. The random walk in environment ω, starting at z, is then the canonical Markov chain $\hat{X} = (X_n)_{n\ge 0}$ with state space $\mathbb{Z}^d$ and satisfying

$$P_z^\omega(X_0 = z) = 1,$$
$$P_z^\omega(X_{n+1} = y \mid X_n = x) = \pi_{xy}(\omega).$$

The process P_z^ω is called the *quenched law*. The *joint annealed law* is then

$$P_z(d\hat{X}, d\omega) = P_z^\omega(d\hat{X})\mathbb{P}(d\omega),$$

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and $P_z(d\hat{X}, \Omega)$ is the *marginal annealed law* or, by *abus de langage*, just the *annealed law*. We will use E_z and E_z^ω for the expectations under, respectively, P_z and P_z^ω.

We begin by considering a special type of random environment. Namely, assume that $d = \nu + 1 \geq 2$ and let $\{e_i\}_{i=1}^d$ be the canonical basis of $\mathbb{R}^d$. Assume then that $\mathbb{P}$ is i.i.d. and, $\mathbb{P}$-a.s.,

$$\pi_{0,e_1+z} = 0, \text{ if } z \notin E = \{x \in \mathbb{Z}^d : x \cdot e_1 = 0\} \sim \mathbb{Z}^\nu. \tag{1}$$

Condition (1) says that at each step the first coordinate increases by one deterministically. One of the reasons for considering such a model comes from the fact that if one views it as a random walk in a ν-dimensional space-time i.i.d. random environment, with $\mathbb{R}e_1$ being the time axis, then it turns out to be a dual process to some surface growth processes. See, for example, [15].

Clearly, the annealed process, in this case, is equivalent to $(ne_1 + Y_n)_{n\geq 0}$, where Y_n is a homogeneous Markovian random walk on E, with transitions

$$(p(x, x+z) = p(0,z) = \mathbb{E}(\pi_{0,e_1+z}))_{x,z\in E}.$$

If p has a first moment, the annealed walk on $\mathbb{Z}^d$ has a law of large numbers with velocity

$$v = e_1 + \sum_{z\in E} p(0,z)z.$$

If, furthermore, it has a second moment, one then has an annealed invariance principle with diffusion matrix

$$\mathfrak{D} = \sum_{z\in E} (e_1 + z - v)(e_1 + z - v)^t p(0,z).$$

Since any quenched central limit type result would imply an annealed one, one can then see that a second moment condition on p has to be assumed, if one wants to prove an a.s. invariance principle for this model. On the other hand, if $\mathbb{P}(\sup_z \pi_{0z} = 1) = 1$, then the quenched walk becomes deterministic and a central limit is out of question. This justifies our hypothesis (M for moment, E for ellipticity):

Hypothesis (ME). *The measure* $\mathbb{P}$ *satisfies the following condition*

$$\sum_{z\in E} |z|^2 \mathbb{E}(\pi_{0,e_1+z}) < \infty \textit{ and } \mathbb{P}(\sup_{z\in E} \pi_{0,e_1+z} < 1) > 0.$$

Define now, for $t \geq 0$ and a given $(X_n)_{n\geq 0}$,

$$B_n(t) = \frac{X_{[nt]} - [nt]v}{\sqrt{n}} \text{ and } \widetilde{B}_n(t) = \frac{X_{[nt]} - E_0^\omega(X_{[nt]})}{\sqrt{n}}. \tag{2}$$

Here, for $x \in \mathbb{R}$, $[x] = \max\{n \in \mathbb{Z} : n \leq x\}$. For a closed interval $I \subset [0,\infty)$ denote by $D_{\mathbb{R}^d}(I)$ the space of right continuous functions on I, taking values in

$\mathbb{R}^d$, and having left limits. The space $D_{\mathbb{R}^d}(I)$ is endowed with the usual Skorohod topology [14]. For $\omega \in \Omega$, let Q_n^ω, respectively $\widetilde{Q}_n^\omega$, denote the distribution of B_n, respectively $\widetilde{B}_n$, induced by P_0^ω, on the Borel sets of $D_{\mathbb{R}^d}([0,\infty))$. In Section 4 we will prove the following theorem:

Theorem 1. *Let $d \geq 2$ and consider a random walk in an i.i.d. random environment satisfying* (1) *and Hypothesis* (ME). *Then, for $\mathbb{P}$-a.e. ω, the distribution Q_n^ω converges weakly to the distribution of a Brownian motion with diffusion matrix given by $\mathfrak{D}$. Moreover, $n^{-1/2} \max_{k\leq n} |E_0^\omega(X_n) - nv|$ converges to 0, $\mathbb{P}$-a.s. and, therefore, the same invariance principle holds also for $\widetilde{Q}_n^\omega$.*

Remark 1. For a symmetric, non-negative definite $d \times d$ matrix Γ, a Brownian motion with diffusion matrix Γ is the $\mathbb{R}^d$-valued process $(W(t))_{t\geq 0}$ such that $W(0) = 0$, W has continuous paths, independent increments, and for $s < t$ the d-vector $W(t) - W(s)$ has Gaussian distribution with mean zero and covariance matrix $(t-s)\Gamma$. If the rank of Γ is m, one can produce such a process by finding a $d \times m$ matrix Λ such that $\Gamma = \Lambda\Lambda^t$, and by defining $W(t) = \Lambda B(t)$ where B is an m-dimensional standard Brownian motion.

In space-time product random environments the invariance principle under the annealed P_0 is just Donsker's classical invariance principle. But in general random environments even the annealed invariance principle is far from immediate. For recent results in this direction see [10] and [26] and the references therein. Moreover, the switch from annealed central limit type results to quenched ones for random walks in random environments is a hard problem that has been subject to a very slow progress.

Quenched results have been proved under specialized assumptions. For example, if the random walk is balanced (i.e. $\pi_{0x} = \pi_{0,-x}$), then X_n becomes a martingale under the quenched measure, and one can show a quenched invariance principle [18]. On the other hand, if the random walk has a sufficiently high-dimensional simple symmetric random walk part, then one can use a natural regeneration structure that arises [8]. If the random environment is a small perturbation of the simple symmetric random walk, then a quenched invariance principle has been proved using renormalization techniques; see [9] and the recent [27]. Even in the basic reversible case of random walks among random conductances, quenched invariance principles have only recently been shown to hold [23]. There the authors adopt the approach developed by Kipnis and Varadhan [16] for reversible Markov chains to get the central limit theorem; see Section 3 below. Also, see [20] and [17] for a review on the approach in [16]. For a non-reversible setting where [16] has also been useful, see [28]. Notably, as it is usually the case in this field, the invariance principle does not immediately follow from the fixed time central limit theorem. In [23], recent Gaussian estimates are used to perform the transition.

In the case of a space-time product random environment the central limit version for $B_n(1)$ has been known in the case of "small noise"; see [3, 6, 25]. When $d = 2$, the walk is nearest-neighbor, and the annealed walk is symmetric, this central limit theorem was recently shown in [2] without any "small noise" assumptions. For $d \geq 2$, the "small noise" assumption of [3] was recently extended in [4] to prove the fixed time central limit theorem under the hypothesis that $\sup_\omega \pi_{0z}(\omega)$ has an exponential moment, as opposed to our second moment assumption in Hypothesis (ME). If one assumes on top of Hypothesis (ME) that $\mathbb{P}$-a.s., $\pi_{0,e_1+z} = 0$, if $z \notin \{e_i, -e_i\}_{i\geq 2}$, and positive otherwise, then the invariance principle for the process B_n in dimension $d \geq 4$ has been proved in [7]. When the space-time random environment is not product but Markovian in the time direction, the marginal central limit theorem has also been shown under a "small noise" assumption [5].

Our arguments differ from those of [2–6,25]. The ideas in [7] and [23] are somewhat related in spirit. Our approach is based on adaptations of the well-known Kipnis-Varadhan method [16] to non-reversible situations developed by Maxwell and Woodroofe [19] and Derriennic and Lin [12]. Maxwell and Woodroofe use fairly concrete probabilistic reasoning. We will refer directly to their paper for some preliminary steps in our proof. The approach of Derriennic and Lin is more abstract and powerful, cast in the framework of Banach space contractions, and ultimately produces stronger results. We apply their results to conclude our proof. All this will be described in Section 2 where we prove a result that holds for general random environments.

In the course of this paper ω, ω_0, and ω_1 will denote generic elements of Ω. We will write A^t for the transpose of a vector or matrix A. An element of $\mathbb{R}^d$ is regarded as a $d \times 1$ matrix, or column vector. The set of whole numbers $\{0, 1, 2, \cdots\}$ will be denoted by $\mathbb{N}$.

2. Quenched invariance principle for general random environments under moment hypotheses

In this section we consider the general random walk in a random environment as formulated in the first two paragraphs of the Introduction. In particular, the special structure of assumption (1) and Hypothesis (ME) are not assumed in this section.

First, let us define the drift

$$D(\omega) = E_0^\omega(X_1) = \sum_z z\pi_{0z}(\omega).$$

For a bounded measurable function h on Ω, define

$$\Pi h(\omega) = \sum_z \pi_{0z}(\omega)h(T_z\omega).$$

In fact, $\Pi - I$ defines the generator of the Markov process of the environment, as seen from the particle. This is the process on Ω with transitions

$$\pi(\omega, A) = P_0^\omega(T_{X_1}\omega \in A).$$

In this section, we will assume there exists a probability measure $\mathbb{P}_\infty$ on Ω that is invariant for the transition Π. Then, the operator Π can be extended to a contraction on $L^p(\mathbb{P}_\infty)$, for every $p \in [1, \infty]$. We will use the notation $\mathbb{E}_\infty$ for the corresponding expectation. When the initial distribution is $\mathbb{P}_\infty$, we will denote this Markov process by $\widetilde{P}_0^\infty$. We will also write $P_0^\infty(d\hat{X}, d\omega) = P_0^\omega(d\hat{X})\mathbb{P}_\infty(d\omega)$ and E_0^∞ for the expectation under P_0^∞. Note that $\widetilde{P}_0^\infty$ is the probability measure induced by P_0^∞ and $(T_{X_n}\omega)$ onto $\Omega^{\mathbb{N}}$. With this notation, the measure

$$\mu_2^\infty(d\omega_0, d\omega_1) = \pi(\omega_0, d\omega_1)\mathbb{P}_\infty(d\omega_0)$$

describes the law of $(\omega, T_{X_1}\omega)$, under P_0^∞.

We will assume in this section that $D \in L^2(\mathbb{P}_\infty)$. Next, for $\varepsilon > 0$, let h_ε be the solution of

$$(1+\varepsilon)h_\varepsilon - \Pi h_\varepsilon = g = D - v,$$

where $v = \mathbb{E}_\infty(D)$. In fact, one can write:

$$h_\varepsilon = \sum_{k=1}^{\infty}(1+\varepsilon)^{-k}\Pi^{k-1}g \in L^2(\mathbb{P}_\infty).$$

Define

$$H_\varepsilon(\omega_0, \omega_1) = h_\varepsilon(\omega_1) - \Pi h_\varepsilon(\omega_0).$$

Then one has the following theorem.

Theorem 2. *Let $d \geq 1$ and let $\mathbb{P}_\infty$ be any probability measure on $(\Omega, \mathfrak{S})$ that is invariant under Π and ergodic for the Markov process on Ω with generator $\Pi - I$. Assume that $\sum_z |z|^2\mathbb{E}_\infty(\pi_{0z}) < \infty$. Assume also that there exists an $\alpha < 1/2$ such that*

$$\sqrt{\mathbb{E}_\infty\left(\left| E_0^\omega(X_n) - nv \right|^2\right)} = \left\|\sum_{k=0}^{n-1}\Pi^k g\right\|_2 = \mathcal{O}(n^\alpha), \tag{3}$$

where $\|\cdot\|_p$ is the $L^p(\mathbb{P}_\infty)$-norm. Then we get the following conclusions: The limit

$$H = \lim_{\varepsilon\to 0^+} H_\varepsilon \tag{4}$$

exists in $L^2(\mu_2^\infty)$. For $\mathbb{P}_\infty$-a.e. ω, the distribution Q_n^ω of the process B_n defined in (2) converges weakly to the distribution of a Brownian motion with diffusion matrix

$$E_0^\infty\left[(X_1 - D(\omega) + H(\omega, T_{X_1}\omega))(X_1 - D(\omega) + H(\omega, T_{X_1}\omega))^t\right], \tag{5}$$

as defined in Remark 1. Moreover, $n^{-1/2}\max_{k\leq n}|E_0^\omega(X_k) - kv|$ converges to 0, $\mathbb{P}_\infty$-a.s. and, therefore, the same invariance principle holds for $\widetilde{Q}_n^\omega$.

Remark 2. When (3) only holds for $\alpha = 1/2$, e.g. when $d = 1$, the quenched central limit theorem may only hold with random centering $E_0^\omega(X_n)$ and not with deterministic centering nv. See Examples 3 and 4 and Proposition 1 of [21].

Proof. The proof uses essentially the strategy of [12] which is an extension of some of the results of [19], to which in turn we will refer the reader for part of the calculations.

First let us give a few more definitions. For $\varepsilon > 0$, let

$$M_n^\varepsilon = \sum_{k=0}^{n-1} H_\varepsilon(T_{X_k}\omega, T_{X_{k+1}}\omega),\ \ \bar{X}_n = X_n - \sum_{k=0}^{n-1} D(T_{X_k}\omega).$$

Furthermore, define

$$S_n^\varepsilon = \sum_{k=0}^{n-1} h_\varepsilon(T_{X_k}\omega), \;\; R_n^\varepsilon = h_\varepsilon(\omega) - h_\varepsilon(T_{X_n}\omega),$$

so that

$$X_n - nv = \bar{X}_n + M_n^\varepsilon + \varepsilon S_n^\varepsilon + R_n^\varepsilon.$$

Now, we proceed with the proof. The existence of the limit in (4) follows from Proposition 1 of [19]. Thus, if one defines

$$M_n = \sum_{k=0}^{n-1} H(T_{X_k}\omega, T_{X_{k+1}}\omega),$$

then, for $\mathbb{P}_\infty$-almost every ω, $(M_n)_{n\geq 1}$ is a P_0^ω-square integrable martingale relative to the filtration $\{\mathcal{F}_n = \sigma(X_0, \cdots, X_n)\}_{n\geq 0}$. It also follows from Lemma 1 of [19] that one has $\|h_\varepsilon\|_2 = \mathcal{O}(\varepsilon^{-\alpha})$. Define the error by

$$R_n = X_n - nv - \bar{X}_n - M_n = M_n^\varepsilon - M_n + \varepsilon S_n^\varepsilon + R_n^\varepsilon. \tag{6}$$

Corollary 4 of [19] shows that

$$E_0^\infty(|\, R_n\,|^2) = \mathcal{O}(n^{2\alpha}). \tag{7}$$

Let $M_n^*(t) = n^{-1/2}(\bar{X}_{[nt]} + M_{[nt]})$. $(M_n^*(t))_{0\leq t\leq 1}$ converges weakly, under P_0^ω for $\mathbb{P}_\infty$-a.e. ω, to a Brownian motion with diffusion matrix as in (5). This follows from a vector-valued version of a well-known invariance principle for martingales. For the convenience of the reader we provide a proof of it in the appendix. The limits needed as hypotheses for this invariance principle follow from ergodicity and the square-integrability of M_1 and X_1. In turn, the assumption $\sum |\, z\,|^2\, \mathbb{E}_\infty(\pi_{0z}) < \infty$ guarantees that X_1 and M_1 are square-integrable for $\mathbb{P}_\infty$-a.e. ω.

We have

$$\sup_{0\leq t\leq 1} \Big|\, B_n(t) - M_n^*(t) \,\Big| \leq n^{-1/2} \max_{k\leq n} |\, R_k\,|\,.$$

Therefore, the invariance principle for $(B_n(t))_{0\leq t\leq 1}$ will follow once we show that

$$n^{-1/2} \max_{k\leq n} |\, R_k\,| \underset{n\to\infty}{\longrightarrow} 0, \;\text{ in } P_0^\omega\text{-probability, for } \mathbb{P}_\infty\text{-a.e. } \omega. \tag{8}$$

By (6), $E_0^\omega(R_n) = E_0^\omega(X_n) - nv$. Hence the invariance principle for the process $(\widetilde{B}_n(t))_{0\leq t\leq 1}$ will follow once we show that

$$n^{-1/2} \max_{k\leq n} \Big|\, E_0^\omega(R_k) \,\Big| \underset{n\to\infty}{\longrightarrow} 0, \;\text{ for } \mathbb{P}_\infty\text{-a.e. } \omega. \tag{9}$$

To prove (8) and (9) we apply the theory of "fractional coboundaries" of Derriennic and Lin [11]. The first application is to the shift map θ on the sequence space $\Omega^{\mathbb{N}}$ which is a contraction on the space $L^2(\widetilde{P}_0^\infty)$. On $\Omega \times \Omega$ define first $f(\omega_0, \omega_1) = g(\omega_0) - H(\omega_0, \omega_1)$. Then, P_0^∞-a.s.

$$R_n = \sum_{k=0}^{n-1} f(T_{X_k}\omega, T_{X_{k+1}}\omega).$$

For sequences $\bar{\omega} = (\omega^{(i)})_{i\in\mathbb{N}} \in \Omega^{\mathbb{N}}$ define $F(\bar{\omega}) = f(\omega^{(0)}, \omega^{(1)})$ and

$$\widetilde{R}_n = \sum_{k=0}^{n-1} F \circ \theta^k.$$

Then $F \in L^2(\widetilde{P}_0^\infty)$ and the process $(\widetilde{R}_n)_{n\geq 1}$ has the same distribution under $\widetilde{P}_0^\infty$ as the process $(R_n)_{n\geq 1}$ has under P_0^∞.

Condition (7) shows that the assumptions of Theorem 2.17 of [11] are satisfied. The conclusion is that $F \in (I-\theta)^\eta L^2(\widetilde{P}_0^\infty)$, for any $\eta \in (0, 1-\alpha)$. Since $\alpha < 1/2$, we can find such an $\eta \in (1/2, 1-\alpha)$. But then (i) in Theorem 3.2 of [11] implies that $n^{-1/2}\widetilde{R}_n$ converges to 0, $\widetilde{P}_0^\infty$-a.s. This implies that $n^{-1/2}R_n$ converges to 0, P_0^∞-a.s. In other words, $n^{-1/2}R_n$ converges to 0, P_0^ω-a.s., for $\mathbb{P}_\infty$-a.e. ω. From this, (8) is immediate.

To prove (9) apply the same results from [11] to the contraction Π on $L^2(\mathbb{P}_\infty)$. Because

$$E_0^\omega(R_n) = E_0^\omega(X_n) - nv = \sum_{k=0}^{n-1} \Pi^k g$$

repeating the above argument with θ replaced by Π and (7) replaced by (3) proves (9).

Once we have the invariance principle on $D_{\mathbb{R}^d}([0, 1])$, the identities

$$B_n(t) = \sqrt{N} B_{nN}(t/N) \text{ and } \widetilde{B}_n(t) = \sqrt{N}\widetilde{B}_{nN}(t/N)$$

show that on $D_{\mathbb{R}^d}([0, N])$ B_n and $\widetilde{B}_n$ converge to the process $\sqrt{N}W(t/N)$ which is the same as W. Then weak convergence on each $D_{\mathbb{R}^d}([0, N])$ implies weak convergence on $D_{\mathbb{R}^d}([0, \infty])$. The proof of Theorem 2 is complete. □

3. On the corrector function

In this section, we will prove an interesting property of the error R_n defined in (6). This property will show the key difference between the one and multi-dimensional cases. Although we will not make use of this property, it establishes a connection with other existing ways of approaching the problem; see [23], for instance.

First, define the functions

$$f_\varepsilon(\omega_0, \omega_1) = g(\omega_0) - H_\varepsilon(\omega_0, \omega_1) - \varepsilon h_\varepsilon(\omega_0),$$

$$f(\omega_0, \omega_1) = g(\omega_0) - H(\omega_0, \omega_1).$$

A calculation shows us that

$$f_\varepsilon(\omega_0, \omega_1) = h_\varepsilon(\omega_0) - h_\varepsilon(\omega_1),$$

$$R_n^\varepsilon = \sum_{k=0}^{n-1} f_\varepsilon(T_{X_k}\omega, T_{X_{k+1}}\omega),$$
$$R_n = \sum_{k=0}^{n-1} f(T_{X_k}\omega, T_{X_{k+1}}\omega).$$

The following proposition establishes a "co-cycle" property of f.

Proposition 1. *For $m, n \in \mathbb{N}$, let $\Big((x_i)_{i=0}^n, (\tilde{x}_j)_{j=0}^m\Big)$ be "an admissible bridge", i.e. such that $x_n = \tilde{x}_m = x$, and*

$$\int P_0^\omega(X_i = x_i, 0 \le i \le n) P_0^\omega(X_j = \tilde{x}_j, 0 \le j \le m)\mathbb{P}_\infty(d\omega) > 0.$$

Then, one has

$$\sum_{i=0}^{n-1} f(T_{x_i}\omega, T_{x_{i+1}}\omega) = \sum_{j=0}^{m-1} f(T_{\tilde{x}_j}\omega, T_{\tilde{x}_{j+1}}\omega),\ \mathbb{P}_\infty\text{-}a.s. \tag{10}$$

Proof. Fix m, n and x, and let $Q_{0,x,n,m}^\omega$ be the measure on the space of double paths $((x_i)_{i=0}^n, (\tilde{x}_j)_{j=0}^m)$ defining two independent random walks driven by the same environment ω, both starting at 0 and ending at x after, respectively, n and m steps. That is,

$$Q_{0,x,n,m}^\omega(((x_i)_{i=0}^n, (\tilde{x}_j)_{j=0}^m)) = \prod_{i=0}^{n-1} \pi_{x_i x_{i+1}}(\omega) \prod_{j=0}^{m-1} \pi_{\tilde{x}_j \tilde{x}_{j+1}}(\omega),$$

if $x_n = \tilde{x}_m = x,\ x_0 = \tilde{x}_0 = 0$, and 0 otherwise. Define also

$$Q_{0,x,n,m} = \int Q_{0,x,n,m}^\omega \mathbb{P}_\infty(d\omega).$$

Then one has, for each $\varepsilon > 0$,

$$\sum_{i=0}^{n-1} f_\varepsilon(T_{X_i}\omega, T_{X_{i+1}}\omega) = \sum_{j=0}^{m-1} f_\varepsilon(T_{\tilde{X}_j}\omega, T_{\tilde{X}_{j+1}}\omega),\ Q_{0,x,n,m}\text{-a.s.}$$

By (4) each term above converges, in $L^2(P_0^\infty)$, to the corresponding term in (10). Notice now that, although $Q_{0,x,n,m}$ is not a probability measure, one still has $\frac{dQ_{0,x,n,m}}{dP_0^\infty} \le 1$. Therefore, the convergence also happens in $L^2(Q_{0,x,n,m})$ and as a result

$$\sum_{i=0}^{n-1} f(T_{X_i}\omega, T_{X_{i+1}}\omega) = \sum_{j=0}^{m-1} f(T_{\tilde{X}_j}\omega, T_{\tilde{X}_{j+1}}\omega),\ Q_{0,x,n,m}^\omega\text{-a.s., for } \mathbb{P}_\infty\text{-a.e. } \omega.$$

Therefore, (10) holds for any admissible bridge. □

Let us, for simplicity, assume that all points in $\mathbb{Z}^d$ can be reached from 0 by some admissible path. The above Lemma tells us then that if one defines the so-called "corrector function" as

$$\chi(x,\omega) = \sum_{i=0}^{m-1} f(T_{x_i}\omega, T_{x_{i+1}}\omega),$$

where $(x_0, \cdots, x_m)$ is any admissible path from 0 to x, then

$$R_n = \chi(X_n, \omega) = \chi([nv], \omega) + \chi(X_n - [nv], T_{[nv]}\omega). \tag{11}$$

Here, for $x \in \mathbb{R}^d$, $[\cdot]$ acts on each coordinate separately, to give $[x]$. Relation (11) has a quite interesting implication. The term $\chi([nv], \omega)$ represents the fluctuations coming from the environment itself. When $d = 1$, $\chi([nv], \omega)$ is of order $\sqrt{n}$ and for a quenched invariance principle one needs to consider $X_n - nv - \chi([nv], \omega)$, i.e. to have a random centering. See [29] and Example 4 of [21] for more details. However, when $d \geq 2$ the walker can see "more" environments. The quantity $\frac{\chi([nv],\omega)}{\sqrt{n}}$ then vanishes at the limit and condition (8) has a chance to hold. See [23] for a result where control of the corrector is key to a quenched invariance principle in a reversible setting.

4. Space-time i.i.d. random environments

In this section, we will prove Theorem 1 via an application of Theorem 2.

Proof of Theorem 1. For $n \geq 0$, let

$$f_n(\omega) = \sum_{x: x\cdot e_1 = -n} P_x^\omega(X_n = 0).$$

By translation invariance of $\mathbb{P}$ one has $\mathbb{E}(f_n) = 1$. One then can check that f_n is a martingale relative to the filtration

$$\{\mathfrak{S}_{-n} = \sigma((\pi_{xy})_y,\ x \cdot e_1 \geq -n)\}_{n \geq 0}.$$

Therefore, there is a probability measure $\mathbb{P}_\infty$ such that

$$\frac{d\mathbb{P}_{\infty|\mathfrak{S}_{-n}}}{d\mathbb{P}_{|\mathfrak{S}_{-n}}} = f_n.$$

An induction on f_n shows also that $\mathbb{P}_\infty$ is invariant for Π. Since $f_0 = 1$, we have that $\mathbb{P} = \mathbb{P}_\infty$ on $\mathfrak{S}_0$. This will be of great use to us. On the one hand, since $(\pi_{0z})_z$ is $\mathfrak{S}_0$-measurable, Hypothesis (ME) implies that $D \in L^2(\mathbb{P}_\infty)$. On the other hand, on $\mathfrak{S}_0$ the i.i.d. structure carries over to $\mathbb{P}_\infty$. Using this, we will show in the following lemma that $\mathbb{P}_\infty$ is also ergodic for the Markov process with generator $\Pi - I$.

Lemma 1. *If $\mathbb{P}$ is i.i.d. and satisfies (1), then the invariant measure $\mathbb{P}_\infty$, constructed above, is ergodic for the Markov process with generator $\Pi - I$.*

Proof. Consider a bounded local function Ψ on Ω that is measurable with respect to $\sigma((\pi_{xy})_y, |x \cdot e_1| \le K)$, for some integer $K \ge 0$. Due to (1), $(\Psi(T_{X_{m_0+3Km}}\omega))_{m\ge 0}$ is a sequence of i.i.d. random variables, under P_0, for any $m_0 \ge K$. Therefore,

$$P_0\left(\forall m_0 \ge K : \lim_{n\to\infty} n^{-1} \sum_{3Km \le n-m_0} \Psi(T_{X_{m_0+3Km}}\omega) = \frac{E_0(\Psi(T_{X_{m_0}}\omega))}{3K}\right) = 1.$$

It then follows that, $\mathbb{P}$-a.s.

$$P_0^\omega\left(\lim_{n\to\infty} n^{-1} \sum_{m=K}^{n-1} \Psi(T_{X_m}\omega) = c\right) = 1,$$

where

$$c = (3K)^{-1} \sum_{m_0=K}^{4K-1} E_0(\Psi(T_{X_{m_0}}\omega)).$$

Since the above quenched probability is $\mathfrak{S}_0$-measurable the convergence also holds $\mathbb{P}_\infty$-a.s. But then c cannot be anything other than $\mathbb{E}_\infty(\Psi)$.

Using bounded convergence one then sees that $n^{-1}\sum_{m=0}^{n-1}\Pi^m\Psi$ converges to $\mathbb{E}_\infty(\Psi)$, $\mathbb{P}_\infty$-a.s. By L^1-approximations we get this same limit in the L^1 sense for all $\Psi \in L^1(\mathbb{P}_\infty)$ and the ergodicity follows from the development in Section IV.2 of [22]. This proves Lemma 1. □

We continue with the proof of Theorem 1. Next, we will show that condition (3) is satisfied.

Lemma 2. *Under the assumptions of Theorem* 1 *condition* (3) *is satisfied with* $\alpha = 1/4$.

Proof. Observe that

$$\left\|\sum_{k=0}^{n-1}\Pi^k g\right\|_2^2 = \sum_{i,j=0}^{n-1}\sum_{x,y\in\mathbb{Z}^d}\int P_0^\omega(X_i = x)P_0^\omega(X_j = y)g(T_x\omega)g(T_y\omega)\mathbb{P}(d\omega).$$

Note that the integral is taken with respect to $\mathbb{P}$ instead of $\mathbb{P}_\infty$. This is because all the integrands depend on ω through $\mathfrak{S}_0$.

Now, since $\mathbb{P}$ is a product measure and g has mean zero, the summands in the above sum vanish unless $i = j$ and $x = y$. Therefore, one has

$$\left\|\sum_{k=0}^{n-1}\Pi^k g\right\|_2^2 = \|g\|_2^2 \sum_{k=0}^{n-1}\int P_{0,0}^\omega(X_k = \tilde{X}_k)\mathbb{P}(d\omega),$$

where X_n and $\tilde{X}_n$ are two independent walkers driven by the same environment ω. We will denote their law by $P_{0,0}^\omega$. To check condition (3) one needs to find the asymptotic behaviour of the above sum.

Notice now that, under $\int P_{0,0}^\omega\mathbb{P}(d\omega)$, the difference $Y_n = X_n - \tilde{X}_n$ performs a random walk on E with the following kernel:

$$q(0, y) = \sum_{z \in E} \mathbb{E}(\pi_{0,e_1+z}\pi_{0,e_1+z+y}),$$
$$q(x, y) = \sum_{z \in E} \mathbb{E}(\pi_{0,e_1+z})\mathbb{E}(\pi_{0,e_1+z+y-x}), \text{ if } x \neq 0. \tag{12}$$

This walk is actually a homogeneous symmetric random walk on E, perturbed at 0. Due to Lemma 3.3 of [15] one then has

$$\sum_{k=0}^{n-1} \int P^{\omega}_{0,0}(X_k = \tilde{X}_k)\mathbb{P}(d\omega) = \mathcal{O}(\sqrt{n}).$$

Therefore, condition (3) is satisfied and we are done. □

Remark 3. Our application of Lemma 3.3 from [15] may appear unjustified because of the additional hypotheses [15] employs. However, (2.2) in [15] is superfluous. Once one notices that

$$q(0, y) > 0 \Rightarrow q_0(0, y) > 0,$$

one can apply **P7.1** on page 65 of Spitzer [24] to reduce the treatment to a situation where (2.2) holds. Here, q_0 is the transition kernel of the unperturbed random walk and is defined by (12), for all x. Also, in Lemma 3.2 of [15] the authors reference **P12.3** of Spitzer's book, which requires more than just two moments on q_0. However, one can instead use (3) of section 12 on page 122 of [24]. Furthermore, Lemma 3.1 of [15] is not needed for our purposes, since we only need the upper bound in (3.22) therein. Lastly, the reference to **P7.9** of [24], in Lemma 3.3 of [15], can be replaced by **P7.6**. This allows to discard "strong aperiodicity".

Now that we have verified all the assumptions of Theorem 2 are satisfied, we can conclude the proof of Theorem 1. Indeed, Theorem 2 implies that the claim of Theorem 1 holds for $\mathbb{P}_\infty$-a.e. ω. But since everything depends on ω only through $\mathfrak{S}_0$, and $\mathbb{P}_\infty = \mathbb{P}$ on $\mathfrak{S}_0$, the same holds for $\mathbb{P}$-a.e. ω.

Of course, Theorem 2 yields a different formula for the diffusion matrix. However, the annealed invariance principle has to have the same diffusion matrix which, as we have mentioned in the introduction, is precisely $\mathfrak{D}$. We leave it for the reader to double-check, with a direct calculation, that the two formulae do coincide. □

Remark 4. Note that $(\bar{X}_1 + M_1) \cdot e_1 = 0$ and, therefore, the Brownian motion in question is actually ν-dimensional or smaller. This is of course also clear from the formula for $\mathfrak{D}$.

A. An invariance principle for a vector-valued martingale difference array

In this appendix we give a proof of the vector-valued martingale invariance principle, needed in the proof of Theorem 2, that is based on the corresponding scalar result. The scalar version appears as Theorem 7.4 in Chapter 7 of Durrett's textbook [13]. It is noteworthy that an invariance principle for general Banach space valued martingale differences that unifies several results in the literature has been proved by [1].

To avoid confusion with time t we write in this section A^T for the transpose of a vector or matrix A, instead of the A^t we have used in the rest of the paper. An element of $\mathbb{R}^d$ is still regarded as a $d \times 1$ matrix.

Let $(\Omega, \mathcal{G}, P)$ be a probability space on which are defined sub-σ-algebras $\mathcal{G}_{n,k} \subset \mathcal{G}$ and $\mathbb{R}^d$-valued random vectors $Y_{n,k}$. We say that $\{Y_{n,k}, \mathcal{G}_{n,k} : n \geq 1, 1 \leq k \leq n\}$ is an $\mathbb{R}^d$-valued square-integrable martingale difference array if the following properties are satisfied:

(i) $Y_{n,k}$ is $\mathcal{G}_{n,k}$-measurable, $\mathcal{G}_{n,k-1} \subset \mathcal{G}_{n,k}$,
(ii) $E(\left| Y_{n,k} \right|^2) < \infty$,
(iii) $E(Y_{n,k}|\mathcal{G}_{n,k-1}) = 0$,

and in the last condition we take $\mathcal{G}_{n,0} = \{\phi, \Omega\}$. Define the $\mathbb{R}^d$-valued processes $S_n(\cdot)$ by

$$S_n(t) = \sum_{k=1}^{[nt]} Y_{n,k}$$

for $0 \leq t \leq 1$. The paths of $S_n(\cdot)$ are in the Skorohod space $D_{\mathbb{R}^d}([0, 1])$ of $\mathbb{R}^d$-valued cadlag paths on $[0, 1]$. Recall now from Remark 1 the definition of a Brownian motion with diffusion matrix Γ. One then has the following:

Theorem 3. *Let $\{Y_{n,k}, \mathcal{G}_{n,k} : n \geq 1, 1 \leq k \leq n\}$ be an $\mathbb{R}^d$-valued square-integrable martingale difference array on a probability space $(\Omega, \mathcal{G}, P)$. Let Γ be a symmetric, non-negative definite $d \times d$ matrix. Assume that*

$$\lim_{n\to\infty} \sum_{k=1}^{[nt]} E(Y_{n,k}Y_{n,k}^T|\mathcal{G}_{n,k-1}) = t\Gamma \text{ in probability,} \tag{13}$$

for each $0 \leq t \leq 1$, and

$$\lim_{n\to\infty} \sum_{k=1}^{n} E(\left| Y_{n,k} \right|^2 \mathbb{I}\{\left| Y_{n,k} \right| \geq \varepsilon\}|\mathcal{G}_{n,k-1}) = 0 \text{ in probability,} \tag{14}$$

for each $\varepsilon > 0$. Then $S_n(\cdot)$ converges weakly to a Brownian motion with diffusion matrix Γ on the Skorohod space $D_{\mathbb{R}^d}([0, 1])$.

Proof. The key to the proof is to apply a scalar martingale invariance principle to one-dimensional projections of S_n, conditional on the past.

Fix $0 \leq s < 1$. Consider k time points $0 \leq s_1 < s_2 < \cdots < s_k \leq s$, and a non-negative, bounded continuous function Ψ on $\mathbb{R}^{kd}$. Abbreviate

$$Z_n = \Psi\left(S_n(s_1), S_n(s_2), \cdots, S_n(s_k)\right).$$

Assume $E(Z_n) > 0$ for all n. Pick also a non-zero vector $\theta \in \mathbb{R}^d$, and a bounded continuous function f on the Skorohod space $D_{\mathbb{R}}([0, 1 - s])$ of scalar-valued paths. Finally, let B_θ denote a one-dimensional Brownian motion with variance $E(B_\theta(t)^2) = \theta^T \Gamma \theta t$.

Lemma 3. *We have the limit*

$$\lim_{n\to\infty} \frac{E\left(f\left(\theta\cdot S_n(s+\cdot)-\theta\cdot S_n(s)\right)Z_n\right)}{E(Z_n)} = E(f(B_\theta)). \tag{15}$$

Proof. Define a probability measure $\widetilde{P}_n$ on Ω by

$$\widetilde{P}_n(A) = \frac{1}{E(Z_n)} E(\mathbb{I}_A \cdot Z_n).$$

$\widetilde{E}_n$ denotes expectation under $\widetilde{P}_n$. Since Z_n is $\mathcal{G}_{n,[ns]}$-measurable, we have

$$\widetilde{E}_n(h|\mathcal{G}_{n,k}) = E(h|\mathcal{G}_{n,k})$$

for any $k \geq [ns]$ and $h \in L^1(P)$.

Define a scalar martingale difference array $\{X_{n,m}, \mathcal{F}_{n,m} : n \geq 1, 1 \leq m \leq n-[ns]\}$ by $X_{n,m} = \theta \cdot Y_{n,[ns]+m}$ and $\mathcal{F}_{n,m} = \mathcal{G}_{n,[ns]+m}$.

Observe first that by assumption (13) one has

$$\begin{aligned}
V_{n,[nt]} &\equiv \sum_{j=1}^{[nt]} \widetilde{E}_n(X_{n,j}^2|\mathcal{F}_{n,j-1}) = \sum_{j=1}^{[nt]} E(X_{n,j}^2|\mathcal{G}_{n,[ns]+j-1}) \\
&= \theta^T \left\{ \sum_{k=1}^{[ns]+[nt]} E(Y_{n,k}Y_{n,k}^T|\mathcal{G}_{n,k-1}) - \sum_{k=1}^{[ns]} E(Y_{n,k}Y_{n,k}^T|\mathcal{G}_{n,k-1}) \right\} \theta \\
&\longrightarrow t\theta^T\Gamma\theta \quad \text{in probability as } n\to\infty.
\end{aligned}$$

In case the reader is concerned that the first sum on the second-last line above goes up to $j = [ns]+[nt]$, we point out that assumption (14) implies

$$\lim_{n\to\infty} \max_{1\leq k\leq n} \left| Y_{n,k} \right| = 0 \quad \text{in probability.} \tag{16}$$

Thus the limits are not affected by finitely many terms. (16) follows from Dvoretsky's Lemma, by an argument that can be found in the proof of Theorem (7.3) in Section 7.7 of Durrett [13] (see part (f) of that proof).

Next, by assumption (14), for any $\varepsilon > 0$ there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned}
&\sum_{j=1}^{n-[ns]} \widetilde{E}_n(X_{n,j}^2 \mathbb{I}\{\left| X_{n,j} \right| \geq \varepsilon\}|\mathcal{F}_{n,j-1}) \\
&\leq |\theta|^2 \sum_{k=1}^{n} E(\left| Y_{n,k} \right|^2 \mathbb{I}\{\left| Y_{n,j} \right| \geq \varepsilon_0\}|\mathcal{G}_{n,k-1}) \\
&\longrightarrow 0 \quad \text{in probability as } n\to\infty.
\end{aligned}$$

We have verified the hypotheses of the Lindeberg-Feller Theorem for martingales that appears as Theorem (7.3) in Section 7.7 of Durrett [13]. Consequently the process

$$\begin{aligned}
U_n(t) &= \sum_{j=1}^{[nt]} X_{n,j} \\
&= \theta\cdot S_n(s+t) - \theta\cdot S_n(s) - \theta\cdot Y_{n,[n(s+t)]}\mathbb{I}\{[ns]+[nt] < [n(s+t)]\}
\end{aligned}$$

satisfies

$$\widetilde{E}_n(f(U_n)) \to E(f(B_\theta)). \tag{17}$$

To be precise, Durrett's theorem treats the continuous process $\theta \cdot \bar{S}_n(\cdot)$ defined by linear interpolation:

$$\bar{S}_n(t) = S_n(t) + (nt - [nt])Y_{n,[nt]+1}\,, \quad 0 \le t \le 1.$$

But by (16),

$$\sup_{0\le t\le 1} \left| S_n(t) - \bar{S}_n(t) \right| \to 0 \quad \text{in probability,} \tag{18}$$

so the cadlag and continuous versions converge weakly together.

(17) is the same as (15), again because by (16) whether $[ns] + [nt]$ differs from $[n(s + t)]$ is immaterial for the limit. Lemma 3 is proved. □

Now we prove Theorem 3 from this lemma. First, by taking $s = 0$ and $\Psi \equiv 1$, $\theta \cdot \bar{S}_n$ converges weakly to the Brownian motion B_θ, for each vector θ. Thus all the scalar processes obtained as projections of $\{\bar{S}_n\}$ are tight, and hence the vector-valued processes $\{\bar{S}_n\}$ themselves are tight on the space $C_{\mathbb{R}^d}([0, 1])$. And then by (18), the vector-valued processes $\{S_n\}$ are tight on the space $D_{\mathbb{R}^d}([0, 1])$. This detour via the continuous processes $\{\bar{S}_n\}$ to get tightness of $\{S_n\}$ was used because tightness of vector-valued processes from projections is not as obvious for cadlag paths as it is for continuous paths (see Exercise 22 from Chapter 3 of Ethier-Kurtz [14]).

Let a process X be a weak limit point of $\{S_n\}$, and let S_{n_j} be the subsequence along which $S_{n_j} \Rightarrow X$. The map $\eta \mapsto \theta \cdot \eta$ from $D_{\mathbb{R}^d}([0, 1])$ into $D_{\mathbb{R}}([0, 1])$ is continuous, hence $\theta \cdot X$ has the distribution of the Brownian motion B_θ. It follows that X has a version with almost surely continuous paths. Then the finite-dimensional marginals converge weakly:

$$\left(S_{n_j}(s_1), S_{n_j}(s_2), \cdots, S_{n_j}(s_k)\right) \Rightarrow (X(s_1), X(s_2), \cdots, X(s_k))\,.$$

From all this we conclude that along $\{n_j\}$ the left-hand side of (15) converges to

$$\frac{E\left(f\left(\theta \cdot X(s + \cdot) - \theta \cdot X(s)\right) \Psi\left(X(s_1), \cdots, X(s_k)\right)\right)}{E\left(\Psi\left(X(s_1), \cdots, X(s_k)\right)\right)}.$$

By (15) this must equal $E(f(B_\theta))$. Since the time points $\{s_i\}$ and the function Ψ are arbitrary, it follows that

$$E\left(e^{i\theta\cdot(X(s+t)-X(s))} \,\middle|\, X(r) : 0 \le r \le s\right) = E\left(e^{iB_\theta(t)}\right) = e^{-\frac{t}{2}\theta^T\Gamma\theta}. \tag{19}$$

Varying the vector θ here implies that the increment $X(s + t) - X(s)$ is independent of the past up to time s, and is distributed like the increment of a Brownian motion with diffusion matrix Γ. Inductively on the number of increments we conclude that X has independent increments, continuous paths and the correct Gaussian finite-dimensional distributions, which makes it a Brownian motion with diffusion matrix Γ.

Note that it is critically important for this argument that in (19) we can condition on $\{X(r) : 0 \leq r \leq s\}$ and not only on $\{\theta \cdot X(r) : 0 \leq r \leq s\}$. This latter would not suffice for the conclusion, as indicated by Exercise 2 in Chapter 7 of [14]. This completes the proof of Theorem 3. □

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