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Almost sure functional central limit theorem for ballistic random walk in random environment

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Abstract. We consider a multidimensional random walk in a product random environment with bounded steps, transience in some spatial direction, and high enough moments on the regeneration time. We prove an invariance principle, or functional central limit theorem, under almost every environment for the diffusively scaled centered walk. The main point behind the invariance principle is that the quenched mean of the walk behaves subdiffusively.

Résumé. Nous considérons une marche aléatoire multidimensionnelle en environnement aléatoire produit. La marche est à pas bornés, transiente dans une direction spatiale donnée, et telle que le temps de régénération posséde un moment suffisamment haut. Nous prouvons un principe d'invariance, ou un théorème limite central fonctionnel, sous presque tout environnement pour la marche centrée et diffusivement normalisée. Le point principal derrière le principe d'invariance est que la moyenne trempée (quenched) de la marche est sous-diffusive.

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1. Introduction and main result

We prove a quenched functional central limit theorem (CLT) for ballistic random walk in random environment (RWRE) on the d-dimensional integer lattice $\mathbb{Z}^d$ in dimensions $d \ge 2$. Here is a general description of the model, fairly standard since quite a while. An environment ω is a configuration of probability vectors $\omega = (\omega_x)_{x\in\mathbb{Z}^d} \in \Omega = \mathcal{P}^{\mathbb{Z}^d}$, where $\mathcal{P} = \{(p_z)_{z\in\mathbb{Z}^d}\colon\ p_z \ge 0, \sum_z p_z = 1\}$ is the simplex of all probability vectors on $\mathbb{Z}^d$. Vector $\omega_x = (\omega_{x,z})_{z\in\mathbb{Z}^d}$ gives the probabilities of jumps out of state x, and the transition probabilities are denoted by $\pi_{x,y}(\omega) = \omega_{x,y-x}$. To run the random walk, fix an environment ω and an initial state $z \in \mathbb{Z}^d$. The random walk $X_{0,\infty} = (X_n)_{n\ge 0}$ in environment ω started at z is then the canonical Markov chain with state space $\mathbb{Z}^d$ whose path measure P_z^ω satisfies

$$P_z^\omega\{X_0 = z\} = 1 \quad\text{and}\quad P_z^\omega\{X_{n+1} = y | X_n = x\} = \pi_{x,y}(\omega).$$

On the space Ω we put its product σ-field $\mathfrak{S}$, natural shifts $\pi_{x,y}(T_z\omega) = \pi_{x+z,y+z}(\omega)$, and a $\{T_z\}$-invariant probability measure $\mathbb{P}$ that makes the system $(\Omega, \mathfrak{S}, (T_z)_{z\in\mathbb{Z}^d}, \mathbb{P})$ ergodic. In this paper $\mathbb{P}$ is an i.i.d. product measure on $\mathcal{P}^{\mathbb{Z}^d}$. In other words, the vectors $(\omega_x)_{x\in\mathbb{Z}^d}$ are i.i.d. across the sites x under $\mathbb{P}$.

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Statements, probabilities and expectations under a fixed environment, such as the distribution P_z^ω above, are called *quenched*. When the environment is also averaged out, the notions are called *averaged*, or also *annealed*. In particular, the averaged distribution $P_z(dx_{0,\infty})$ of the walk is the marginal of the joint distribution $P_z(dx_{0,\infty}, d\omega) = P_z^\omega(dx_{0,\infty})\mathbb{P}(d\omega)$ on paths and environments.

Several excellent expositions on RWRE exist, and we refer the reader to the lectures [3,20] and [23].

This paper investigates the directionally transient situation. That is, we assume that there exists a vector $\hat{u} \in \mathbb{Z}^d$ such that

$$P_0\{X_n \cdot \hat{u} \to \infty\} = 1. \tag{1.1}$$

The key moment assumption (M) below is also expressed in terms of $\hat{u}$ so this vector needs to be fixed for the rest of the paper. There is no essential harm in assuming $\hat{u} \in \mathbb{Z}^d$ and this is convenient. Appendix B shows that at the expense of a larger moment, an arbitrary $\hat{u}$ can be replaced by an integer vector $\hat{u}$.

The transience assumption provides regeneration times, first defined and studied in the multidimensional setting by Sznitman and Zerner [22]. As a function of the path $X_{0,\infty}$ regeneration time τ_1 is the first time at which

$$\sup_{n<\tau_1} X_n \cdot \hat{u} < X_{\tau_1} \cdot \hat{u} = \inf_{n\geq\tau_1} X_n \cdot \hat{u}. \tag{1.2}$$

The benefit here is that the past and the future of the walk lie in separate half-spaces. Transience (1.1) is equivalent to $P_0(\tau_1 < \infty) = 1$ (Proposition 1.2 in [22]).

To be precise, [22] is written under assumptions of uniform ellipticity and nearest-neighbor jumps. In an i.i.d. environment many properties established for uniformly elliptic nearest-neighbor walks extend immediately to walks with bounded steps without ellipticity assumptions, the above-mentioned equivalence among them. In such cases we treat the point simply as having been established in earlier literature.

In addition to the product form of $\mathbb{P}$, the following three assumptions are used in this paper: a high moment (M) on τ_1, bounded steps (S), and some regularity (R).

Hypothesis (M). $E_0(\tau_1^{p_0}) < \infty$ *for some* $p_0 > 176d$.

Hypothesis (S). *There exists a finite, deterministic, positive constant* r_0 *such that* $\mathbb{P}\{\pi_{0,z} = 0\} = 1$ *whenever* $|z| > r_0$.

Hypothesis (R). *Let* $\mathcal{J} = \{z\colon \mathbb{E}\pi_{0,z} > 0\}$ *be the set of admissible steps under* $\mathbb{P}$. *Then* $\mathcal{J} \not\subset \mathbb{R}u$ *for all* $u \in \mathbb{R}^d$, *and*

$$\mathbb{P}\{\exists z\colon \pi_{0,0} + \pi_{0,z} = 1\} < 1. \tag{1.3}$$

The bound on p_0 in Hypothesis (M) is of course meaningless and only indicates that our result is true if p_0 is large enough. We have not sought to tighten the exponent because in any case the final bound would not be small with our current arguments. After the theorem we return to discuss the hypotheses further. These assumptions are strong enough to imply a law of large numbers: there exists a velocity $v \neq 0$ such that

$$P_0\Big\{\lim_{n\to\infty} n^{-1} X_n = v\Big\} = 1. \tag{1.4}$$

Representations for v are given in (2.6) and Lemma 5.1. Define the (approximately) centered and diffusively scaled process

$$B_n(t) = \frac{X_{[nt]} - [nt]v}{\sqrt{n}}. \tag{1.5}$$

As usual $[x] = \max\{n \in \mathbb{Z}\colon n \leq x\}$ is the integer part of a real x. Let $D_{\mathbb{R}^d}[0,\infty)$ be the standard Skorohod space of $\mathbb{R}^d$-valued cadlag paths (see [8] for the basics). Let $Q_n^\omega = P_0^\omega(B_n \in \cdot)$ denote the quenched distribution of the process B_n on $D_{\mathbb{R}^d}[0,\infty)$.

The result of this paper concerns the limit of the process B_n as $n \to \infty$. As expected, the limit process is a Brownian motion with correlated coordinates. For a symmetric, nonnegative definite $d \times d$ matrix $\mathfrak{D}$, a *Brownian motion with* *diffusion matrix* $\mathfrak{D}$ is the $\mathbb{R}^d$-valued process $\{B(t)\colon t \geq 0\}$ with continuous paths, independent increments, and such that for $s < t$ the d-vector $B(t) - B(s)$ has Gaussian distribution with mean zero and covariance matrix $(t-s)\mathfrak{D}$. The matrix $\mathfrak{D}$ is *degenerate* in direction $u \in \mathbb{R}^d$ if $u^t\mathfrak{D}u = 0$. Equivalently, $u \cdot B(t) = 0$ almost surely.

Here is the main result.

Theorem 1.1. *Let $d \geq 2$ and consider a random walk in an i.i.d. product random environment that satisfies transience* (1.1), *moment assumption* (M) *on the regeneration time*, *bounded step-size hypothesis* (S), *and the regularity required by* (R). *Then for $\mathbb{P}$-almost every ω distributions Q_n^ω converge weakly on $D_{\mathbb{R}^d}[0,\infty)$ to the distribution of a Brownian motion with a diffusion matrix $\mathfrak{D}$ that is independent of ω. $u^t\mathfrak{D}u = 0$ iff u is orthogonal to the span of* $\{x - y\colon \mathbb{E}(\pi_{0x})\mathbb{E}(\pi_{0y}) > 0\}$.

Equation (2.7) gives the expression for the diffusion matrix $\mathfrak{D}$, familiar for example from [18].

We turn to a discussion of the hypotheses. Obviously (S) is only for technical convenience, while (M) and (R) are the serious assumptions.

Moment assumption (M) is difficult to check. Yet it is a sensible hypothesis because it is known to follow from many concrete assumptions.

A RWRE is called *non-nestling* if for some $\delta > 0$

$$\mathbb{P}\left\{\sum_{z\in\mathbb{Z}^d} z\cdot\hat{u}\pi_{0,z} \geq \delta\right\} = 1. \tag{1.6}$$

This terminology was introduced by Zerner [24]. Together with (S), non-nestling implies even uniform quenched exponential moment bounds on the regeneration times. See Lemma 3.1 in [14].

Most work on RWRE takes as standing assumptions that $\pi_{0,z}$ is supported by the $2d$ nearest neighbors of the origin, and *uniform ellipticity*: for some $\kappa > 0$,

$$\mathbb{P}\{\pi_{0,e} \geq \kappa\} = 1 \quad \text{for all unit vectors } e. \tag{1.7}$$

Nearest-neighbor jumps with uniform ellipticity of course imply Hypotheses (S) and (R). In the uniformly elliptic case, the moment bound (M) on τ_1 follows from the easily testable condition (see [19])

$$\mathbb{E}\left[\left(\sum_{z\in\mathbb{Z}^d} z\cdot\hat{u}\pi_{0,z}\right)^+\right] > \kappa^{-1}\mathbb{E}\left[\left(\sum_{z\in\mathbb{Z}^d} z\cdot\hat{u}\pi_{0,z}\right)^-\right].$$

A more general condition that implies Hypothesis (M) is *Sznitman's condition* (T′), see Proposition 3.1 in [19]. Condition (T′) cannot be checked by examining the environment ω_0 at the origin. But it is still an "effective" condition in the sense that it can be checked by examining the environment in finite cubes. Moreover, in condition (T′) the direction vector $\hat{u}$ can be replaced by a vector in a neighborhood. Consequently the vector can be taken rational, and then also integral. Thus our assumption that $\hat{u} \in \mathbb{Z}^d$ entails no loss in generality.

Hypothesis (M) is further justified by a currently accepted assumption about uniformly elliptic RWRE. Namely, it is believed that once a uniformly elliptic walk is *ballistic* ($v \neq 0$) the regeneration time has all moments (see [19]). Thus conditional on this supposition, the present work settles the question of quenched CLT for uniformly elliptic, multidimensional ballistic RWRE with bounded steps.

Hypotheses (M) and (S) are used throughout the paper. Hypothesis (R) on the other hand makes only one important appearance: to guarantee the nondegeneracy of a certain Markov chain (Lemma 7.13). Yet it is Hypothesis (R) that is actually necessary for the quenched CLT.

Hypothesis (R) can be violated in two ways: (a) the walk lies in a one-dimensional linear subspace, or (b) assumption (1.3) is false in which case the walk follows a sequence of steps completely determined by ω and the only quenched randomness is in the time taken to leave a site (call this the "restricted path" case). In case (b) the walk is bounded if there is a chance that the walk intersects itself. This is ruled out by transience (1.1).

In the unbounded situation in case (b) the quenched CLT breaks down because the scaled variable $n^{-1/2}(X_n - nv)$ is not even tight under P_0^ω. There is still a quenched CLT for the walk centered at its quenched mean, that is, for the process $\widetilde{B}_n(t) = n^{-1/2}\{X_{[nt]} - E_0^\omega(X_{[nt]})\}$. Furthermore, the quenched mean itself satisfies a CLT. Process B_n does satisfy an averaged CLT, which comes from the combination of the diffusive fluctuations of $\widetilde{B}_n$ and of the quenched mean. (See [13] for these results.) The same situation should hold in one dimension also, and has been proved in some cases [10,13,23].

Next a brief discussion of the current situation in this area of probability and the place of the present work in this context. Several themes appear in recent work on quenched CLT's for multidimensional RWRE.

(i) Small perturbations of classical random walk have been studied by many authors. The most significant results include the early work of Bricmont and Kupiainen [4] and more recently Sznitman and Zeitouni [21] for small perturbations of Brownian motion in dimension $d \geq 3$.

(ii) An averaged CLT can be turned into a quenched CLT by bounding the variances of quenched expectations of test functions on the path space. This idea was applied by Bolthausen and Sznitman [2] to nearest-neighbor, uniformly elliptic non-nestling walks in dimension $d \geq 4$ under a small noise assumption. Berger and Zeitouni [1] developed the approach further to cover more general ballistic walks without the small noise assumption, but still in dimension $d \geq 4$.

After the appearance of the first version of the present paper, Berger and Zeitouni combined some ideas from our Section 6 with their own approach to bounding intersections. This resulted in an alternative proof of Theorem 1.1 in the uniformly elliptic nearest-neighbor case that appeared in a revised version of article [1]. The proof in [1] has the virtue that it does not require the ergodic invariant distribution that we utilize to reduce the proof to a bound on the variance of the quenched mean.

(iii) Our approach is based on the subdiffusivity of the quenched mean of the walk. That is, we show that the variance of $E_0^\omega(X_n)$ is of order $n^{2\alpha}$ for some $\alpha < 1/2$. This is achieved through intersection bounds. We introduced this line of reasoning in [12], subsequently applied it to walks with a forbidden direction in [15], and recently to non-nestling walks in [14]. Theorem 2.1 summarizes the general principle for application in the present paper.

It is common in this field to look for an invariant distribution $\mathbb{P}_\infty$ for the environment process that is mutually absolutely continuous with the original $\mathbb{P}$, at least on the part of the space Ω to which the drift points. Instead of absolute continuity, we use bounds on the variation distance between $\mathbb{P}_\infty$ and $\mathbb{P}$. This distance decays polynomially in direction $\hat{u}$, at a rate that depends on the strength of the moment assumption (M). From this we also get an ergodic theorem for functions of the environment that are local in direction $-\hat{u}$. This in turn would give the absolute continuity if it were needed for the paper.

The remainder of the paper is for the proofs. The next section collects preliminary material and finishes with an outline of the rest of the paper.

2. Preliminaries for the proof

Recall that we assume $\hat{u} \in \mathbb{Z}^d$. This is convenient because the lattice $\mathbb{Z}^d$ decomposes into *levels* identified by the integer value $x \cdot \hat{u}$. See Appendix B for the step from a general $\hat{u}$ to an integer vector $\hat{u}$.

Let us summarize notation for the reader's convenience. Constants whose exact values are not important and can change from line to line are often denoted by C. The set of nonnegative integers is $\mathbb{N} = \{0, 1, 2, \ldots\}$. Vectors and sequences are abbreviated $x_{m,n} = (x_m, x_{m+1}, \ldots, x_n)$ and $x_{m,\infty} = (x_m, x_{m+1}, x_{m+2}, \ldots)$. Similar notation is used for finite and infinite random paths: $X_{m,n} = (X_m, X_{m+1}, \ldots, X_n)$ and $X_{m,\infty} = (X_m, X_{m+1}, X_{m+2}, \ldots)$. $X_{[0,n]} = \{X_k\colon 0 \leq k \leq n\}$ denotes the set of sites visited by the walk. $\mathfrak{D}^t$ is the transpose of a vector or matrix $\mathfrak{D}$. An element of $\mathbb{R}^d$ is regarded as a $d \times 1$ column vector. The left shift on the path space $(\mathbb{Z}^d)^{\mathbb{N}}$ is $(\theta^k x_{0,\infty})_n = x_{n+k}$. $|\cdot|$ denotes Euclidean norm on $\mathbb{R}^d$.

$\mathbb{E}$, E_0, and E_0^ω denote expectations under, respectively, $\mathbb{P}$, P_0, and P_0^ω. $\mathbb{P}_\infty$ will denote an invariant measure on Ω, with expectation $\mathbb{E}_\infty$. Abbreviate $P_0^\infty(\cdot) = \mathbb{E}_\infty P_0^\omega(\cdot)$ and $E_0^\infty(\cdot) = \mathbb{E}_\infty E_0^\omega(\cdot)$ to indicate that the environment of a quenched expectation is averaged under $\mathbb{P}_\infty$. A family of σ-algebras on Ω that in a sense look towards the future is defined by $\mathfrak{S}_\ell = \sigma\{\omega_x\colon x \cdot \hat{u} \geq \ell\}$.

Define the *drift*

$$D(\omega) = E_0^\omega[X_1] = \sum_z z\pi_{0,z}(\omega).$$

The *environment process* is the Markov chain on Ω with transition kernel

$$\Pi(\omega, A) = P_0^\omega\{T_{X_1}\omega \in A\}.$$

The proof of the quenched CLT Theorem 1.1 utilizes crucially the environment process and its invariant distribution. A preliminary part of the proof is summarized in the next theorem quoted from [12]. This Theorem 2.1 was proved by applying the arguments of Maxwell and Woodroofe [11] and Derriennic and Lin [6] to the environment process.

Theorem 2.1 [12]. *Let $d \ge 1$. Suppose the probability measure $\mathbb{P}_\infty$ on $(\Omega, \mathfrak{S})$ is invariant and ergodic for the Markov transition Π. Assume that $\sum_z |z|^2 \mathbb{E}_\infty[\pi_{0,z}] < \infty$ and that there exists an $\alpha < 1/2$ such that as $n \to \infty$*

$$\mathbb{E}_\infty\big[\big|E_0^\omega(X_n) - n\mathbb{E}_\infty(D)\big|^2\big] = \mathrm{O}\big(n^{2\alpha}\big). \tag{2.1}$$

Then as $n \to \infty$ the following weak limit happens for $\mathbb{P}_\infty$-a.e. ω: distributions Q_n^ω converge weakly on the space $D_{\mathbb{R}^d}[0, \infty)$ to the distribution of a Brownian motion with a symmetric, non-negative definite diffusion matrix $\mathfrak{D}$ that is independent of ω.

Proceeding with further definitions, we already defined above the first Sznitman–Zerner regeneration time τ_1 as the first time at which

$$\sup_{n<\tau_1} X_n \cdot \hat{u} < X_{\tau_1} \cdot \hat{u} = \inf_{n \ge \tau_1} X_n \cdot \hat{u}.$$

The first backtracking time is defined by

$$\beta = \inf\{n \ge 0\colon\ X_n \cdot \hat{u} < X_0 \cdot \hat{u}\}. \tag{2.2}$$

P_0-a.s. transience in direction $\hat{u}$ guarantees that

$$P_0(\beta = \infty) > 0. \tag{2.3}$$

Otherwise the walk would return below level 0 infinitely often (see Proposition 1.2 in [22]). Furthermore, a walk transient in direction $\hat{u}$ will reach infinitely many levels. At each new level it has a fresh chance to regenerate. This implies that τ_1 is P_0-a.s. finite (Proposition 1.2. in [22]). Consequently we can iterate to define $\tau_0 = 0$, and for $k \ge 1$

$$\tau_k = \tau_{k-1} + \tau_1 \circ \theta^{\tau_{k-1}}.$$

For i.i.d. environments Sznitman and Zerner [22] proved that the *regeneration slabs*

$$\mathcal{S}_k = \big(\tau_{k+1} - \tau_k, (X_{\tau_k+n} - X_{\tau_k})_{0\le n\le \tau_{k+1}-\tau_k}, \big\{\omega_{X_{\tau_k}+z}\colon\ 0 \le z \cdot \hat{u} < (X_{\tau_{k+1}} - X_{\tau_k}) \cdot \hat{u}\big\}\big) \tag{2.4}$$

are i.i.d. for $k \ge 1$, each distributed as the initial slab $(\tau_1, (X_n)_{0\le n\le\tau_1}, \{\omega_z\colon\ 0 \le z \cdot \hat{u} < X_{\tau_1} \cdot \hat{u}\})$ under $P_0(\cdot|\beta = \infty)$. Strictly speaking, uniform ellipticity and nearest-neighbor jumps were standing assumptions in [22], but these assumptions are not needed for the proof of the i.i.d. structure. From this and assumptions (1.1) and (M) it then follows for $k \ge 1$ that

$$E_0\big[(\tau_{k+1} - \tau_k)^{p_0}\big] = E_0\big[\tau_1^{p_0}|\beta = \infty\big] \le \frac{E_0(\tau_1^{p_0})}{P_0(\beta = \infty)} < \infty. \tag{2.5}$$

From the renewal structure and moment estimates a law of large numbers (1.4) and an averaged functional central limit theorem follow, along the lines of Theorem 2.3 in [22] and Theorem 4.1 in [18]. These references treat walks that satisfy Kalikow's condition, less general than Hypothesis (M). But the proofs only rely on the existence of moments of τ_1, now ensured by Hypothesis (M). The limiting velocity for the law of large numbers is

$$v = \frac{E_0[X_{\tau_1}|\beta = \infty]}{E_0[\tau_1|\beta = \infty]}. \tag{2.6}$$

The averaged CLT states that the distributions $P_0\{B_n \in \cdot\}$ converge to the distribution of a Brownian motion with diffusion matrix

$$\mathfrak{D} = \frac{E_0[(X_{\tau_1} - \tau_1 v)(X_{\tau_1} - \tau_1 v)^t | \beta = \infty]}{E_0[\tau_1 | \beta = \infty]}. \tag{2.7}$$

Once we know that the $\mathbb{P}$-a.s. quenched CLT holds with a constant diffusion matrix, this diffusion matrix must be the same $\mathfrak{D}$ as for the averaged CLT. We prove here the degeneracy statement of Theorem 1.1.

Lemma 2.1. *Define $\mathfrak{D}$ by (2.7) and let $u \in \mathbb{R}^d$. Then $u^t \mathfrak{D} u = 0$ iff u is orthogonal to the span of $\{x - y\colon \mathbb{E}[\pi_{0,x}]\mathbb{E}[\pi_{0,y}] > 0\}$.*

Proof. The argument is a minor embellishment of that given for a similar degeneracy statement on pp. 123–124 of [13] for the forbidden-direction case where $\pi_{0,z}$ is supported by $z \cdot \hat{u} \geq 0$. We spell out enough of the argument to show how to adapt that proof to the present case.

Again, the intermediate step is to show that $u^t \mathfrak{D} u = 0$ iff u is orthogonal to the span of $\{x - v\colon \mathbb{E}[\pi_{0,x}] > 0\}$. The argument from orthogonality to $u^t \mathfrak{D} u = 0$ goes as in [13], p. 124.

Suppose $u^t \mathfrak{D} u = 0$ which is the same as

$$P_0\{X_{\tau_1} \cdot u = \tau_1 v \cdot u | \beta = \infty\} = 1. \tag{2.8}$$

Take x such that $\mathbb{E}\pi_{0,x} > 0$. Several cases need to be considered.

If $x \cdot \hat{u} \geq 0$ but $x \neq 0$ a small modification of the argument in [13], p. 123, works to show that $x \cdot u = v \cdot u$.

Suppose $x \cdot \hat{u} < 0$. Then take y such that $y \cdot \hat{u} > 0$ and $\mathbb{E}\pi_{0,y} > 0$. Such y must exist by the transcience assumption (1.1).

If y is collinear with x and there is no other noncollinear vector y with $y \cdot \hat{u} > 0$, then, since the one-dimensional case is excluded by Hypothesis (R), there must exist another vector z that is not collinear with x or y and such that $z \cdot \hat{u} \leq 0$ and $\mathbb{E}\pi_{0,z} > 0$.

Now for any $n \geq 1$, let m_n be the positive integer such that

$$(m_n y + 2z + nx) \cdot \hat{u} \geq 0 \quad \text{but} \quad \big((m_n - 1)y + 2z + nx\big) \cdot \hat{u} < 0.$$

Let the walk first take m_n y-steps, followed by one z-step, then n x-steps, followed by another z-step, again m_n y-steps, and then regenerate (meaning that $\beta \circ \theta^{2m_n + n + 2} = \infty$). This path is non-self-intersecting and, by the minimality of m_n, backtracks enough to ensure that the first regeneration time is $\tau_1 = 2m_n + n + 2$. Hence

$$P_0\{X_{\tau_1} = 2m_n y + nx + 2z, \tau_1 = 2m_n + n + 2 | \beta = \infty\} \geq (\mathbb{E}\pi_{0,y})^{2m_n} (\mathbb{E}\pi_{0,x})^n (\mathbb{E}\pi_{0,z})^2 > 0$$

and then by (2.8)

$$(nx + 2m_n y + 2z) \cdot u = (n + 2m_n + 2) v \cdot u. \tag{2.9}$$

Since $y \cdot \hat{u} > 0$ we have already shown that $y \cdot u = v \cdot u$. Taking $n \nearrow \infty$ implies $x \cdot u = v \cdot u$.

If y is not collinear with x, repeat the above argument, but without using any z-steps and hence with simply $n = 1$.

When $x = 0$ making the walk take an extra step of size 0 along the path, an almost identical argument to the above can be repeated. Since we have shown that $y \cdot u = v \cdot u$ for any $y \neq 0$ with $\mathbb{E}\pi_{0,y} > 0$, this allows to also conclude that $0 \cdot u = v \cdot u$.

Given $u^t \mathfrak{D} u = 0$, we have established $x \cdot u = v \cdot u$ for any x with $\mathbb{E}\pi_{0,x} > 0$. Now follow the proof in [13], p. 123–124, to its conclusion. □

Here is an outline of the proof of Theorem 1.1. It all goes via Theorem 2.1.

(i) After some basic estimates in Section 3, we prove in Section 4 the existence of the ergodic invariant distribution $\mathbb{P}_\infty$ required for Theorem 2.1. $\mathbb{P}_\infty$ is not convenient to work with so we still need to do computations with $\mathbb{P}$.

For this purpose Section 4 proves that in the direction $\hat{u}$ the measures $\mathbb{P}_\infty$ and $\mathbb{P}$ come polynomially close in variation distance and that the environment process satisfies a P_0-a.s. ergodic theorem. In Section 5 we show that $\mathbb{P}_\infty$ and $\mathbb{P}$ are interchangeable both in the hypotheses that need to be checked and in the conclusions obtained. In particular, the $\mathbb{P}_\infty$-a.s. quenched CLT coming from Theorem 2.1 holds also $\mathbb{P}$-a.s. Then we know that the diffusion matrix $\mathfrak{D}$ is the one in (2.7).

The bulk of the work goes towards verifying condition (2.1), but under $\mathbb{P}$ instead of $\mathbb{P}_\infty$. There are two main stages to this argument.

(ii) By a decomposition into martingale increments the proof of (2.1) reduces to bounding the number of common points of two independent walks in a common environment (Section 6).

(iii) The intersections are controlled by introducing levels at which both walks regenerate. These joint regeneration levels are reached fast enough and the relative positions of the walks from one joint regeneration level to the next are a Markov chain. When this Markov chain drifts away from the origin it can be approximated well enough by a symmetric random walk. This approximation enables us to control the growth of the Green function of the Markov chain, and thereby the number of common points. This is in Section 7 and in Appendix A devoted to the Green function bound.

Appendix B shows that the assumption that $\hat{u}$ has integer coordinates entails no loss of generality if the moment required is doubled. The proof given in Appendix B is from Berger and Zeitouni [1]. Appendix C contains a proof (Lemma 7.13) that requires a systematic enumeration of a large number of cases.

The end result of the development is the bound

$$\mathbb{E}\big[\big|E_0^\omega(X_n) - E_0(X_n)\big|^2\big] = \mathrm{O}\big(n^{2\alpha}\big) \tag{2.10}$$

on the variance of the quenched mean, for some $\alpha \in (1/4, 1/2)$. The parameter α can be taken arbitrarily close to $1/4$ if the exponent p_0 in (M) can be taken arbitrarily large. The same is also true under the invariant measure $\mathbb{P}_\infty$, namely (2.1) is valid for some $\alpha \in (1/4, 1/2)$. Based on the behavior of the Green function of a symmetric random walk, optimal orders in (2.10) should be $n^{1/2}$ in $d = 2$, $\log n$ in $d = 3$, and constant in $d \ge 4$. Getting an optimal bound in each dimension is not a present goal, so in the end we bound all dimensions with the two-dimensional case.

The requirement $p_0 > 176d$ of Hypothesis (M) is derived from the bounds established along the way. There is room in the estimates for us to take one simple and lax route to a sufficient bound. Start from (A.3) with $p_1 = p_2 = p_0/6$ as dictated by Proposition 7.10 and (7.29). Taking $p_0 = 220$ gives the bound $Cn^{22/32}$. Feed this bound into Proposition 6.1 where it sets $\bar{\alpha} = 11/32$. Next in (6.3) take $\alpha - \bar{\alpha} = 1/8$ to get the requirement $p_0 > 176d$. Finally in (5.3) take $\alpha - \bar{\alpha} = 1/32$ which places the demand $p_0 > 160d$. With $d \ge 2$ all are satisfied with $p_0 > 176d$. (Actually $11/32 + 1/8 + 1/32 = 1/2$ but since the inequalities are strict there is room to keep α strictly below $1/2$.)

Sections 3–6 are valid for all dimensions $d \ge 1$, but Section 7 requires $d \ge 2$.

3. Basic estimates for ballistic RWRE

In addition to the regeneration times already defined, let

$$J_m = \inf\{i \ge 0:\ \tau_i \ge m\}.$$

Lemma 3.1. *Let $\mathbb{P}$ be an i.i.d. product measure and satisfy Hypotheses (S) and (M). We have these bounds*:

$$E_0\big[\tau_\ell^{p_0}\big] \le C\ell^{p_0} \qquad \textit{for all } \ell \ge 1, \tag{3.1}$$

$$\sup_{m\ge 0} E_0\big[|\tau_{J_m} - m|^p\big] \le C \qquad \textit{for } 1 \le p \le p_0 - 1, \tag{3.2}$$

$$\sup_{m\ge 0} E_0\bigg[\Big|\inf_{n\ge 0}(X_{m+n} - X_m)\cdot\hat{u}\Big|^p\bigg] \le C \qquad \textit{for } 1 \le p \le p_0 - 1, \tag{3.3}$$

$$\sup_{m\ge 0} P_0\big\{(X_{n+m} - X_m)\cdot\hat{u} \le \sqrt{n}\big\} \le Cn^{-p} \qquad \textit{for } 1 \le p \le \frac{p_0 - 1}{2}. \tag{3.4}$$

Proof. Equation (3.1) follows from (2.5) and Jensen's inequality.

The proof of (3.2) comes by a renewal argument. Let $Y_j = \tau_{j+1} - \tau_j$ for $j \geq 1$ and $V_0 = 0$, $V_m = Y_1 + \cdots + Y_m$. The forward recurrence time of this pure renewal process is $g_n = \min\{k \geq 0\colon\ n + k \in \{V_m\}\}$. A decomposition according to the value of τ_1 gives

$$\tau_{J_n} - n = (\tau_1 - n)^+ + \sum_{k=1}^{n-1} \mathbb{1}\{\tau_1 = k\} g_{n-k}. \tag{3.5}$$

First we bound the moment of g_n. For this write a renewal equation

$$g_n = (Y_1 - n)^+ + \sum_{k=1}^{n-1} \mathbb{1}\{Y_1 = k\} g_{n-k} \circ \theta,$$

where θ shifts the sequence $\{Y_k\}$ so that $g_{n-k} \circ \theta$ is independent of Y_1. Only one term on the right can be nonzero, so for any $p \geq 1$

$$g_n^p = \bigl((Y_1 - n)^+\bigr)^p + \sum_{k=1}^{n-1} \mathbb{1}\{Y_1 = k\} (g_{n-k} \circ \theta)^p.$$

Set $z(n) = E_0[((Y_1 - n)^+)^p]$. Assumption $p \leq p_0 - 1$ and (2.5) give $E_0[Y_1^{p+1}] < \infty$ which implies $\sum z(n) < \infty$. Taking expectations and using independence gives the equation

$$E_0 g_n^p = z(n) + \sum_{k=1}^{n-1} P_0\{Y_1 = k\} E_0 g_{n-k}^p.$$

Induction on n shows that

$$E_0 g_n^p \leq \sum_{k=1}^{n} z(k) \leq C \quad \text{for all } n.$$

Raise (3.5) to the power p, take expectations, use Hypothesis (M), and substitute this last bound in there to complete the proof of (3.2).

Equation (3.3) follows readily. Since the walk does not backtrack after time τ_{J_m} and steps are bounded by Hypothesis (S),

$$\Bigl|\inf_{n \geq 0} (X_{m+n} - X_m) \cdot \hat{u}\Bigr| = \Bigl|\inf_{n\colon\, m \leq n \leq \tau_{J_m}} (X_n - X_m) \cdot \hat{u}\Bigr| \leq r_0 |\hat{u}| (\tau_{J_m} - m).$$

Apply (3.2) to this last quantity.

Lastly we show (3.4). For $a < b$ define

$$V_{a,b} = \sum_{i \geq 1} \mathbb{1}\{a < \tau_i < b\}.$$

Then $(X_{m+n} - X_m) \cdot \hat{u} \leq \sqrt{n}$ implies $V_{m,m+n} \leq \sqrt{n}$. Recall the i.i.d. structure of slabs $(\mathcal{S}_k)_{k \geq 1}$ defined in (2.4). For the first inequality note that either there are no regeneration times in $[m, m+n)$, or there is one and we restart at the first one.

$$\begin{aligned} &P_0\bigl\{V_{m,m+n} \leq \sqrt{n}\bigr\} \\ &\quad \leq P_0\{\tau_{J_m} - m \geq n\} + P_0\bigl\{V_{0,n} \leq \sqrt{n} \,|\, \beta = \infty\bigr\} + \sum_{k=1}^{n-1} P_0\{\tau_{J_m} - m = k\} P_0\bigl\{V_{0,n-k} \leq \sqrt{n} - 1 \,|\, \beta = \infty\bigr\} \end{aligned}$$

$$\le P_0\{\tau_{J_m} - m \ge n\} + P_0\{\tau_{[\sqrt{n}]+1} \ge n | \beta = \infty\} + C\sum_{k=1}^{n-1} k^{-2p} P_0\{\tau_{[\sqrt{n}]} \ge n - k | \beta = \infty\}$$

$$\le \frac{C}{n^p} + Cn^p \sum_{k=1}^{n-1} \frac{1}{k^{2p}(n-k)^{2p}} \le \frac{C}{n^p}.$$

We used (3.2) in the second inequality and then again in the third inequality, along with (3.1). For the last inequality split the sum according to $k \le n/2$ and $k > n/2$, in the former case bound $1/(n-k)$ by $2/n$, and in the latter case bound $1/k$ by $2/n$. □

4. Invariant measure and ergodicity

For integers ℓ define the σ-algebras $\mathfrak{S}_\ell = \sigma\{\omega_x\colon\ x \cdot \hat{u} \ge \ell\}$ on Ω. Denote the restriction of the measure $\mathbb{P}$ to the σ-algebra $\mathfrak{S}_\ell$ by $\mathbb{P}_{|\mathfrak{S}_\ell}$. In this section we prove the next two theorems. The variation distance of two probability measures is $d_{\mathrm{Var}}(\mu, \nu) = \sup\{\mu(A) - \nu(A)\}$ with the supremum taken over measurable sets A. $\mathbb{E}_\infty$ denotes expectation under the invariant measure $\mathbb{P}_\infty$ whose existence is established below. The corresponding joint measure on environments and paths is denoted by $P_0^\infty(d\omega, dx_{0,\infty}) = \mathbb{P}_\infty(d\omega) P_0^\omega(dx_{0,\infty})$ with expectation E_0^∞.

Theorem 4.1. *Assume $\mathbb{P}$ is product and satisfies Hypotheses (S) and (M), with $p_0 > 4d + 1$. Then there exists a probability measure $\mathbb{P}_\infty$ on Ω with these properties.*

(a) *Hypothesis (S) holds $\mathbb{P}_\infty$-almost surely.*
(b) *$\mathbb{P}_\infty$ is invariant and ergodic for the Markov transition kernel Π.*
(c) *For all $\ell \ge 1$*

$$d_{\mathrm{Var}}(\mathbb{P}_{\infty|\mathfrak{S}_\ell}, \mathbb{P}_{|\mathfrak{S}_\ell}) \le C\ell^{1-p_0}. \tag{4.1}$$

(d) *Under P_0^∞ the walk has these properties*:
  (i) *For $1 \le p \le p_0 - 1$*

$$E_0^\infty\Big[\Big|\inf_{n\ge 0} X_n \cdot \hat{u}\Big|^p\Big] \le C. \tag{4.2}$$

  (ii) *For $1 \le p \le (p_0 - 1)/2$ and $n \ge 1$,*

$$P_0^\infty\{X_n \cdot \hat{u} \le n^{1/2}\} \le Cn^{-p}. \tag{4.3}$$

More could be said about $\mathbb{P}_\infty$. For example, following [22], one can show that $\mathbb{P}_\infty$ comes as a limit, and has a renewal-type representation that involves the regeneration times. But we cover only properties needed in the sequel. Along the way we establish this ergodic theorem under the original environment measure.

Theorem 4.2. *Assumptions as in the above Theorem 4.1. Let Ψ be a bounded $\mathfrak{S}_{-a}$-measurable function on Ω, for some $0 < a < \infty$. Then*

$$\lim_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} \Psi(T_{X_j}\omega) = \mathbb{E}_\infty \Psi \quad P_0\text{-almost surely.} \tag{4.4}$$

Theorem 4.2 tells us that there is a unique invariant $\mathbb{P}_\infty$ in a natural relationship to $\mathbb{P}$, and also gives the absolute continuity $\mathbb{P}_{\infty|\mathfrak{S}_{-a}} \ll \mathbb{P}_{|\mathfrak{S}_{-a}}$. Limit (4.4) cannot hold for all bounded measurable Ψ on Ω because this would imply the absolute continuity $\mathbb{P}_\infty \ll \mathbb{P}$ on the entire space Ω. A counterexample that satisfies (M) and (S) but where the quenched walk is degenerate was given by Bolthausen and Sznitman [2], Proposition 1.5. Whether regularity assumption (R) or ellipticity will make a difference here is not presently clear. For the simpler case of space–time walks (see description of model in [12]) with nondegenerate P_0^ω absolute continuity $\mathbb{P}_\infty \ll \mathbb{P}$ does hold on the entire space. Theorem 3.1 in [2] proves this for nearest-neighbor jumps with some weak ellipticity. The general case is no harder.

Proof of Theorems 4.1 and 4.2. Let $\mathbb{P}_n(A) = P_0\{T_{X_n}\omega \in A\}$. A computation shows that

$$f_n(\omega) = \frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}(\omega) = \sum_x P_x^\omega\{X_n = 0\}.$$

By Hypothesis (S) we can replace the state space $\Omega = \mathcal{P}^{\mathbb{Z}^d}$ with the compact space $\Omega_0 = \mathcal{P}_0^{\mathbb{Z}^d}$, where

$$\mathcal{P}_0 = \bigl\{(p_z) \in \mathcal{P}:\ p_z = 0 \text{ if } |z| > r_0\bigr\}. \tag{4.5}$$

Compactness gives a subsequence $\{n_j\}$ along which $n_j^{-1}\sum_{m=1}^{n_j}\mathbb{P}_m$ converges weakly to a probability measure $\mathbb{P}_\infty$ on Ω_0. Hypothesis (S) transfers to $\mathbb{P}_\infty$ by virtue of having been included in the state space Ω_0. We have verified part (a) of Theorem 4.1.

Due to Hypothesis (S) Π is Feller-continuous. Consequently the weak limit $n_j^{-1}\sum_{m=1}^{n_j}\mathbb{P}_m \to \mathbb{P}_\infty$ together with $\mathbb{P}_{n+1} = \mathbb{P}_n\Pi$ implies the Π-invariance of $\mathbb{P}_\infty$.

Next we derive the bound on the variation distance. On metric spaces total variation distance can be characterized in terms of continuous functions:

$$d_{\mathrm{Var}}(\mu,\nu) = \frac{1}{2}\sup\biggl\{\int f\,\mathrm{d}\mu - \int f\,\mathrm{d}\nu:\ f \text{ continuous},\ \sup|f| \le 1\biggr\}.$$

This makes $d_{\mathrm{Var}}(\mu,\nu)$ lower semicontinuous which we shall find convenient below.

Fix $\ell > 0$. Then

$$\begin{aligned}\frac{\mathrm{d}\mathbb{P}_{n|\mathfrak{S}_\ell}}{\mathrm{d}\mathbb{P}_{|\mathfrak{S}_\ell}} &= \mathbb{E}\biggl[\sum_x P_x^\omega\Bigl\{X_n = 0, \max_{j\le n} X_j\cdot\hat{u} \le \frac{\ell}{2}\Bigr\}\Bigm|\mathfrak{S}_\ell\biggr]\\ &\quad + \sum_x \mathbb{E}\biggl[P_x^\omega\Bigl\{X_n = 0, \max_{j\le n} X_j\cdot\hat{u} > \frac{\ell}{2}\Bigr\}\Bigm|\mathfrak{S}_\ell\biggr].\end{aligned} \tag{4.6}$$

The $L^1(\mathbb{P})$-norm of the second term is

$$\sum_x P_x\Bigl\{X_n = 0, \max_{j\le n} X_j\cdot\hat{u} > \frac{\ell}{2}\Bigr\} = P_0\Bigl\{\max_{j\le n} X_j\cdot\hat{u} > X_n\cdot\hat{u} + \frac{\ell}{2}\Bigr\} \equiv I_{n,\ell}.$$

The integrand in the first term on the right-hand side of (4.6) is measurable with respect to $\sigma(\omega_x:\ x\cdot\hat{u} \le \ell/2)$ and therefore independent of $\mathfrak{S}_\ell$. So this term is equal to the nonrandom constant

$$\begin{aligned}&\sum_x P_x\Bigl\{X_n = 0, \max_{j\le n} X_j\cdot\hat{u} \le \frac{\ell}{2}\Bigr\}\\ &\quad = 1 - P_0\Bigl\{\max_{j\le n} X_j\cdot\hat{u} > X_n\cdot\hat{u} + \frac{\ell}{2}\Bigr\}\\ &\quad = 1 - I_{n,\ell}.\end{aligned}$$

Altogether,

$$d_{\mathrm{Var}}(\mathbb{P}_{n|\mathfrak{S}_\ell}, \mathbb{P}_{|\mathfrak{S}_\ell}) \le \frac{1}{2}\int\biggl|\frac{\mathrm{d}\mathbb{P}_{n|\mathfrak{S}_\ell}}{\mathrm{d}\mathbb{P}_{|\mathfrak{S}_\ell}} - 1\biggr|\,\mathrm{d}\mathbb{P} \le I_{n,\ell}.$$

Now write

$$
\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} I_{k,\ell} &= \frac{1}{n}\sum_{k=1}^{n} P_0\Big\{\max_{j\le k} X_j\cdot\hat u > X_k\cdot\hat u + \frac{\ell}{2}\Big\} \\
&\le \frac{1}{n}E_0\Bigg[\sum_{k=1}^{\tau_1\wedge n}\mathbb{1}\Big\{\max_{j\le k} X_j\cdot\hat u > X_k\cdot\hat u + \frac{\ell}{2}\Big\}\Bigg] + \frac{1}{n}\sum_{k=2}^{n} E_0\Big[(\tau_k-\tau_{k-1})\mathbb{1}\Big\{X_{\tau_k}\cdot\hat u - X_{\tau_{k-1}}\cdot\hat u > \frac{\ell}{2}\Big\}\Big] \\
&\le n^{-1}E_0[\tau_1\wedge n] + \frac{n-1}{n}E_0\Big[\tau_1\mathbb{1}\Big\{\tau_1 > \frac{\ell}{2r_0}\Big\}\,\Big|\,\beta=\infty\Big] \\
&\le Cn^{-1} + C\ell^{1-p_0}.
\end{aligned}
$$

The last inequality came from Hypothesis (M) and Hölder's inequality. Let $n\to\infty$ along the relevant subsequence and use lower semicontinuity and convexity of the variation distance. This proves part (c).

Concerning backtracking: notice first that due to (3.3) we have

$$
\mathbb{E}_k\Big[E_0^\omega\Big(\Big|\inf_{n\ge 0} X_n\cdot\hat u\Big|^p\Big)\Big] = E_0\Big[E_0^{T_{X_k}\omega}\Big(\Big|\inf_{n\ge 0} X_n\cdot\hat u\Big|^p\Big)\Big] = E_0\Big[\Big|\inf_{n\ge 0}(X_{n+k}-X_k)\cdot\hat u\Big|^p\Big] \le C_p.
$$

Since $E_0^\omega(|\inf_{0\le n\le N} X_n\cdot\hat u|^p)$ is a continuous function of ω, the definition of $\mathbb{P}_\infty$ along with the above estimate and monotone convergence imply (4.2). (e.i) has been proved.

Write once again, using (3.4)

$$
\mathbb{E}_k\big[P_0^\omega\{X_n\cdot\hat u\le\sqrt{n}\}\big] = E_0\big[P_0^{T_{X_k}\omega}\{X_n\cdot\hat u\le\sqrt{n}\}\big] = P_0\big\{(X_{n+k}-X_k)\cdot\hat u\le\sqrt{n}\big\}\le Cn^{-p}.
$$

Since $P_0^\omega\{X_n\cdot\hat u\le\sqrt{n}\}$ is a continuous function of ω, the definition of $\mathbb{P}_\infty$ along with the above estimate imply (4.3) and proves (e.ii).

As the last point we prove the ergodicity. Let Ψ be a bounded local function on Ω. It suffices to prove that for some constant b

$$
\lim_{n\to\infty} E_0^\infty\Bigg|n^{-1}\sum_{j=0}^{n-1}\Psi(T_{X_j}\omega) - b\Bigg| = 0. \tag{4.7}
$$

By an approximation it follows from this that for all $F\in L^1(\mathbb{P}_\infty)$

$$
n^{-1}\sum_{j=0}^{n-1}\Pi^j F(\omega)\to\mathbb{E}_\infty F\quad\text{in } L^1(\mathbb{P}_\infty). \tag{4.8}
$$

By standard theory (Section IV.2 in [16]) this is equivalent to ergodicity of $\mathbb{P}_\infty$ for the transition Π.

We combine the proof of Theorem 4.2 with the proof of (4.7). For this purpose let a be a positive integer and Ψ a bounded $\mathfrak{S}_{-a+1}$-measurable function. Let

$$
\varphi_i = \sum_{j=\tau_{ai}}^{\tau_{a(i+1)}-1}\Psi(T_{X_j}\omega).
$$

From the i.i.d. regeneration slabs and the moment bound (3.1) follows the limit

$$
\lim_{m\to\infty} m^{-1}\sum_{j=0}^{\tau_{am}-1}\Psi(T_{X_j}\omega) = \lim_{m\to\infty} m^{-1}\sum_{i=0}^{m-1}\varphi_i = b_0\quad P_0\text{-almost surely,} \tag{4.9}
$$

where the constant b_0 is defined by the limit.

To justify limit (4.9) more explicitly, recall the definition of regeneration slabs given in (2.4). Define a function Φ of the regeneration slabs by

$$\Phi(\mathcal{S}_0,\mathcal{S}_1,\mathcal{S}_2,\dots)=\sum_{j=\tau_a}^{\tau_{2a}-1}\Psi(T_{X_j}\omega).$$

Since each regeneration slab has thickness in $\hat{u}$-direction at least 1, the Ψ-terms in the sum do not read the environments below level zero and consequently the sum is a function of $(\mathcal{S}_0,\mathcal{S}_1,\mathcal{S}_2,\dots)$. Next one can check for $k\ge 1$ that

$$\begin{aligned}&\Phi(\mathcal{S}_{a(k-1)},\mathcal{S}_{a(k-1)+1},\mathcal{S}_{a(k-1)+2},\dots)\\&\quad=\sum_{j=\tau_a(X_{\tau_{a(k-1)}+\cdot}-X_{\tau_{a(k-1)}})}^{\tau_{2a}(X_{\tau_{a(k-1)}+\cdot}-X_{\tau_{a(k-1)}})-1}\Psi\bigl(T_{X_{\tau_{a(k-1)}+j}-X_{\tau_{a(k-1)}}}(T_{X_{\tau_{a(k-1)}}}\omega)\bigr)=\varphi_k.\end{aligned}$$

Now the sum of φ-terms in (4.9) can be decomposed into

$$\varphi_0+\varphi_1+\sum_{k=1}^{m-2}\Phi(\mathcal{S}_{ak},\mathcal{S}_{ak+1},\mathcal{S}_{ak+2},\dots).$$

The limit (4.9) follows because the slabs $(\mathcal{S}_k)_{k\ge 1}$ are i.i.d. and the finite initial terms $\varphi_0+\varphi_1$ are eliminated by the m^{-1} factor.

Let $\alpha_n=\inf\{k\colon\ \tau_{ak}\ge n\}$. Bound (3.1) implies that $n^{-1}(\tau_{a(\alpha_n-1)}-\tau_{a\alpha_n})\to 0$ P_0-almost surely. Consequently (4.9) yields the next limit, for another constant b:

$$\lim_{n\to\infty}n^{-1}\sum_{j=0}^{n-1}\Psi(T_{X_j}\omega)=b\quad P_0\text{-almost surely.}\tag{4.10}$$

By boundedness this limit is valid also in $L^1(P_0)$ and the initial point of the walk is immaterial by shift-invariance of $\mathbb{P}$. Let $\ell>0$ and abbreviate

$$G_{n,x}(\omega)=E_x^\omega\Biggl[\Biggl|n^{-1}\sum_{j=0}^{n-1}\Psi(T_{X_j}\omega)-b\Biggr|\mathbb{1}\Biggl\{\inf_{j\ge 0}X_j\cdot\hat{u}\ge X_0\cdot\hat{u}-\frac{\ell^{1/2}}{2}\Biggr\}\Biggr].$$

Let

$$\mathcal{I}=\bigl\{x\in\mathbb{Z}^d\colon\ x\cdot\hat{u}\ge\ell^{1/2},\ |x|\le r_0\ell\bigr\}.$$

If ℓ is large enough relative to a, then for $x\in\mathcal{I}$ the function $G_{n,x}$ is $\mathfrak{S}_{\ell^{1/2}/3}$-measurable. Use the bound (4.1) on the variation distance and the fact that the functions $G_{n,x}(\omega)$ are uniformly bounded over all x,n,ω.

$$\begin{aligned}\mathbb{P}_\infty\Biggl\{\sum_{x\in\mathcal{I}}P_0^\omega[X_\ell=x]G_{n,x}(\omega)\ge\varepsilon_1\Biggr\}&\le\sum_{x\in\mathcal{I}}\mathbb{P}_\infty\Biggl\{G_{n,x}(\omega)\ge\frac{\varepsilon_1}{C\ell^d}\Biggr\}\\&\le C\ell^d\varepsilon_1^{-1}\sum_{x\in\mathcal{I}}\mathbb{E}_\infty G_{n,x}\le C\ell^d\varepsilon_1^{-1}\sum_{x\in\mathcal{I}}\mathbb{E}G_{n,x}+C\ell^{2d}\varepsilon_1^{-1}\ell^{(1-p_0)/2}.\end{aligned}$$

By (4.10) $\mathbb{E}G_{n,x}\to 0$ for any fixed x. Thus from above we get for any fixed ℓ,

$$\overline{\lim}_{n\to\infty}E_0^\infty\bigl[\mathbb{1}\{X_\ell\in\mathcal{I}\}G_{n,X_\ell}\bigr]\le\varepsilon_1+C\ell^{2d}\varepsilon_1^{-1}\ell^{(1-p_0)/2}.$$

The reader should bear in mind that the constant C is changing from line to line. Finally, take $p \le (p_0-1)/2$ and use (4.2) and (4.3) to write

$$\begin{aligned}
&\overline{\lim}_{n\to\infty} E_0^\infty\left|n^{-1}\sum_{j=0}^{n-1}\Psi(T_{X_j}\omega)-b\right| \\
&\quad\le \overline{\lim}_{n\to\infty} E_0^\infty\left[\mathbb{1}\{X_\ell\in\mathcal{I}\}\left|n^{-1}\sum_{j=\ell}^{n+\ell-1}\Psi(T_{X_j}\omega)-b\right|\mathbb{1}\left\{\inf_{j\ge\ell}X_j\cdot\hat{u}\ge X_\ell\cdot\hat{u}-\frac{\ell^{1/2}}{2}\right\}\right] \\
&\qquad+CP_0^\infty\{X_\ell\notin\mathcal{I}\}+CP_0^\infty\left\{\inf_{j\ge\ell}X_j\cdot\hat{u}<X_\ell\cdot\hat{u}-\frac{\ell^{1/2}}{2}\right\} \\
&\quad\le \overline{\lim}_{n\to\infty} E_0^\infty\big[\mathbb{1}\{X_\ell\in\mathcal{I}\}G_{n,X_\ell}\big] \\
&\qquad+CP_0^\infty\big\{X_\ell\cdot\hat{u}<\ell^{1/2}\big\}+CP_0^\infty\left\{\inf_{j\ge0}X_j\cdot\hat{u}<-\frac{\ell^{1/2}}{2}\right\} \\
&\quad\le \varepsilon_1+C\ell^{2d}\varepsilon_1^{-1}\ell^{(1-p_0)/2}+C\ell^{-p}+C\ell^{-p/2}.
\end{aligned}$$

Consequently, if we first pick ε_1 small enough then ℓ large, we will have shown (4.7). For the second term on the last line we need $p_0>4d+1$. Ergodicity of $\mathbb{P}_\infty$ has been shown. This concludes the proof of Theorem 4.1.

Theorem 4.2 has also been established. It follows from the combination of (4.7) and (4.10). □

5. Change of measure

There are several stages in the proof where we need to check that a desired conclusion is not affected by choice between $\mathbb{P}$ and $\mathbb{P}_\infty$. We collect all instances of such transfers in this section. The standing assumptions of this section are that $\mathbb{P}$ is an i.i.d. product measure that satisfies Hypotheses (M) and (S), and that $\mathbb{P}_\infty$ is the measure given by Theorem 4.1. We show first that $\mathbb{P}_\infty$ can be replaced with $\mathbb{P}$ in the key condition (2.1) of Theorem 2.1.

Lemma 5.1. *The velocity v defined by (2.6) satisfies $v=\mathbb{E}_\infty(D)$. There exists a constant C such that*

$$\big|E_0(X_n)-n\mathbb{E}_\infty(D)\big|\le C\quad\text{for all } n\ge1. \tag{5.1}$$

Proof. We start by showing $v=\mathbb{E}_\infty(D)$. The finite step-size condition in the definition of (4.5) of $\mathcal{P}_0$ makes the function $D(\omega)$ bounded and continuous on Ω_0. By the Cesàro definition of $\mathbb{P}_\infty$,

$$\mathbb{E}_\infty(D)=\lim_{j\to\infty}\frac{1}{n_j}\sum_{k=0}^{n_j-1}\mathbb{E}_k(D)=\lim_{j\to\infty}\frac{1}{n_j}\sum_{k=0}^{n_j-1}E_0\big[D(T_{X_k}\omega)\big].$$

Hypothesis (S) implies that the law of large numbers $n^{-1}X_n\to v$ holds also in $L^1(P_0)$. From this and the Markov property

$$v=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}E_0[X_{k+1}-X_k]=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}E_0\big[D(T_{X_k}\omega)\big].$$

We have proved $v=\mathbb{E}_\infty(D)$.

The variables $(X_{\tau_{j+1}}-X_{\tau_j},\tau_{j+1}-\tau_j)_{j\ge1}$ are i.i.d. with sufficient moments by Hypotheses (M) and (S). With $\alpha_n=\inf\{j\ge1:\ \tau_j-\tau_1\ge n\}$ Wald's identity gives

$$\begin{aligned}
&E_0[X_{\tau_{\alpha_n}}-X_{\tau_1}]=E_0[\alpha_n]E_0[X_{\tau_1}|\beta=\infty], \\
&E_0[\tau_{\alpha_n}-\tau_1]=E_0[\alpha_n]E_0[\tau_1|\beta=\infty].
\end{aligned}$$

Consequently, by the definition (2.6) of v,

$$E_0[X_n] - nv = vE_0[\tau_{\alpha_n} - \tau_1 - n] - E_0[X_{\tau_{\alpha_n}} - X_{\tau_1} - X_n].$$

The right-hand side is bounded by a constant again by Hypotheses (M) and (S) and by (3.2). □

Proposition 5.2. *Assume that there exists an* $\bar\alpha < 1/2$ *such that*

$$\mathbb{E}\big[\big|E_0^\omega(X_n) - E_0(X_n)\big|^2\big] = \mathrm{O}\big(n^{2\bar\alpha}\big). \tag{5.2}$$

Let $\alpha \in (\bar\alpha, 1)$ *and assume that*

$$p_0 > \frac{5d}{\alpha - \bar\alpha}. \tag{5.3}$$

Then condition (2.1) *is satisfied with* α.

Proof. Assumption (5.3) permits us to choose p such that

$$2d\frac{1-\bar\alpha}{\alpha-\bar\alpha} < p \le \frac{p_0 - 1}{2}.$$

Due to the strict inequality above there is room to choose $0 < \varepsilon < d^{-1}(\alpha - \bar\alpha)$ such that $p > 2d + 2\varepsilon^{-1}(1-\alpha)$. Let $\ell = n^\varepsilon$ and $j = \ell^2$.

By (5.1) assumption (5.2) turns into

$$\mathbb{E}\big[\big|E_0^\omega(X_n) - nv\big|^2\big] = \mathrm{O}\big(n^{2\bar\alpha}\big). \tag{5.4}$$

Define $A_\ell = \{\inf_{n\ge 0} X_n \cdot \hat u \ge \ell\}$. The next calculation starts with Π-invariance of $\mathbb{P}_\infty$.

$$\begin{aligned}
&\mathbb{E}_\infty\big[\big|E_0^\omega(X_n) - nv\big|^2\big] \\
&\quad = E_0^\infty\big[\big|E_0^{T_{X_j}\omega}(X_n - nv)\big|^2\big] \\
&\quad \le E_0^\infty\big[\big|E_0^{T_{X_j}\omega}(X_n - nv)\big|^2, X_j\cdot\hat u > \ell\big] + 4r_0^2n^2P_0^\infty\{X_j\cdot\hat u \le \ell\} \\
&\quad \le 2E_0^\infty\big[\big|E_0^{T_{X_j}\omega}(X_n - nv, A_{-\ell/2})\big|^2, X_j\cdot\hat u > \ell\big] \\
&\qquad + 8r_0^2n^2E_0^\infty\big[P_0^{T_{X_j}\omega}\big(A_{-\ell/2}^c\big), X_j\cdot\hat u > \ell\big] + 4r_0^2n^2P_0^\infty\{X_j\cdot\hat u \le \ell\} \\
&\quad \le 2\sum_{\substack{x:\,|x|\le r_0 j\\ \text{and } x\cdot\hat u > \ell}} \mathbb{E}_\infty\big[\big|E_0^{T_x\omega}(X_n - nv, A_{-\ell/2})\big|^2\big] \\
&\qquad + 8r_0^2n^2\sum_{\substack{x:\,|x|\le r_0 j\\ \text{and } x\cdot\hat u > \ell}} \mathbb{E}_\infty\big[P_0^{T_x\omega}\big(A_{-\ell/2}^c\big)\big] + 4r_0^2n^2P_0^\infty\{X_j\cdot\hat u \le \ell\}
\end{aligned}$$

(switch from $\mathbb{E}_\infty$ back to $\mathbb{E}$ by (4.1))

$$\begin{aligned}
&\quad \le 2\sum_{\substack{x:\,|x|\le r_0 j\\ \text{and } x\cdot\hat u > \ell}} \mathbb{E}\big[\big|E_0^{T_x\omega}(X_n - nv, A_{-\ell/2})\big|^2\big] + 8r_0^2n^2\sum_{\substack{x:\,|x|\le r_0 j\\ \text{and } x\cdot\hat u > \ell}} \mathbb{E}\big[P_0^{T_x\omega}\big(A_{-\ell/2}^c\big)\big] \\
&\qquad + C(r_0 j)^d r_0^2 n^2 \ell^{-p} + 4r_0^2n^2P_0^\infty\{X_j\cdot\hat u \le \ell\} \\
&\quad \le 2\sum_{\substack{x:\,|x|\le r_0 j\\ \text{and } x\cdot\hat u > \ell}} \mathbb{E}\big[|X_n - nv|^2\big] + 16r_0^2n^2\sum_{\substack{x:\,|x|\le r_0 j\\ \text{and } x\cdot\hat u > \ell}} P_0\big(A_{-\ell/2}^c\big) + C(r_0 j)^d r_0^2n^2\ell^{-p} + 4r_0^2n^2P_0^\infty\{X_j\cdot\hat u \le \ell\}
\end{aligned}$$

(use form (5.4) of the assumption; apply (3.3) to $P_0(A^c_{-\ell/2})$ and (4.3) to $P_0^\infty\{X_j \cdot \hat{u} \le \ell\}$; recall that $j = \ell^2 = n^{2\varepsilon}$)

$$\le Cj^d n^{2\bar{\alpha}} + Cj^d n^2 \ell^{-p} + Cn^2 j^{-p} \le C\bigl(n^{2\bar{\alpha}+2d\varepsilon} + n^{2d\varepsilon+2-p\varepsilon} + n^{2-2p\varepsilon}\bigr).$$

The first two exponents are $< 2\alpha$ by the choice of p and ε, and the last one is less than the second one. □

Once we have verified the assumptions of Theorem 2.1 we have the CLT under $\mathbb{P}_\infty$-almost every ω. But the goal is the CLT under $\mathbb{P}$-almost every ω. As the final point of this section we prove the transfer of the central limit theorem from $\mathbb{P}_\infty$ to $\mathbb{P}$. This is where we use the ergodic theorem, Theorem 4.2. Let W be the probability distribution of the Brownian motion with diffusion matrix $\mathfrak{D}$.

Lemma 5.3. *Suppose the weak convergence $Q_n^\omega \Rightarrow W$ holds for $\mathbb{P}_\infty$-almost every ω. Then the same is true for $\mathbb{P}$-almost every ω.*

Proof. It suffices to show that for any $\delta > 0$ and any bounded uniformly continuous F on $D_{\mathbb{R}^d}[0,\infty)$

$$\varlimsup_{n\to\infty} E_0^\omega\bigl[F(B_n)\bigr] \le \int F\,\mathrm{d}W + \delta \quad \mathbb{P}\text{-a.s.}$$

By considering also $-F$ this gives $E_0^\omega[F(B_n)] \to \int F\,\mathrm{d}W$ $\mathbb{P}$-a.s. for each such function. A countable collection of them determines weak convergence.

Fix such an F and assume $|F| \le 1$. Let $c = \int F\,\mathrm{d}W$ and

$$\bar{h}(\omega) = \varlimsup_{n\to\infty} E_0^\omega\bigl[F(B_n)\bigr].$$

For $\ell > 0$ recall the events

$$A_{-\ell} = \Bigl\{\inf_{n\ge 0} X_n \cdot \hat{u} \ge -\ell\Bigr\}$$

and define

$$\bar{h}_\ell(\omega) = \varlimsup_{n\to\infty} E_0^\omega\bigl[F(B_n), A_{-\ell}\bigr]$$

and

$$\Psi_\ell(\omega) = \mathbb{1}\biggl\{\omega\colon\ \bar{h}_\ell(\omega) \le c + \frac{1}{2}\delta,\ P_0^\omega\bigl(A^c_{-\ell}\bigr) \le \frac{1}{2}\delta\biggr\}.$$

The assumed quenched CLT under $\mathbb{P}_\infty$ gives $\mathbb{P}_\infty\{\bar{h} = c\} = 1$. Therefore, $\mathbb{P}_\infty$-a.s.

$$\Psi_\ell(\omega) = \mathbb{1}\biggl\{\omega\colon\ P_0^\omega\bigl(A^c_{-\ell}\bigr) \le \frac{1}{2}\delta\biggr\}.$$

From (4.2) we know that if ℓ is fixed large enough, then $\mathbb{E}_\infty \Psi_\ell > 0$. Since Ψ_ℓ is $\mathfrak{S}_{-\ell}$-measurable Theorem 4.2 implies that

$$n^{-1}\sum_{j=1}^n \Psi_\ell(T_{X_j}\omega) \to \mathbb{E}_\infty \Psi_\ell > 0 \quad P_0\text{-a.s.}$$

But $\{\bar{h}_\ell \le c + \frac{1}{2}\delta,\ P_0^\omega(A^c_{-\ell}) \le \frac{1}{2}\delta\} \subset \{\bar{h} \le c + \delta\}$. We conclude that the stopping time

$$\zeta = \inf\bigl\{n \ge 0\colon\ \bar{h}(T_{X_n}\omega) \le c + \delta\bigr\}$$

is P_0-a.s. finite. From the definitions we now have

$$\overline{\lim}_{n\to\infty} E_0^{T_{X_\zeta}\omega}\big[F(B_n)\big] \le \int F\,\mathrm{d}W + \delta \quad P_0\text{-a.s.}$$

Then by bounded convergence

$$\overline{\lim}_{n\to\infty} E_0^{\omega} E_0^{T_{X_\zeta}\omega}\big[F(B_n)\big] \le \int F\,\mathrm{d}W + \delta \quad \mathbb{P}\text{-a.s.}$$

Since ζ is a finite stopping time, the strong Markov property, the uniform continuity of F and bounded step size Hypothesis (S) imply

$$\overline{\lim}_{n\to\infty} E_0^{\omega}\big[F(B_n)\big] \le \int F\,\mathrm{d}W + \delta \quad \mathbb{P}\text{-a.s.}$$

This concludes the proof. □

6. Reduction to path intersections

The preceding sections have reduced the proof of the main result Theorem 1.1 to proving the estimate

$$\mathbb{E}\big[\big|E_0^\omega(X_n) - E_0(X_n)\big|^2\big] = \mathrm{O}\big(n^{2\alpha}\big) \quad \text{for some } \alpha < \frac{1}{2}. \tag{6.1}$$

The next reduction takes us to the expected number of intersections of the paths of two independent walks X and $\widetilde{X}$ in the same environment. The argument uses a decomposition into martingale differences through an ordering of lattice sites. This idea for bounding a variance is natural and has been used in RWRE earlier by Bolthausen and Sznitman [2].

Let $P_{0,0}^\omega$ be the quenched law of the walks $(X, \widetilde{X})$ started at $(X_0, \widetilde{X}_0) = (0, 0)$ and $P_{0,0} = \int P_{0,0}^\omega \mathbb{P}(\mathrm{d}\omega)$ the averaged law with expectation operator $E_{0,0}$. The set of sites visited by a walk is denoted by $X_{[0,n)} = \{X_k\colon 0 \le k < n\}$ and $|A|$ is the number of elements in a discrete set A.

Proposition 6.1. *Let $\mathbb{P}$ be an i.i.d. product measure and satisfy Hypotheses (M) and (S). Assume that there exists an $\bar\alpha < 1/2$ such that*

$$E_{0,0}\big[|X_{[0,n)} \cap \widetilde{X}_{[0,n)}|\big] = \mathrm{O}\big(n^{2\bar\alpha}\big). \tag{6.2}$$

Let $\alpha \in (\bar\alpha, 1/2)$. Assume

$$p_0 > \frac{22d}{\alpha - \bar\alpha}. \tag{6.3}$$

Then condition (6.1) is satisfied for α.

Proof. For $L \ge 0$, define $\mathcal{B}(L) = \{x \in \mathbb{Z}^d\colon |x| \le L\}$. Fix $n \ge 1$ and let $(x_j)_{j\ge1}$ be some fixed ordering of $\mathcal{B}(r_0 n)$ satisfying

$$\forall i \ge j\colon \quad x_i \cdot \hat{u} \ge x_j \cdot \hat{u}.$$

For $B \subset \mathbb{Z}^d$ let $\mathfrak{S}_B = \sigma\{\omega_x\colon x \in B\}$. Let $A_j = \{x_1, \dots, x_j\}$, $\zeta_0 = E_0(X_n)$, and for $j \ge 1$

$$\zeta_j = \mathbb{E}\big[E_0^\omega(X_n) | \mathfrak{S}_{A_j}\big].$$

$(\zeta_j - \zeta_{j-1})_{j\ge 1}$ is a sequence of $L^2(\mathbb{P})$-martingale differences. By Hypothesis (S) $X_n \in \mathcal{B}(r_0 n)$ and so

$$\mathbb{E}\bigl[\bigl|E_0^\omega(X_n) - E_0(X_n)\bigr|^2\bigr] = \sum_{j=1}^{|\mathcal{B}(r_0 n)|} \mathbb{E}\bigl[|\zeta_j - \zeta_{j-1}|^2\bigr]. \tag{6.4}$$

For $z \in \mathbb{Z}^d$ define half-spaces

$$\mathcal{H}(z) = \bigl\{x \in \mathbb{Z}^d:\ x\cdot\hat{u} > z\cdot\hat{u}\bigr\}.$$

Since $A_{j-1} \subset A_j \subset \mathcal{H}(x_j)^c$,

$$\begin{aligned}
&\mathbb{E}\bigl[|\zeta_j - \zeta_{j-1}|^2\bigr] \\
&\quad = \int \mathbb{P}(\mathrm{d}\omega_{A_j}) \left| \iint \mathbb{P}(\mathrm{d}\omega_{A_j^c})\mathbb{P}(\mathrm{d}\tilde{\omega}_{x_j})\bigl\{E_0^\omega(X_n) - E_0^{\langle\omega,\tilde{\omega}_{x_j}\rangle}(X_n)\bigr\}\right|^2 \\
&\quad \le \iint \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(x_j)^c})\mathbb{P}(\mathrm{d}\tilde{\omega}_{x_j}) \left| \int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(x_j)})\bigl\{E_0^\omega(X_n) - E_0^{\langle\omega,\tilde{\omega}_{x_j}\rangle}(X_n)\bigr\}\right|^2. \qquad (6.5)
\end{aligned}$$

Above $\langle\omega,\tilde{\omega}_{x_j}\rangle$ denotes an environment obtained from ω by replacing ω_{x_j} with $\tilde{\omega}_{x_j}$.

We fix a point $z = x_j$ to develop a bound for the expression above, and then return to collect the estimates. Abbreviate $\tilde{\omega} = \langle\omega,\tilde{\omega}_{x_j}\rangle$. Consider two walks that both start at 0, one obeys environment ω and the other obeys $\tilde{\omega}$. Couple them so that they stay together until the first time they visit z. Until a visit to z happens, the walks are identical. Let

$$H_z = \min\{n \ge 1:\ X_n = z\}$$

be the first hitting time of site z and write

$$\begin{aligned}
&\int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)})\bigl(E_0^\omega(X_n) - E_0^{\tilde{\omega}}(X_n)\bigr) \qquad (6.6)\\
&\quad = \int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)}) \sum_{m=0}^{n-1} P_0^\omega\{H_z = m\}\bigl(E_z^\omega[X_{n-m} - z] - E_z^{\tilde{\omega}}[X_{n-m} - z]\bigr) \\
&\quad = \int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)}) \sum_{m=0}^{n-1}\sum_{\ell>0} P_0^\omega\Bigl\{H_z = m, \ell - 1 \le \max_{0\le j\le m} X_j\cdot\hat{u} - z\cdot\hat{u} < \ell\Bigr\} \qquad (6.7)\\
&\qquad \times \bigl(E_z^\omega[X_{n-m} - z] - E_z^{\tilde{\omega}}[X_{n-m} - z]\bigr).
\end{aligned}$$

Decompose $\mathcal{H}(z) = \mathcal{H}_\ell(z) \cup \mathcal{H}'_\ell(z)$, where

$$\mathcal{H}_\ell(z) = \bigl\{x \in \mathbb{Z}^d:\ z\cdot\hat{u} < x\cdot\hat{u} < z\cdot\hat{u} + \ell\bigr\} \quad \text{and} \quad \mathcal{H}'_\ell(z) = \bigl\{x \in \mathbb{Z}^d:\ x\cdot\hat{u} \ge z\cdot\hat{u} + \ell\bigr\}.$$

Take a single (ℓ, m) term from the sum in (6.7) and only the expectation $E_z^\omega[X_{n-m} - z]$, and split it further into two terms:

$$\begin{aligned}
&\int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)}) P_0^\omega\Bigl\{H_z = m, \ell - 1 \le \max_{0\le j\le m} X_j\cdot\hat{u} - z\cdot\hat{u} < \ell\Bigr\} E_z^\omega[X_{n-m} - z] \\
&\quad = \int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)}) P_0^\omega\Bigl\{H_z = m, \ell - 1 \le \max_{0\le j\le m} X_j\cdot\hat{u} - z\cdot\hat{u} < \ell\Bigr\} E_z^\omega[X_{\tau_\ell + n - m} - X_{\tau_\ell}] \qquad (6.8)\\
&\qquad + \int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)}) P_0^\omega\Bigl\{H_z = m, \ell - 1 \le \max_{0\le j\le m} X_j\cdot\hat{u} - z\cdot\hat{u} < \ell\Bigr\} E_z^\omega[X_{n-m} - X_{\tau_\ell + n - m} + X_{\tau_\ell} - z]. \qquad (6.9)
\end{aligned}$$

Regeneration time τ_ℓ with index ℓ is used simply to guarantee that the post-regeneration walk $X_{\tau_\ell+\cdot}$ stays in $\mathcal{H}'_\ell(z)$. Below we make use of this to get independence from the environments in $\mathcal{H}'_\ell(z)^c$.

Integral (6.8) is developed further as follows.

$$\begin{aligned}
&\int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)}) P_0^\omega\Big\{H_z = m, \ell-1 \le \max_{0\le j\le m} X_j\cdot\hat u - z\cdot\hat u < \ell\Big\} E_z^\omega[X_{\tau_\ell+n-m} - X_{\tau_\ell}] \\
&\quad= \int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}_\ell(z)}) P_0^\omega\Big\{H_z = m, \ell-1 \le \max_{0\le j\le m} X_j\cdot\hat u - z\cdot\hat u < \ell\Big\} \int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}'_\ell(z)}) E_z^\omega[X_{\tau_\ell+n-m} - X_{\tau_\ell}] \\
&\quad= \int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}_\ell(z)}) P_0^\omega\Big\{H_z = m, \ell-1 \le \max_{0\le j\le m} X_j\cdot\hat u - z\cdot\hat u < \ell\Big\} E_z[X_{\tau_\ell+n-m} - X_{\tau_\ell} | \mathfrak{S}_{\mathcal{H}'_\ell(z)^c}] \\
&\quad= \int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}_\ell(z)}) P_0^\omega\Big\{H_z = m, \ell-1 \le \max_{0\le j\le m} X_j\cdot\hat u - z\cdot\hat u < \ell\Big\} E_0[X_{n-m} | \beta = \infty]. \qquad (6.10)
\end{aligned}$$

The last equality above comes from the regeneration structure, see Theorem 1.4 in [22]. The σ-algebra $\mathfrak{S}_{\mathcal{H}'_\ell(z)^c}$ is contained in the σ-algebra $\mathcal{G}_\ell$ defined by (1.29) of [22] for the walk starting at z.

The last quantity (6.10) above reads the environment only until the first visit to z, hence does not see the distinction between ω and $\tilde\omega$. Consequently when integral (6.7) is developed separately for ω and $\tilde\omega$ into the sum of integrals (6.8) and (6.9), integrals (6.8) first develop into (6.10) separately for ω and $\tilde\omega$ and then cancel each other.

We are left with two instances of integral (6.9), one for both ω and $\tilde\omega$. Put these back into the (ℓ, m) sum in (6.7). Include also the square around this expression from line (6.5). These expressions for ω and $\tilde\omega$ are bounded separately with identical steps and added together in the end. Thus we first separate the two by an application of $(a+b)^2 \le 2(a^2+b^2)$. We continue the argument for the expression for ω with this bound on the square of (6.7):

$$2\Bigg\{\sum_{\ell>0}\sum_{m=0}^{n-1}\int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)}) P_0^\omega\Big\{H_z = m, \ell-1 \le \max_{0\le k\le m} X_k\cdot\hat u - z\cdot\hat u < \ell\Big\}\Big|E_z^\omega(X_{n-m} - X_{\tau_\ell+n-m} + X_{\tau_\ell} - z)\Big|\Bigg\}^2$$

(apply the step bound (S))

$$\le 8r_0^2 \int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)})\Bigg\{\sum_{\ell>0} P_0^\omega\Big\{H_z < n, \ell-1 \le \max_{0\le k\le H_z} X_k\cdot\hat u - z\cdot\hat u < \ell\Big\} E_z^\omega(\tau_\ell)\Bigg\}^2$$

(introduce $\varepsilon = (\alpha - \bar\alpha)/4 > 0$)

$$\begin{aligned}
&\le 16r_0^2 n^\varepsilon \sum_{\ell\le n^\varepsilon}\int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)}) P_0^\omega\{H_z<n\}^2 E_z^\omega\big(\tau_\ell^2\big) \\
&\quad + 16r_0^2 \sum_{\ell>n^\varepsilon}\int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)}) P_0^\omega\Big\{H_z<n, \ell-1 \le \max_{0\le k\le H_z} X_k\cdot\hat u - z\cdot\hat u < \ell\Big\} E_z^\omega\big(\tau_\ell^2\big)
\end{aligned}$$

[pick conjugate exponents $p>1$ and $q>1$]

$$\begin{aligned}
&\le 16r_0^2 n^\varepsilon \sum_{\ell\le n^\varepsilon}\bigg(\int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)}) P_0^\omega\{H_z<n\}^{2q}\bigg)^{1/q}\bigg(\int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)}) E_z^\omega\big[\tau_\ell^{2p}\big]\bigg)^{1/p} \\
&\quad + 16r_0^2 \sum_{\ell>n^\varepsilon}\bigg(\int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)}) E_z^\omega\big[\tau_\ell^{2p}\big]\bigg)^{1/p} \\
&\qquad \times\bigg(\int \mathbb{P}(\mathrm{d}\omega_{\mathcal{H}(z)}) P_0^\omega\Big\{H_z<n, \ell-1 \le \max_{0\le k\le H_z} X_k\cdot\hat u - z\cdot\hat u < \ell\Big\}^q\bigg)^{1/q}.
\end{aligned}$$

The step above requires $p_0 \ge 2p$. This and what is needed below can be achieved by choosing

$$p = \frac{d}{\alpha - \bar\alpha} \quad \text{and} \quad q = \frac{d}{d - (\alpha - \bar\alpha)}.$$

Now put the above bound and its counterpart for $\tilde\omega$ back into (6.5), and continue with another application of Hölder's inequality:

$$\begin{aligned}
&\mathbb{E}\big[|\zeta_j - \zeta_{j-1}|^2\big] \\
&\quad\le 32 r_0^2 n^\varepsilon \sum_{\ell \le n^\varepsilon} \mathbb{E}\big[P_0^\omega\{H_{x_j} < n\}^{2q}\big]^{1/q} E_0\big[\tau_\ell^{2p}\big]^{1/p} \\
&\qquad + 32 r_0^2 \sum_{\ell > n^\varepsilon} E_0\big[\tau_\ell^{2p}\big]^{1/p} \mathbb{E}\Big[P_0^\omega\Big\{H_{x_j} < n, \ell - 1 \le \max_{0 \le k \le H_{x_j}} X_k \cdot \hat u - x_j \cdot \hat u < \ell\Big\}^q\Big]^{1/q}
\end{aligned}$$

(apply (3.1))

$$\begin{aligned}
&\quad\le C n^{4\varepsilon} \mathbb{E}\big[P_0^\omega\{H_{x_j} < n\}^{2q}\big]^{1/q} \\
&\qquad + C \sum_{\ell > n^\varepsilon} \ell^2 \mathbb{E}\Big[P_0^\omega\Big\{H_{x_j} < n, \ell - 1 \le \max_{0 \le k \le H_{x_j}} X_k \cdot \hat u - x_j \cdot \hat u < \ell\Big\}^q\Big]^{1/q}
\end{aligned}$$

(utilize $q > 1$)

$$\begin{aligned}
&\quad\le C n^{4\varepsilon} \mathbb{E}\big[P_0^\omega\{H_{x_j} < n\}^2\big]^{1/q} \\
&\qquad + C \sum_{\ell > n^\varepsilon} \ell^2 \sum_{k=0}^{n-1} \sum_{|x| \le r_0 n} E_0\Big[P_0^\omega\{X_k = x\} P_x^\omega\Big\{\Big|\inf_{m \ge 0} X_m \cdot \hat u - x \cdot \hat u\Big| \ge \ell - 1\Big\}\Big]^{1/q} \\
&\quad\le C n^{4\varepsilon} P_{0,0}\{x_j \in X_{[0,n)} \cap \widetilde X_{[0,n)}\}^{1/q} + C n^{d+1} \sum_{\ell > n^\varepsilon} \ell^2 P_0\Big\{\Big|\inf_{m \ge 0} X_m \cdot \hat u\Big| \ge \ell - 1\Big\}^{1/q} \\
&\quad\le C n^{4\varepsilon} P_{0,0}\{x_j \in X_{[0,n)} \cap \widetilde X_{[0,n)}\}^{1/q} + C n^{2\alpha - d}.
\end{aligned}$$

In the last step we used (3.3) with an exponent $\tilde p = 3q + q\varepsilon^{-1}(2d + 1 - 2\alpha)$. This requires $\tilde p \le p_0 - 1$ which follows from (6.3). Finally put these bounds in the sum in (6.4) and develop the last bound:

$$\begin{aligned}
\mathbb{E}\big[\big|E_0^\omega(X_n) - E_0(X_n)\big|^2\big] &= \sum_{j=1}^{|\mathcal{B}(r_0 n)|} \mathbb{E}\big[|\zeta_j - \zeta_{j-1}|^2\big] \\
&\le C n^{4\varepsilon} \sum_{j=1}^{|\mathcal{B}(r_0 n)|} P_{0,0}\{x_j \in X_{[0,n)} \cap \widetilde X_{[0,n)}\}^{1/q} + C n^{2\alpha} \\
&\le C n^{4\varepsilon} \big(n^d\big)^{1-1/q} \Bigg(\sum_{j=1}^{|\mathcal{B}(r_0 n)|} P_{0,0}\{x_j \in X_{[0,n)} \cap \widetilde X_{[0,n)}\}\Bigg)^{1/q} + C n^{2\alpha} \\
&\le C n^{4\varepsilon + d - d/q + 2\bar\alpha/q} + C n^{2\alpha},
\end{aligned}$$

where we used the assumption (6.2) in the last inequality. With $q = d(d - (\alpha - \bar\alpha))^{-1}$ and $\varepsilon = (\alpha - \bar\alpha)/4$ as chosen above, the last line is $O(n^{2\alpha})$. Equation (6.1) has been verified. □

7. Bound on intersections

The remaining piece of the proof of Theorem 1.1 is this estimate:

$$E_{0,0}\big[|X_{[0,n)} \cap \widetilde{X}_{[0,n)}|\big] = \mathrm{O}\big(n^{2\alpha}\big) \quad \text{for some } \alpha < \frac{1}{2}, \tag{7.1}$$

where X and $\widetilde{X}$ are two independent walks driven by a common environment with quenched distribution $P^\omega_{x,y}[X_{0,\infty} \in A, \widetilde{X}_{0,\infty} \in B] = P^\omega_x(A)P^\omega_y(B)$ and averaged distribution $E_{x,y}(\cdot) = \mathbb{E}P^\omega_{x,y}(\cdot)$.

To deduce the sublinear bound we introduce joint regeneration times at which both walks regenerate on the same level in space (but not necessarily at the same time). Intersections happen only within the joint regeneration slabs, and the expected number of intersections decays at a polynomial rate in the distance between the points of entry into the slab. From joint regeneration to regeneration the difference of the two walks is a Markov chain. This Markov chain can be approximated by a symmetric random walk. Via this preliminary work the required bound boils down to deriving a Green function estimate for a Markov chain that can be suitably approximated by a symmetric random walk. This part is relegated to Appendix A. Except for the appendices, we complete the proof of the functional central limit theorem in this section.

To aid our discussion of a pair of walks $(X, \widetilde{X})$ we introduce some new notation. We write $\theta^{m,n}$ for the shift on pairs of paths: $\theta^{m,n}(x_{0,\infty}, y_{0,\infty}) = (\theta^m x_{0,\infty}, \theta^n y_{0,\infty})$. If we write separate expectations for X and $\widetilde{X}$ under $P^\omega_{x,y}$, these are denoted by E^ω_x and $\widetilde{E}^\omega_y$.

By a *joint stopping time* we mean a pair $(\alpha, \tilde{\alpha})$ that satisfies $\{\alpha = m, \tilde{\alpha} = n\} \in \sigma\{X_{0,m}, \widetilde{X}_{0,n}\}$. Under the distribution $P^\omega_{x,y}$ the walks X and $\widetilde{X}$ are independent. Consequently if $\alpha \vee \tilde{\alpha} < \infty$ $P^\omega_{x,y}$-almost surely then for any events A and B,

$$\begin{aligned} &P^\omega_{x,y}\big\{(X_{0,\alpha}, \widetilde{X}_{0,\tilde{\alpha}}) \in A, (X_{\alpha,\infty}, \widetilde{X}_{\tilde{\alpha},\infty}) \in B\big\} \\ &\quad = E^\omega_{x,y}\big[\mathbb{1}\big\{(X_{0,\alpha}, \widetilde{X}_{0,\tilde{\alpha}}) \in A\big\} P^\omega_{X_\alpha, \widetilde{X}_{\tilde{\alpha}}}\big\{(X_{0,\infty}, \widetilde{X}_{0,\infty}) \in B\big\}\big]. \end{aligned}$$

This type of joint restarting will be used without comment in the sequel.

The backtracking time β is as before in (2.2) and for the $\widetilde{X}$ walk it is $\tilde{\beta} = \inf\{n \ge 1\colon \widetilde{X}_n \cdot \hat{u} < \widetilde{X}_0 \cdot \hat{u}\}$. When the walks are on a common level their difference lies in the hyperplane

$$\mathbb{V}_d = \big\{z \in \mathbb{Z}^d\colon z \cdot \hat{u} = 0\big\}. \tag{7.2}$$

From a common level there is a uniform positive chance for simultaneously never backtracking.

Lemma 7.1. *Assume $\hat{u}$-transience* (1.1) *and the bounded step Hypothesis* (S). *Then*

$$\eta \equiv \inf_{x-y \in \mathbb{V}_d} P_{x,y}\{\beta \wedge \tilde{\beta} = \infty\} > 0. \tag{7.3}$$

Proof. By shift-invariance it is enough to consider the case $P_{0,x}$ for $x \in \mathbb{V}_d$. By the independence of environments and the bound r_0 on the step size,

$$P_{0,x}\{\beta = \tilde{\beta} = \infty\} \ge P_0\left\{\beta > \frac{|x|}{4r_0}\right\}^2 - 2P_0\left\{\frac{|x|}{4r_0} < \beta < \infty\right\}.$$

As $|x| \to \infty$ the right-hand side above converges to $2\eta_1 = P_0\{\beta = \infty\}^2 > 0$. Then we can find $L > 0$ such that

$$|x| > L \quad \Longrightarrow \quad P_{0,x}\{\beta \wedge \tilde{\beta} = \infty\} > \eta_1 > 0. \tag{7.4}$$

It remains to check that $P_{0,x}\{\beta \wedge \tilde{\beta} = \infty\} > 0$ for any fixed $x \le |L|$. The case $x = 0$ is immediate because $P_{0,0}\{\beta = \tilde{\beta} = \infty\} = 0$ implies $P^\omega_0\{\beta = \infty\}^2 = 0$ $\mathbb{P}$-a.s. and therefore contradicts transience (2.3).

Let us assume that $x \neq 0$.

If $\mathcal{J}=\{z\colon \mathbb{E}\pi_{0,z}>0\}\subset\mathbb{R}u$, transience implies $u\cdot\hat{u}>0$. Then $x+\mathbb{R}u$ and $\mathbb{R}u$ do not intersect and independence gives $P_{0,x}\{\beta=\tilde{\beta}=\infty\}=P_0\{\beta=\infty\}^2>0$. (We did not invoke Hypothesis (R) to rule out this case to avoid appealing to (R) unnecessarily.)

Let us now assume that $\mathcal{J}\not\subset\mathbb{R}u$ for any u.

The proof is completed by constructing two finite walks that start at 0 and x with these properties: the walks do not backtrack below level 0, they reach a common fresh level ℓ at entry points that are as far apart as desired, and this pair of walks has positive probability. Then if additionally the walks regenerate at level ℓ (an event independent of the one just described) the event $\beta\wedge\tilde{\beta}=\infty$ has been realized. We also make these walks reach level ℓ in such a manner that no lower level can serve as a level for joint regeneration. This construction will be helpful later on in the proof of Lemma 7.13.

To construct the paths let z and w be two nonzero noncollinear vectors such that $z\cdot\hat{u}>0$, $\mathbb{E}\pi_{0z}>0$, and $\mathbb{E}\pi_{0w}>0$. Such exist: the assumption that $\mathcal{J}$ not be one-dimensional implies the existence of some pair of noncollinear vectors $w,\tilde{w}\in\mathcal{J}$. Then transience (1.1) implies the existence of $z\in\mathcal{J}$ with $z\cdot\hat{u}>0$. Either w or $\tilde{w}$ must be noncollinear with z.

The case $w\cdot\hat{u}>0$ is easy: let one walk repeat z-steps and the other one repeat w-steps suitably many times. We provide more detail for the case $w\cdot\hat{u}\le 0$.

Let $n>0$ and $m\ge 0$ be the minimal integers such that $-nw\cdot\hat{u}=mz\cdot\hat{u}$. Since $mz+nw\ne 0$ by noncollinearity but $(mz+nw)\cdot\hat{u}=0$ there must exist a vector $\tilde{u}$ such that $\tilde{u}\cdot\hat{u}=0$ and $mz\cdot\tilde{u}+nw\cdot\tilde{u}>0$. Replacing x by $-x$ if necessary we can then assume that

$$nw\cdot\tilde{u}+mz\cdot\tilde{u}>0\ge x\cdot\tilde{u}. \tag{7.5}$$

Interchangeability of x and $-x$ comes from symmetry and shift-invariance:

$$P_{0,x}\{\beta\wedge\tilde{\beta}=\infty\}=P_{0,-x}\{\beta\wedge\tilde{\beta}=\infty\}.$$

The point of (7.5) is that the path $\{(iz)_{i=0}^m,(mz+jw)_{j=0}^n\}$ points away from x in direction $\tilde{u}$.

Pick k large enough to have $|x-kmz-knw|>L$. Let the X walk start at 0 and take km z-steps followed by kn w-steps (returning back to level 0) and then $km+1$ z-steps (ending at a fresh level). Let the $\widetilde{X}$ walk start at x and take $km+1$ z-steps. These two paths do not self-intersect or intersect each other, as can be checked routinely though somewhat tediously.

The endpoints of the paths are $2kmz+z+knw$ and $x+kmz+z$ which are on a common level, but further than L apart. After these paths let the two walks regenerate, with probability controlled by (7.4). This joint evolution implies $\beta\wedge\tilde{\beta}=\infty$ so by independence of environments

$$P_{0,x}\{\beta\wedge\tilde{\beta}=\infty\}\ge(\mathbb{E}\pi_{0z})^{3km+2}(\mathbb{E}\pi_{0w})^{kn}\eta_1>0. \qquad\square$$

We now begin the development towards joint regeneration times for the walks X and $\widetilde{X}$. Define the stopping time

$$\gamma_\ell=\inf\{n\ge 0\colon\ X_n\cdot\hat{u}\ge\ell\}$$

and the running maximum

$$M_n=\sup\{X_i\cdot\hat{u}\colon\ i\le n\}.$$

We write $\gamma(\ell)$ when subscripts or superscripts become complicated. $\widetilde{M}_n$ and $\tilde{\gamma}_\ell$ are the corresponding quantities for the $\widetilde{X}$ walk.

Let h be the greatest common divisor of

$$\mathcal{L}=\bigl\{\ell\ge 0\colon\ P_0(\exists n\colon\ X_n\cdot\hat{u}=\ell)>0\bigr\}. \tag{7.6}$$

First we observe that all high enough multiples of h are accessible levels from 0.

Lemma 7.2. *There exists a finite ℓ_0 such that for all $\ell \ge \ell_0$*

$$P_0\{\exists n\colon\ X_n \cdot \hat u = h\ell\} > 0.$$

Proof. The point is that $\mathcal{L}$ is closed under addition. Indeed, if ℓ_1 and ℓ_2 are in $\mathcal{L}$, then let $x^{(i)}_{0,n_i}$, $i \in \{1,2\}$, be two paths such that $x^{(i)}_0 = 0$, $x^{(i)}_{n_i} \cdot \hat u = \ell_i$, and $P_0\{X_{0,n_i} = x^{(i)}_{0,n_i}\} > 0$. Let k_1 be the smallest index such that $x^{(1)}_{k_1} = x^{(1)}_{n_1} + x^{(2)}_{k_2}$ for some $k_2 \in [0,n_2]$. The set of such k_1 is not empty because $k_1 = n_1$ and $k_2 = 0$ satisfy this equality. Now the path $(x^{(1)}_{0,k_1}, x^{(1)}_{n_1} + x^{(2)}_{k_2+1,n_2})$ starts at 0, ends on level $\ell_1+\ell_2$ and has positive P_0-probability.

The familiar argument [7], Lemma 5.4, Chapter 5, shows that all large enough multiples of h lie in $\mathcal{L}$. □

Next we show that all high enough multiples of h can be reached as fresh levels without backtracking.

Lemma 7.3. *There exists a finite ℓ_1 such that for all $\ell \ge \ell_1$*

$$P_0\{X_{\gamma_{h\ell}} \cdot \hat u = h\ell, \beta > \gamma_{h\ell}\} > 0. \tag{7.7}$$

Proof. Pick and fix a step x such that $\mathbb{E}\pi_{0,x} > 0$ and $x \cdot \hat u > 0$. Then $x \cdot \hat u = kh$ for some $k > 0$. For any $0 \le j \le k-1$, by appeal to Lemma 7.2, we find a path $\sigma^{(j)}$, with positive P_0-probability, going from 0 to a level ℓh with $\ell = j \bmod k$. By deleting initial and final segments if necessary and by shifting the reduced path, we can assume that $\sigma^{(j)}$ visits a level in $kh\mathbb{Z}$ only at the beginning and a level in $jh + kh\mathbb{Z}$ only at the end. In particular, $\sigma^{(0)}$ is the single point 0.

Let $y^{(j)}$ be the endpoint of $\sigma^{(j)}$. Pick $m = m^{(j)}$ large enough so that the path $\tilde\sigma^{(j)} = ((ix)_{0\le i<m}, mx + \sigma^{(j)}, mx + y^{(j)} + (ix)_{1\le i\le m})$ stays at or above level 0 and ends at a fresh level. It has positive P_0-probability because its constituent pieces all do. Note that the only self-intersections are those that possibly exist within the piece $mx + \sigma^{(j)}$, and even these can be removed by erasing loops from $\sigma^{(j)}$ as part of its construction if so desired. Let ℓ_1 be the maximal level attained by $\tilde\sigma^{(0)}, \dots, \tilde\sigma^{(k-1)}$.

Given $\ell \ge \ell_1$ let $j = \ell \bmod k$. Path $\tilde\sigma^{(j)}$ followed by appropriately many x-steps realizes the event in (7.7) and has positive P_0-probability. □

Next we extend the estimation to joint fresh levels of two walks reached without backtracking.

Lemma 7.4. *Let $\ell_2 h$ be the next multiple of h after $r_0|\hat u| + \ell_1 h$ with ℓ_1 as in Lemma 7.3. There exists $\eta > 0$ with this property: uniformly over all x and y such that $x \cdot \hat u, y \cdot \hat u \in [0, r_0|\hat u|] \cap h\mathbb{Z}$,*

$$P_{x,y}\bigl\{\exists i\colon\ ih \in [0,\ell_2 h], X_{\gamma_{ih}} \cdot \hat u = \widetilde X_{\tilde\gamma_{ih}} \cdot \hat u = ih, \beta > \gamma_{ih}, \tilde\beta > \tilde\gamma_{ih}\bigr\} \ge \eta. \tag{7.8}$$

Proof. Let $x \cdot \hat u = \ell h$ and $y \cdot \hat u = \tilde\ell h$. Lemma 7.3 gives a positive P_0-probability path $\sigma = z_{0,n}$ that connects 0 to level $\ell_2 h - \ell h$ and stays above level 0. Choose $\tilde\sigma = \tilde z_{0,\tilde n}$ similarly for $\tilde\ell$. If the paths $x + \sigma$ and $y + \tilde\sigma$ intersect, redefine $x + \sigma$ to follow $y + \tilde\sigma$ from the first time it intersects $y + \tilde\sigma$. The probability in (7.8) is bounded below by

$$P_{x,y}\{X_{0,n} = x + \sigma, \widetilde X_{0,\tilde n} = y + \tilde\sigma\} > 0.$$

Uniformity over x, y comes from observing that there are finitely many possible such positive lower bounds because we have finitely many admissible initial levels ℓ and $\tilde\ell$ and finitely many ways to intersect the shifts of the corresponding paths. □

Define the first common fresh level to be

$$L = \inf\{\ell\colon\ X_{\gamma_\ell} \cdot \hat u = \widetilde X_{\tilde\gamma_\ell} \cdot \hat u = \ell\}.$$

If the walks start on a common level then this initial level is L. Iteration of Lemma 7.4 shows that L is always a.s. finite provided the walks start on levels in $h\mathbb{Z}$. (This and more is proved in Lemma 7.5.)

Next we define, in stages, the first joint regeneration level of two walks $(X, \widetilde{X})$ that start at initial points $X_0, \widetilde{X}_0$ on a common level $\lambda_0 \in h\mathbb{Z}$. First define

$$J = \begin{cases} M_{\beta\wedge\tilde\beta} \vee \tilde M_{\beta\wedge\tilde\beta} + h & \text{if } \beta\wedge\tilde\beta < \infty, \\ \infty & \text{if } \beta\wedge\tilde\beta = \infty, \end{cases}$$

and then

$$\lambda = \begin{cases} L\circ\theta^{\gamma_J,\tilde\gamma_J} = \inf\{\ell \ge J\colon\ X_{\gamma_\ell}\cdot\hat u = \widetilde{X}_{\tilde\gamma_\ell}\cdot\hat u = \ell\} & \text{if } J<\infty, \\ \infty & \text{if } J=\infty. \end{cases}$$

If $\lambda < \infty$, then λ is the first common fresh level after at least one walk backtracked. Also, $\lambda = \infty$ iff neither walk backtracked. Let

$$\lambda_1 = L\circ\theta^{\gamma(\lambda_0+h),\tilde\gamma(\lambda_0+h)}$$

which is the first common fresh level strictly above the initial level λ_0. For $n \ge 2$ as long as $\lambda_{n-1} < \infty$ define successive common fresh levels

$$\lambda_n = \lambda\circ\theta^{\gamma_{\lambda_{n-1}},\tilde\gamma_{\lambda_{n-1}}}.$$

Joint regeneration at level λ_n is signaled by $\lambda_{n+1} = \infty$. Consequently the first joint regeneration level is

$$\Lambda = \sup\{\lambda_n\colon\ \lambda_n < \infty\}.$$

$\Lambda < \infty$ a.s. because by Lemma 7.1 at each common fresh level λ_n the walks have at least chance $\eta > 0$ to simultaneously not backtrack. The first joint regeneration times are

$$(\mu_1, \tilde\mu_1) = (\gamma_\Lambda, \tilde\gamma_\Lambda). \tag{7.9}$$

The present goal is to get moment bounds on μ_1 and $\tilde\mu_1$. To be able to shift levels back to level 0 we fix representatives from all non-empty levels. For all $j \in \mathcal{L}_0 = \{z\cdot\hat u\colon\ z\in\mathbb{Z}^d\}$ pick and fix $\hat v(j) \in \mathbb{Z}^d$ such that $\hat v(j)\cdot\hat u = j$. By the definition of h as the greatest common divisor of $\mathcal{L}$ in (7.6) and the group structure of $\mathcal{L}_0$, $\hat v(j)$ is defined for all $j \in h\mathbb{Z}$.

Lemma 7.5. *For $m \ge 1$ and $p \le p_0$*

$$\sup_{x,y\in\mathbb{V}_d} P_{x,y}\{\Lambda > m\} \le C_p m^{-p}. \tag{7.10}$$

Proof. Recall ℓ_2 from Lemma 7.4. Consider $m > 2\ell_2 h$ and let $n_0 = [m/(2\ell_2 h)]$.

Iterations of (7.3) utilized below proceed as follows: for $k \ge 2$ and any event B that depends on the paths $(X_{0,\gamma(\lambda_{k-1})}, \widetilde{X}_{0,\tilde\gamma(\lambda_{k-1})})$,

$$\begin{aligned} &P_{x,y}\{\lambda_k<\infty, \lambda_{k-1}<\infty, B\} \\ &\quad= P_{x,y}\bigl\{(\beta\wedge\tilde\beta)\circ\theta^{\gamma_{\lambda_{k-1}},\tilde\gamma_{\lambda_{k-1}}} < \infty, \lambda_{k-1}<\infty, B\bigr\} \\ &\quad= \sum_{z,w} P_{x,y}\{X_{\gamma(\lambda_{k-1})} = z, \widetilde{X}_{\tilde\gamma(\lambda_{k-1})} = w, \lambda_{k-1}<\infty, B\} P_{z,w}\{\beta\wedge\tilde\beta<\infty\} \\ &\quad\le P_{x,y}\{\lambda_{k-1}<\infty, B\}(1-\eta). \end{aligned}$$

The product comes from dependence on disjoint environments: the event $\{\beta\wedge\tilde\beta<\infty\}$ does not need environments below the starting level $z\cdot\hat u = w\cdot\hat u$, while the event $\{X_{\gamma(\lambda_{k-1})} = z, \widetilde{X}_{\tilde\gamma(\lambda_{k-1})} = w, B\}$ only reads environments strictly below this level.

After the sum decomposition below iterate (7.3) to bound $P_{x,y}\{\lambda_{n_0}<\infty\}$ and to go from $\lambda_n<\infty$ down to $\lambda_{k+1}<\infty$ inside the sum. Then weaken $\lambda_{k+1}<\infty$ to $\lambda_k<\infty$. Note that $\lambda_1<\infty$ a.s. so this event does not contribute a $1-\eta$ factor and hence there is only a power $(1-\eta)^{n_0-1}$ for the middle term.

$$\begin{aligned}
&P_{x,y}\{\Lambda>2m\}\\
&\quad\le P_{x,y}\{\lambda_1>m\}+P_{x,y}\{\lambda_{n_0}<\infty\}+\sum_{n=2}^{n_0-1}\sum_{k=1}^{n-1}P_{x,y}\left\{\lambda_n<\infty,\frac{m}{n}<\lambda\circ\theta^{\gamma_{\lambda_k},\tilde\gamma_{\lambda_k}}-\lambda_k<\infty\right\}\\
&\quad\le P_{x,y}\{\lambda_1>m\}+(1-\eta)^{n_0-1} \qquad (7.11)\\
&\qquad+\sum_{n=2}^{n_0-1}\sum_{k=1}^{n-1}(1-\eta)^{n-k-1}P_{x,y}\left\{\lambda_k<\infty,\frac{m}{n}<\lambda\circ\theta^{\gamma_{\lambda_k},\tilde\gamma_{\lambda_k}}-\lambda_k<\infty\right\}. \qquad (7.12)
\end{aligned}$$

Separate probability (7.12) into two parts:

$$\begin{aligned}
&P_{x,y}\left\{\lambda_k<\infty,\frac{m}{n}<\lambda\circ\theta^{\gamma_{\lambda_k},\tilde\gamma_{\lambda_k}}-\lambda_k<\infty\right\}\\
&\quad\le 2P_{x,y}\left\{\lambda_k<\infty,\frac{m}{2n}<M_{\beta\wedge\tilde\beta}\circ\theta^{\gamma_{\lambda_k},\tilde\gamma_{\lambda_k}}+h-\lambda_k<\infty\right\} \qquad (7.13)\\
&\qquad+P_{x,y}\left\{\lambda_k<\infty, J\circ\theta^{\gamma_{\lambda_k},\tilde\gamma_{\lambda_k}}<\infty,\frac{m}{2n}<(L\circ\theta^{\gamma_J,\tilde\gamma_J}-J)\circ\theta^{\gamma_{\lambda_k},\tilde\gamma_{\lambda_k}}\right\}. \qquad (7.14)
\end{aligned}$$

For probability (7.13)

$$\begin{aligned}
&P_{x,y}\left(\lambda_k<\infty,\frac{m}{2n}<M_{\beta\wedge\tilde\beta}\circ\theta^{\gamma_{\lambda_k},\tilde\gamma_{\lambda_k}}+h-\lambda_k<\infty\right)\\
&\quad=\sum_{z\cdot\hat u=\tilde z\cdot\hat u=0}P_{x,y}\{\lambda_k<\infty,X_{\gamma_{\lambda_k}}=z+\hat v(\lambda_k),\widetilde X_{\tilde\gamma_{\lambda_k}}=\tilde z+\hat v(\lambda_k)\}P_{z,\tilde z}\left\{\frac{m}{2n}<M_{\beta\wedge\tilde\beta}+h<\infty\right\}\\
&\quad\le CP_{x,y}\{\lambda_k<\infty\}\left(\frac{n}{m}\right)^p\le\cdots\le C(1-\eta)^{k-1}\left(\frac{n}{m}\right)^p. \qquad (7.15)
\end{aligned}$$

The independence above came from the fact that the variable $M_{\beta\wedge\tilde\beta}$ needs environments only on levels at or above the initial level. Starting at level 0, on the event $\beta\wedge\tilde\beta<\infty$ we have

$$M_{\beta\wedge\tilde\beta}+h\le r_0|\hat u|\beta\wedge\tilde\beta+h\le C(\tau_1+\tilde\tau_1).$$

Then we invoked Hypothesis (M) for the moments of τ_1 and $\tilde\tau_1$. Finally iterate (7.3) again as prior to (7.12).

Probability (7.14) does not develop as conveniently because L needs environments below the starting level. To remove this dependence we use the event $\mathcal{E}$ defined below. Start by rewriting (7.14) as follows.

$$\begin{aligned}
&P_{x,y}\left\{\lambda_k<\infty, J\circ\theta^{\gamma_{\lambda_k},\tilde\gamma_{\lambda_k}}<\infty,\frac{m}{2n}<(L\circ\theta^{\gamma_J,\tilde\gamma_J}-J)\circ\theta^{\gamma_{\lambda_k},\tilde\gamma_{\lambda_k}}\right\}\\
&\quad=\sum_{j\in h\mathbb{Z}}\sum_{z,\tilde z}E_{x,y}\left[\lambda_k<\infty, J\circ\theta^{\gamma(\lambda_k),\tilde\gamma(\lambda_k)}=j, X_{\gamma_j}=z,\widetilde X_{\tilde\gamma_j}=\tilde z, P^\omega_{z,\tilde z}\left\{\frac{m}{2n}<L-j\right\}\right]. \qquad (7.16)
\end{aligned}$$

Fix j for the moment. We bound the probability in (7.16). Let s_0 and s_1 be the integers defined by

$$(s_0-1)\ell_2h<j\le s_0\ell_2h<\cdots<s_1\ell_2h\le j+\frac{m}{2n}<(s_1+1)\ell_2h.$$

In the beginning of the proof we assured that $\frac{m}{2n} > \ell_2 h$ so s_0 and s_1 are well defined. Define

$$\mathcal{E} = \{\exists i\colon ih \in [0, \ell_2 h], X_{\gamma_{ih}} \cdot \hat{u} = \widetilde{X}_{\tilde{\gamma}_{ih}} \cdot \hat{u} = ih, \beta > \gamma_{ih}, \tilde{\beta} > \tilde{\gamma}_{ih}\},$$

an event that guarantees a common fresh level in a zone of height $\ell_2 h$ without backtracking. We use $\mathcal{E}$ in situations where the levels of the initial points are in $[0, r_0|\hat{u}|] \cap h\mathbb{Z}$ and then $\mathcal{E}$ only needs environments $\{\omega_a\colon a \cdot \hat{u} \in [0, \ell_2 h)\}$. For any integer $s \in [s_0, s_1 - 1]$ we do the following decomposition.

$$\begin{aligned} &P^{\omega}_{z,\tilde{z}}\{L > (s+1)\ell_2 h\} \\ &\quad\le P^{\omega}_{z,\tilde{z}}\{L > s\ell_2 h, (X_{\gamma(s\ell_2 h)+\cdot} - \hat{v}(s\ell_2 h), \widetilde{X}_{\tilde{\gamma}(s\ell_2 h)+\cdot} - \hat{v}(s\ell_2 h)) \in \mathcal{E}^c\} \\ &\quad\le \sum_{w,\tilde{w}} P^{\omega}_{z,\tilde{z}}\{L > s\ell_2 h, X_{\gamma(s\ell_2 h)} = w, \widetilde{X}_{\tilde{\gamma}(s\ell_2 h)} = \tilde{w}\} P^{T_{\hat{v}(s\ell_2 h)}\omega}_{w - \hat{v}(s\ell_2 h), \tilde{w} - \hat{v}(s\ell_2 h)}\{\mathcal{E}^c\}. \end{aligned}$$

To begin the iterative factoring write $P^{\omega}_{z,\tilde{z}}\{\frac{m}{2n} < L - j\} \le P^{\omega}_{z,\tilde{z}}\{L > s_1 \ell_2 h\}$ and substitute the above decomposition with $s = s_1 - 1$ into (7.16). Notice that for each $(w, \tilde{w})$, the quenched probability

$$P^{T_{\hat{v}((s_1-1)\ell_2 h)}\omega}_{w - \hat{v}((s_1-1)\ell_2 h), \tilde{w} - \hat{v}((s_1-1)\ell_2 h)}\{\mathcal{E}^c\}$$

is a function of environments $\{\omega_a\colon a \cdot \hat{u} \in [(s_1 - 1)\ell_2 h, s_1 \ell_2 h)\}$ and thereby independent of everything else inside the expectation $E_{x,y}$ in (7.16), as long as $s_0 \le s_1 - 1$. By Lemma 7.4

$$P_{w - \hat{v}((s_1-1)\ell_2 h), \tilde{w} - \hat{v}((s_1-1)\ell_2 h)}\{\mathcal{E}^c\} \le 1 - \eta.$$

After this first round probability (7.14) is bounded, via (7.16), by

$$\sum_{j \in h\mathbb{Z}} \sum_{z,\tilde{z}} E_{x,y}\bigl[\lambda_k < \infty, J \circ \theta^{\gamma(\lambda_k), \tilde{\gamma}(\lambda_k)} = j, X_{\gamma_j} = z, \widetilde{X}_{\tilde{\gamma}_j} = \tilde{z}, P^{\omega}_{z,\tilde{z}}\{L > (s_1 - 1)\ell_2 h\}\bigr](1 - \eta).$$

This procedure is repeated $s_1 - s_0 - 1$ times to arrive at the upper bound

$$\begin{aligned} &P_{x,y}\left\{\lambda_k < \infty, J \circ \theta^{\gamma_{\lambda_k}, \tilde{\gamma}_{\lambda_k}} < \infty, \frac{m}{2n} < (L \circ \theta^{\gamma_J, \tilde{\gamma}_J} - J) \circ \theta^{\gamma_{\lambda_k}, \tilde{\gamma}_{\lambda_k}}\right\} \\ &\quad\le P_{x,y}\{\lambda_k < \infty\}(1 - \eta)^{s_1 - s_0 - 1} \\ &\quad\le C P_{x,y}\{\lambda_k < \infty\}(1 - \eta)^{m/(2\ell_2 h n)} \\ &\quad\le C P_{x,y}\{\lambda_k < \infty\}\left(\frac{n}{m}\right)^p \le C(1 - \eta)^{k-1}\left(\frac{n}{m}\right)^p. \end{aligned}$$

In the last step we iterated (7.3) as earlier.

Substitute this upper bound and (7.15) back to lines (7.13) and (7.14). These in turn go back into the sum on line (7.12). The remaining probability $P_{x,y}\{\lambda_1 > m\}$ on line (7.11) is bounded by $C\mathrm{e}^{-cm}$, by another iteration of Lemma 7.4 with the help of event $\mathcal{E}$.

To summarize, we have shown

$$P_{x,y}\{\Lambda > 2m\} \le C\mathrm{e}^{-cm} + C\sum_{n \ge 1} n(1 - \eta)^{n-2}\left(\frac{n}{m}\right)^p \le Cm^{-p}.$$

□

Next we extend the tail bound to the regeneration times.

Lemma 7.6. *Suppose $p_0 > 3$. Then*

$$\sup_{x,y \in \mathbb{V}_d} P_{x,y}[\mu_1 \vee \tilde{\mu}_1 \ge m] \le Cm^{-p_0/3}. \tag{7.17}$$

In particular, for any $p < p_0/3$,

$$\sup_{x,y\in\mathbb{V}_d} E_{x,y}\big[|\mu_1 \vee \tilde{\mu}_1|^p\big] \le C. \tag{7.18}$$

Proof. By (3.1), since $x \cdot \hat{u} = 0$ for $x \in \mathbb{V}_d$, we can bound

$$P_{x,y}\{\gamma_\ell \ge m\} = P_0\{\gamma_\ell \ge m\} \le P_0\{\tau_\ell \ge m\} \le C\left(\frac{\ell}{m}\right)^{p_0}.$$

Pick conjugate exponents $s = 3$ and $t = 3/2$.

$$\begin{aligned}
P_{x,y}\{\mu_1 \ge m\} &\le \sum_{\ell\ge 1} P_{x,y}\{\gamma_\ell \ge m, \Lambda = \ell\} \\
&\le C\sum_{\ell\ge 1} P_0\{\gamma_\ell \ge m\}^{1/s} P_{x,y}\{\Lambda = \ell\}^{1/t} \\
&\le C\sum_{\ell\ge 1} \frac{\ell^{p_0/3}}{m^{p_0/3}} \frac{1}{\ell^{2p_0/3}} \le Cm^{-p_0/3}.
\end{aligned}$$

The same holds for $\tilde{\mu}_1$. □

After these preliminaries define the sequence of joint regeneration times by $\mu_0 = \tilde{\mu}_0 = 0$ and

$$(\mu_{i+1}, \tilde{\mu}_{i+1}) = (\mu_i, \tilde{\mu}_i) + (\mu_1, \tilde{\mu}_1) \circ \theta^{\mu_i, \tilde{\mu}_i}. \tag{7.19}$$

The previous estimates, Lemmas 7.5 and 7.6, show that common regeneration levels come fast enough. The next tasks are to identify suitable Markovian structures and to develop a coupling. Recall again the definition (7.2) of $\mathbb{V}_d$.

Proposition 7.7. *Under the averaged measure* $P_{x,y}$ *with* $x, y \in \mathbb{V}_d$, *the process* $(\widetilde{X}_{\tilde{\mu}_i} - X_{\mu_i})_{i\ge 1}$ *is a Markov chain on* $\mathbb{V}_d$ *with transition probability*

$$q(x, y) = P_{0,x}\{\widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1} = y | \beta = \tilde{\beta} = \infty\}. \tag{7.20}$$

Note that the time-homogeneous Markov chain does not start from $\widetilde{X}_0 - X_0$ because the transition to $\widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1}$ does not include the condition $\beta = \tilde{\beta} = \infty$.

Proof. Let $n \ge 2$ and $z_1, \ldots, z_n \in \mathbb{V}_d$. The proof comes from iterating the following steps.

$$\begin{aligned}
&P_{0,z}\{\widetilde{X}_{\tilde{\mu}_i} - X_{\mu_i} = z_i \text{ for } 1 \le i \le n\} \\
&\quad = \sum_{\tilde{w}-w=z_{n-1}} P_{0,z}\{\widetilde{X}_{\tilde{\mu}_i} - X_{\mu_i} = z_i \text{ for } 1 \le i \le n-2, X_{\mu_{n-1}} = w, \widetilde{X}_{\tilde{\mu}_{n-1}} = \tilde{w}\} \\
&\qquad \times P_{w,\tilde{w}}\{\widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1} = z_n | \beta = \tilde{\beta} = \infty\} \\
&\quad = \sum_{\tilde{w}-w=z_{n-1}} P_{0,z}\{\widetilde{X}_{\tilde{\mu}_i} - X_{\mu_i} = z_i \text{ for } 1 \le i \le n-2, X_{\mu_{n-1}} = w, \widetilde{X}_{\tilde{\mu}_{n-1}} = \tilde{w}\} \\
&\qquad \times P_{0,z_{n-1}}\{\widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1} = z_n | \beta = \tilde{\beta} = \infty\} \\
&\quad = P_{0,z}\{\widetilde{X}_{\tilde{\mu}_i} - X_{\mu_i} = z_i \text{ for } 1 \le i \le n-1\} q(z_{n-1}, z_n).
\end{aligned}$$

Factoring in the first equality above is justified by the fact that

$$P^{\omega}_{0,z}\{\widetilde{X}_{\tilde{\mu}_i} - X_{\mu_i} = z_i \text{ for } 1 \le i \le n-2, X_{\mu_{n-1}} = w, \widetilde{X}_{\tilde{\mu}_{n-1}} = \tilde{w}, \widetilde{X}_{\tilde{\mu}_n} - X_{\mu_n} = z_n\}$$
$$= P^{\omega}_{0,z}(A) P^{\omega}_{w,\tilde{w}}(B),$$

where A is a collection of paths staying below level $w \cdot \hat{u} = \tilde{w} \cdot \hat{u}$, while

$$B = \{\widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1} = z_n, \beta = \tilde{\beta} = \infty\}$$

is a collection of paths that stay at or above their initial level. □

The Markov chain $Y_k = \widetilde{X}_{\tilde{\mu}_k} - X_{\mu_k}$ will be compared to a random walk obtained by performing the same construction of joint regeneration times to two independent walks in independent environments. To indicate the difference in construction we change notation. Let the pair of walks $(X, \bar{X})$ obey $P_0 \otimes P_z$ with $z \in \mathbb{V}_d$, and denote the first backtracking time of the $\bar{X}$ walk by $\bar{\beta} = \inf\{n \ge 1\colon\ \bar{X}_n \cdot \hat{u} < \bar{X}_0 \cdot \hat{u}\}$. Construct the joint regeneration times $(\rho_k, \bar{\rho}_k)_{k\ge 1}$ for $(X, \bar{X})$ by the same recipe ((7.9), (7.19), and the equations leading to them) as was used to construct $(\mu_k, \tilde{\mu}_k)_{k\ge 1}$ for $(X, \widetilde{X})$. Define $\bar{Y}_k = \bar{X}_{\bar{\rho}_k} - X_{\rho_k}$. An analog of the previous proposition, which we will not spell out, shows that $(\bar{Y}_k)_{k\ge 1}$ is a Markov chain with transition

$$\bar{q}(x, y) = P_0 \otimes P_x[\bar{X}_{\bar{\rho}_1} - X_{\rho_1} = y | \beta = \bar{\beta} = \infty]. \tag{7.21}$$

In the next two proofs we make use of the following decomposition. Suppose $x \cdot \hat{u} = y \cdot \hat{u} = 0$, and let (x_1, y_1) be another pair of points on a common, higher level: $x_1 \cdot \hat{u} = y_1 \cdot \hat{u} = \ell > 0$. Then we can write

$$\begin{aligned}&\big\{(X_0, \widetilde{X}_0) = (x, y), \beta = \tilde{\beta} = \infty, (X_{\mu_1}, \widetilde{X}_{\tilde{\mu}_1}) = (x_1, y_1)\big\}\\ &\quad = \bigcup_{(\gamma,\tilde{\gamma})} \big\{X_{0,n(\gamma)} = \gamma, \widetilde{X}_{0,n(\tilde{\gamma})} = \tilde{\gamma}, \beta \circ \theta^{n(\gamma)} = \tilde{\beta} \circ \theta^{n(\tilde{\gamma})} = \infty\big\}.\end{aligned} \tag{7.22}$$

Here $(\gamma, \tilde{\gamma})$ range over all pairs of paths that connect (x, y) to (x_1, y_1), that stay between levels 0 and $\ell - 1$ before the final points, and for which a joint regeneration fails at all levels before ℓ. $n(\gamma)$ is the index of the final point along the path, so for example $\gamma = (x = z_0, z_1, \dots, z_{n(\gamma)-1}, z_{n(\gamma)} = x_1)$.

Proposition 7.8. *The process $(\bar{Y}_k)_{k\ge 1}$ is a symmetric random walk on $\mathbb{V}_d$ and its transition probability satisfies*

$$\begin{aligned}\bar{q}(x, y) &= \bar{q}(0, y - x) = \bar{q}(0, x - y)\\ &= P_0 \otimes P_0\{\bar{X}_{\bar{\rho}_1} - X_{\rho_1} = y - x | \beta = \bar{\beta} = \infty\}.\end{aligned}$$

Proof. It remains to show that for independent $(X, \bar{X})$ the transition (7.21) reduces to a symmetric random walk. This becomes obvious once probabilities are decomposed into sums over paths because the events of interest are insensitive to shifts by $z \in \mathbb{V}_d$.

$$\begin{aligned}&P_0 \otimes P_x\{\beta = \bar{\beta} = \infty, \bar{X}_{\bar{\rho}_1} - X_{\rho_1} = y\}\\ &\quad = \sum_w P_0 \otimes P_x\{\beta = \bar{\beta} = \infty, X_{\rho_1} = w, \bar{X}_{\bar{\rho}_1} = y + w\}\\ &\quad = \sum_w \sum_{(\gamma,\bar{\gamma})} P_0\big\{X_{0,n(\gamma)} = \gamma, \beta \circ \theta^{n(\gamma)} = \infty\big\} P_x\big\{X_{0,n(\bar{\gamma})} = \bar{\gamma}, \beta \circ \theta^{n(\bar{\gamma})} = \infty\big\}\\ &\quad = \sum_w \sum_{(\gamma,\bar{\gamma})} P_0\{X_{0,n(\gamma)} = \gamma\} P_x\{X_{0,n(\bar{\gamma})} = \bar{\gamma}\}\big(P_0\{\beta = \infty\}\big)^2.\end{aligned} \tag{7.23}$$

Above we used the decomposition idea from (7.22). Here $(\gamma, \bar{\gamma})$ range over the appropriate class of pairs of paths in $\mathbb{Z}^d$ such that γ goes from 0 to w and $\bar{\gamma}$ goes from x to $y + w$. The independence for the last equality above comes from noticing that the quenched probabilities $P_0^\omega\{X_{0,n(\gamma)}=\gamma\}$ and $P_w^\omega\{\beta=\infty\}$ depend on independent collections of environments.

The probabilities on the last line of (7.23) are not changed if each pair $(\gamma,\bar\gamma)$ is replaced by $(\gamma,\gamma')=(\gamma,\bar\gamma-x)$. These pairs connect $(0,0)$ to $(w,y-x+w)$. Because $x\in\mathbb{V}_d$ satisfies $x\cdot\hat u=0$, the shift has not changed regeneration levels. This shift turns $P_x\{X_{0,n(\bar\gamma)}=\bar\gamma\}$ on the last line of (7.23) into $P_0\{X_{0,n(\gamma')}=\gamma'\}$. We can reverse the steps in (7.23) to arrive at the probability

$$P_0\otimes P_0\{\beta=\bar\beta=\infty,\bar X_{\bar\rho_1}-X_{\rho_1}=y-x\}.$$

This proves $\bar q(x,y)=\bar q(0,y-x)$.

Once both walks start at 0 it is immaterial which is labeled X and which $\bar X$, hence symmetry holds. □

It will be useful to know that $\bar q$ inherits all possible transitions from q.

Lemma 7.9. *If $q(z,w)>0$ then also $\bar q(z,w)>0$.*

Proof. By the decomposition from (7.22) we can express

$$P_{x,y}\{(X_{\mu_1},\widetilde X_{\tilde\mu_1})=(x_1,y_1)|\beta=\tilde\beta=\infty\}=\sum_{(\gamma,\tilde\gamma)}\frac{\mathbb{E}P^\omega(\gamma)P^\omega(\tilde\gamma)P^\omega_{x_1}\{\beta=\infty\}P^\omega_{y_1}\{\beta=\infty\}}{P_{x,y}\{\beta=\tilde\beta=\infty\}}.$$

If this probability is positive, then at least one pair $(\gamma,\tilde\gamma)$ must satisfy $\mathbb{E}P^\omega(\gamma)P^\omega(\tilde\gamma)>0$. This implies that $P(\gamma)P(\tilde\gamma)>0$ so that also

$$P_x\otimes P_y\{(X_{\mu_1},\widetilde X_{\tilde\mu_1})=(x_1,y_1)|\beta=\tilde\beta=\infty\}>0.$$ □

In the sequel we detach the notations $Y=(Y_k)$ and $\bar Y=(\bar Y_k)$ from their original definitions in terms of the walks X, $\widetilde X$ and $\bar X$, and use (Y_k) and $(\bar Y_k)$ to denote canonical Markov chains with transitions q and $\bar q$. Now we construct a coupling.

Proposition 7.10. *The single-step transitions $q(x,y)$ for Y and $\bar q(x,y)$ for $\bar Y$ can be coupled in such a way that, when the processes start from a common state $x\neq 0$,*

$$P_{x,x}\{Y_1\neq\bar Y_1\}\le C|x|^{-p_0/6}$$

for all $x\in\mathbb{V}_d$. Here C is a finite positive constant independent of x.

Proof. We start by constructing a coupling of three walks $(X,\widetilde X,\bar X)$ such that the pair $(X,\widetilde X)$ has distribution $P_{x,y}$ and the pair $(X,\bar X)$ has distribution $P_x\otimes P_y$.

First let $(X,\widetilde X)$ be two independent walks in a common environment ω as before. Let $\bar\omega$ be an environment independent of ω. Define the walk $\bar X$ as follows. Initially $\bar X_0=\widetilde X_0$. On the sites $\{X_k\colon 0\le k<\infty\}$ $\bar X$ obeys environment $\bar\omega$, and on all other sites $\bar X$ obeys ω. $\bar X$ is coupled to agree with $\widetilde X$ until the time

$$T=\inf\{n\ge 0\colon\ \bar X_n\in\{X_k\colon 0\le k<\infty\}\}$$

when it hits the path of X.

The coupling between $\bar X$ and $\widetilde X$ can be achieved simply as follows. Given ω and $\bar\omega$, for each x create two independent i.i.d. sequences $(z_k^x)_{k\ge1}$ and $(\bar z_k^x)_{k\ge1}$ with distributions

$$Q^{\omega,\bar\omega}\{z_k^x=y\}=\pi_{x,x+y}(\omega)\quad\text{and}\quad Q^{\omega,\bar\omega}\{\bar z_k^x=y\}=\pi_{x,x+y}(\bar\omega).$$

Do this independently at each x. Each time the $\widetilde X$-walk visits state x, it uses a new z_k^x variable as its next step, and never reuses the same z_k^x again. The $\bar X$ walk operates the same way except that it uses the variables $\bar z_k^x$ when $x\in\{X_k\}$ and the z_k^x variables when $x\notin\{X_k\}$. Now $\bar X$ and $\widetilde X$ follow the same steps z_k^x until $\bar X$ hits the set $\{X_k\}$.

It is intuitively obvious that the walks X and $\bar{X}$ are independent because they never use the same environment. The following calculation verifies this. Let $X_0 = x_0 = x$ and $\widetilde{X} = \bar{X} = y_0 = y$ be the initial states, and $\mathbf{P}_{x,y}$ the joint measure created by the coupling. Fix finite vectors $x_{0,n} = (x_0, \dots, x_n)$ and $y_{0,n} = (y_0, \dots, y_n)$ and recall also the notation $X_{0,n} = (X_0, \dots, X_n)$. The description of the coupling tells us to start as follows.

$$
\begin{aligned}
&\mathbf{P}_{x,y}\{X_{0,n} = x_{0,n}, \bar{X}_{0,n} = y_{0,n}\} \\
&\quad = \int \mathbb{P}(\mathrm{d}\omega) \int \mathbb{P}(\mathrm{d}\bar{\omega}) \int P_x^{\omega}(\mathrm{d}z_{0,\infty}) \mathbb{1}\{z_{0,n} = x_{0,n}\} \\
&\qquad \times \prod_{i:\ y_i \notin \{z_k:\ 0 \le k < \infty\}} \pi_{y_i, y_{i+1}}(\omega) \prod_{i:\ y_i \in \{z_k:\ 0 \le k < \infty\}} \pi_{y_i, y_{i+1}}(\bar{\omega})
\end{aligned}
$$

(by dominated convergence)

$$
\begin{aligned}
&\quad = \lim_{N\to\infty} \int \mathbb{P}(\mathrm{d}\omega) \int \mathbb{P}(\mathrm{d}\bar{\omega}) \int P_x^{\omega}(\mathrm{d}z_{0,N}) \mathbb{1}\{z_{0,n} = x_{0,n}\} \\
&\qquad \times \prod_{i:\ y_i \notin \{z_k:\ 0 \le k \le N\}} \pi_{y_i, y_{i+1}}(\omega) \prod_{i:\ y_i \in \{z_k:\ 0 \le k \le N\}} \pi_{y_i, y_{i+1}}(\bar{\omega}) \\
&\quad = \lim_{N\to\infty} \sum_{z_{0,N}:\ z_{0,n} = x_{0,n}} \int \mathbb{P}(\mathrm{d}\omega) P_x^{\omega}[X_{0,N} = z_{0,N}] \prod_{i:\ y_i \notin \{z_k:\ 0 \le k \le N\}} \pi_{y_i, y_{i+1}}(\omega) \\
&\qquad \times \int \mathbb{P}(\mathrm{d}\bar{\omega}) \prod_{i:\ y_i \in \{z_k:\ 0 \le k \le N\}} \pi_{y_i, y_{i+1}}(\bar{\omega})
\end{aligned}
$$

(by independence of the two functions of ω)

$$
\begin{aligned}
&\quad = \lim_{N\to\infty} \sum_{z_{0,N}: z_{0,n} = x_{0,n}} \int \mathbb{P}(\mathrm{d}\omega) P_x^{\omega}\{X_{0,N} = z_{0,N}\} \\
&\qquad \times \int \mathbb{P}(\mathrm{d}\omega) \prod_{i:\ y_i \notin \{z_k:\ 0 \le k \le N\}} \pi_{y_i, y_{i+1}}(\omega) \int \mathbb{P}(\mathrm{d}\bar{\omega}) \prod_{i:\ y_i \in \{z_k:\ 0 \le k \le N\}} \pi_{y_i, y_{i+1}}(\bar{\omega}) \\
&\quad = P_x\{X_{0,n} = x_{0,n}\} P_y\{X_{0,n} = y_{0,n}\}.
\end{aligned}
$$

Thus at this point the coupled pairs $(X, \widetilde{X})$ and $(X, \bar{X})$ have the desired marginals $P_{x,y}$ and $P_x \otimes P_y$.

Construct the joint regeneration times $(\mu_1, \tilde{\mu}_1)$ for $(X, \widetilde{X})$ and $(\rho_1, \bar{\rho}_1)$ for $(X, \bar{X})$ by the earlier recipes. Define two pairs of walks stopped at their joint regeneration times:

$$
(\Gamma, \bar{\Gamma}) \equiv \bigl((X_{0,\mu_1}, \widetilde{X}_{0,\tilde{\mu}_1}), (X_{0,\rho_1}, \bar{X}_{0,\bar{\rho}_1})\bigr). \tag{7.24}
$$

Suppose the sets $X_{[0,\mu_1 \vee \rho_1)}$ and $\widetilde{X}_{[0,\tilde{\mu}_1 \vee \bar{\rho}_1)}$ do not intersect. Then the construction implies that the path $\bar{X}_{0,\tilde{\mu}_1 \vee \bar{\rho}_1}$ agrees with $\widetilde{X}_{0,\tilde{\mu}_1 \vee \bar{\rho}_1}$, and this forces the equalities $(\mu_1, \tilde{\mu}_1) = (\rho_1, \bar{\rho}_1)$ and $(X_{\mu_1}, \widetilde{X}_{\tilde{\mu}_1}) = (X_{\rho_1}, \bar{X}_{\bar{\rho}_1})$. We insert an estimate on this event.

Lemma 7.11. *For $x \neq y$ in $\mathbb{V}_d$,*

$$
P_{x,y}\{X_{[0,\mu_1 \vee \rho_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1 \vee \bar{\rho}_1)} \neq \emptyset\} \le C|x - y|^{-p_0/3}. \tag{7.25}
$$

Proof. Write

$$
P_{x,y}\{X_{[0,\mu_1 \vee \rho_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1 \vee \bar{\rho}_1)} \neq \emptyset\} \le P_{x,y}\left\{\mu_1 \vee \tilde{\mu}_1 \vee \rho_1 \vee \bar{\rho}_1 > \frac{|x - y|}{2r_0}\right\}.
$$

The conclusion follows from (7.17), extended to cover also $(\rho_1, \bar\rho_1)$. □

From (7.25) we obtain

$$\mathbf{P}_{x,y}\big\{(X_{\mu_1}, \widetilde X_{\tilde\mu_1}) \neq (X_{\rho_1}, \bar X_{\bar\rho_1})\big\} \le \mathbf{P}_{x,y}\{\Gamma \neq \bar\Gamma\} \le C|x-y|^{-p_0/3}. \tag{7.26}$$

But we are not finished yet. To represent the transitions q and $\bar q$ we must also include the conditioning on no backtracking. For this generate an i.i.d. sequence $(X^{(m)}, \widetilde X^{(m)}, \bar X^{(m)})_{m\ge1}$, each triple constructed as $(X, \widetilde X, \bar X)$ above. Continue to write $\mathbf{P}_{x,y}$ for the probability measure of the entire sequence. Let also again

$$\Gamma^{(m)} = \big(X^{(m)}_{0,\mu_1^{(m)}}, \widetilde X^{(m)}_{0,\tilde\mu_1^{(m)}}\big) \quad\text{and}\quad \bar\Gamma^{(m)} = \big(X^{(m)}_{0,\rho_1^{(m)}}, \bar X^{(m)}_{0,\bar\rho_1^{(m)}}\big)$$

be the pairs of paths run up to their joint regeneration times.

Let M be the first m such that the paths $(X^{(m)}, \widetilde X^{(m)})$ do not backtrack, which means that

$$X^{(m)}_k \cdot \hat u \ge X^{(m)}_0 \cdot \hat u \quad\text{and}\quad \widetilde X^{(m)}_k \cdot \hat u \ge \widetilde X^{(m)}_0 \cdot \hat u \quad \text{for all } k \ge 1.$$

Similarly define $\bar M$ for $(X^{(m)}, \bar X^{(m)})_{m\ge1}$. Both M and $\bar M$ are stochastically bounded by geometric random variables by (7.3).

The pair of walks $(X^{(M)}, \widetilde X^{(M)})$ is now distributed as a pair of walks under the measure $P_{x,y}\{\cdot|\beta = \tilde\beta = \infty\}$, while $(X^{(\bar M)}, \bar X^{(\bar M)})$ is distributed as a pair of walks under $P_x \otimes P_y\{\cdot|\beta = \bar\beta = \infty\}$. Consider the two pairs of paths $(\Gamma^{(M)}, \bar\Gamma^{(\bar M)})$ chosen by the random indices $(M, \bar M)$. We insert one more lemma.

Lemma 7.12. *For $x \neq y$ in $\mathbb{V}_d$,*

$$\mathbf{P}_{x,y}\big\{\Gamma^{(M)} \neq \bar\Gamma^{(\bar M)}\big\} \le C|x-y|^{-p_0/6}. \tag{7.27}$$

Proof. Let $\mathcal{A}_m$ be the event that the walks $\widetilde X^{(m)}$ and $\bar X^{(m)}$ agree up to the maximum $\tilde\mu_1^{(m)} \vee \bar\rho_1^{(m)}$ of their regeneration times. The equalities $M = \bar M$ and $\Gamma^{(M)} = \bar\Gamma^{(\bar M)}$ are a consequence of the event

$$\{\mathcal{A}_1 \cap \cdots \cap \mathcal{A}_M\} = \bigcup_{m\ge1} \{M = m\} \cap \mathcal{A}_1 \cap \cdots \cap \mathcal{A}_m,$$

for the following reason. As pointed out earlier, on the event $\mathcal{A}_m$ we have the equality of the regeneration times $\tilde\mu_1^{(m)} = \bar\rho_1^{(m)}$ and of the stopped paths $\widetilde X^{(m)}_{0,\tilde\mu_1^{(m)}} = \bar X^{(m)}_{0,\bar\rho_1^{(m)}}$. By definition, these walks do not backtrack after the regeneration time. Since the walks $\widetilde X^{(m)}$ and $\bar X^{(m)}$ agree up to this time, they must backtrack or fail to backtrack together. If this is true for each $m = 1, \ldots, M$, it forces $\bar M = M$, since the other factor in deciding M and $\bar M$ are the paths $X^{(m)}$ that are common to both. And since the paths agree up to the regeneration times, we have $\Gamma^{(M)} = \bar\Gamma^{(\bar M)}$.

Estimate (7.27) follows:

$$\begin{aligned}
\mathbf{P}_{x,y}\big\{\Gamma^{(M)} \neq \bar\Gamma^{(\bar M)}\big\} &\le \mathbf{P}_{x,y}\big\{\mathcal{A}_1^c \cup \cdots \cup \mathcal{A}_M^c\big\} \\
&\le \sum_{m=1}^\infty \mathbf{P}_{x,y}\big\{M \ge m, \mathcal{A}_m^c\big\} \le \sum_{m=1}^\infty \big(\mathbf{P}_{x,y}\{M \ge m\}\big)^{1/2}\big(\mathbf{P}_{x,y}\big(\mathcal{A}_m^c\big)\big)^{1/2} \\
&\le C|x-y|^{-p_0/6}.
\end{aligned}$$

The last step comes from the estimate in (7.25) for each $\mathcal{A}_m^c$ and the geometric bound on M. □

We are ready to finish the proof of Proposition 7.10. To create initial conditions $Y_0 = \bar Y_0 = x$ let the walks start at $(X_0^{(m)}, \widetilde X_0^{(m)}) = (X_0^{(m)}, \bar X_0^{(m)}) = (0, x)$. Let the final outcome of the coupling be the pair

$$(Y_1, \bar Y_1) = \big(\widetilde X^{(M)}_{\tilde\mu_1^{(M)}} - X^{(M)}_{\mu_1^{(M)}}, \bar X^{(\bar M)}_{\bar\rho_1^{(\bar M)}} - X^{(\bar M)}_{\rho_1^{(\bar M)}}\big)$$

under the measure $\mathbf{P}_{0,x}$. The marginal distributions of Y_1 and $\bar{Y}_1$ are correct (namely, given by the transitions (7.20) and (7.21)) because, as argued above, the pairs of walks themselves have the right marginal distributions. The event $\Gamma^{(M)} = \bar{\Gamma}^{(\bar{M})}$ implies $Y_1 = \bar{Y}_1$, so estimate (7.27) gives the bound claimed in Proposition 7.10. □

The construction of the Markov chain is complete, and we return to the main development of the proof. It remains to prove a sublinear bound on the expected number $E_{0,0}|X_{[0,n)} \cap \widetilde{X}_{[0,n)}|$ of common points of two independent walks in a common environment. Utilizing the joint regeneration times, write

$$E_{0,0}|X_{[0,n)} \cap \widetilde{X}_{[0,n)}| \le \sum_{i=0}^{n-1} E_{0,0}|X_{[\mu_i,\mu_{i+1})} \cap \widetilde{X}_{[\tilde{\mu}_i,\tilde{\mu}_{i+1})}|. \tag{7.28}$$

The term $i = 0$ is a finite constant by bound (7.17) because the number of common points is bounded by the number μ_1 of steps. For each $0 < i < n$ apply a decomposition into pairs of paths from $(0,0)$ to given points (x_1, y_1) in the style of (7.22): $(\gamma, \tilde{\gamma})$ are the pairs of paths with the property that

$$\bigcup_{(\gamma,\tilde{\gamma})} \{X_{0,n(\gamma)} = \gamma, \widetilde{X}_{0,n(\tilde{\gamma})} = \tilde{\gamma}, \beta \circ \theta^{n(\gamma)} = \tilde{\beta} \circ \theta^{n(\tilde{\gamma})} = \infty\} = \{X_0 = \widetilde{X}_0 = 0, X_{\mu_i} = x_1, \widetilde{X}_{\tilde{\mu}_i} = y_1\}.$$

Each term $i > 0$ in (7.28) we rearrange as follows.

$$\begin{aligned}
&E_{0,0}|X_{[\mu_i,\mu_{i+1})} \cap \widetilde{X}_{[\tilde{\mu}_i,\tilde{\mu}_{i+1})}| \\
&\quad = \sum_{x_1,y_1} \sum_{(\gamma,\tilde{\gamma})} P_{0,0}\{X_{0,n(\gamma)} = \gamma, \widetilde{X}_{0,n(\tilde{\gamma})} = \tilde{\gamma}\} E_{x_1,y_1}\big[\mathbb{1}\{\beta = \tilde{\beta} = \infty\}|X_{[0,\mu_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1)}|\big] \\
&\quad = \sum_{x_1,y_1} \sum_{(\gamma,\tilde{\gamma})} P_{0,0}\{X_{0,n(\gamma)} = \gamma, \widetilde{X}_{0,n(\tilde{\gamma})} = \tilde{\gamma}\} P_{x_1,y_1}\{\beta = \tilde{\beta} = \infty\} E_{x_1,y_1}\big[|X_{[0,\mu_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1)}| \,|\, \beta = \tilde{\beta} = \infty\big] \\
&\quad = \sum_{x_1,y_1} P_{0,0}\{X_{\mu_i} = x_1, \widetilde{X}_{\tilde{\mu}_i} = y_1\} E_{x_1,y_1}\big[|X_{[0,\mu_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1)}| \,|\, \beta = \tilde{\beta} = \infty\big].
\end{aligned}$$

We have used the product structure of $\mathbb{P}$ in the first and and last equalities. The last conditional expectation above is handled by estimates (7.3), (7.17), (7.25) and Schwarz inequality:

$$\begin{aligned}
&E_{x_1,y_1}\big[|X_{[0,\mu_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1)}| \,|\, \beta = \tilde{\beta} = \infty\big] \le \eta^{-1} E_{x_1,y_1}\big[|X_{[0,\mu_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1)}|\big] \\
&\quad \le \eta^{-1} E_{x_1,y_1}\big[\mu_1 \cdot \mathbb{1}\{X_{[0,\mu_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1)} \ne \emptyset\}\big] \\
&\quad \le \eta^{-1}\big(E_{x_1,y_1}[\mu_1^2]\big)^{1/2}\big(P_{x_1,y_1}\{X_{[0,\mu_1)} \cap \widetilde{X}_{[0,\tilde{\mu}_1)} \ne \emptyset\}\big)^{1/2} \\
&\quad \le C\big(1 \vee |x_1 - y_1|\big)^{-p_0/6} \le h(x_1 - y_1).
\end{aligned}$$

On the last line we defined

$$h(x) = C\big(|x| \vee 1\big)^{-p_0/6}. \tag{7.29}$$

Insert the last bound back up, and appeal to the Markov property established in Proposition 7.7:

$$\begin{aligned}
E_{0,0}|X_{[\mu_i,\mu_{i+1})} \cap \widetilde{X}_{[\tilde{\mu}_i,\tilde{\mu}_{i+1})}| &\le E_{0,0}\big[h(\widetilde{X}_{\tilde{\mu}_i} - X_{\mu_i})\big] \\
&= \sum_x P_{0,0}\{\widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1} = x\} \sum_y q^{i-1}(x,y)h(y).
\end{aligned}$$

In order to apply Theorem A.1 from Appendix A, we check its hypotheses in the next lemma. Part (1.3) of Hypothesis (R) enters here crucially to guarantee that the transition q has enough irreducibility.

Lemma 7.13. *The Markov chain $(Y_k)_{k\ge 0}$ with transition $q(x,y)$ and the symmetric random walk $(\bar Y_k)_{k\ge 0}$ with transition $\bar q(x,y)$ satisfy assumptions* (A.i), (A.ii), (A.iii) *and* (A.iv) *stated in the beginning of Appendix A. To ensure that $p_1 > 15$ as required by* (A.iv), *we assume $p_0 > 90$.*

Proof. From (7.18) and Hypothesis (S) we get moment bounds

$$E_{0,x}|\bar X_{\bar\rho_k}|^p + E_{0,x}|X_{\rho_k}|^p < \infty$$

for $p < p_0/3$. With $p_0 > 9$ this gives assumption (A.i), namely that $E_0|\bar Y_1|^3 < \infty$. (Lemma 7.6 is applied here to $(X,\bar X)$ even though we wrote the proof only for $(X,\widetilde X)$.) Assumption (A.iii) comes from Lemma 7.9. Assumption (A.iv) comes from Proposition 7.10.

The only part that needs work is assumption (A.ii). The required exponential exit time bound is achieved through a combination of the following three steps, for constants $\delta > 0$, $L > 0$ and a fixed vector $\hat b \ne 0$:

$$P_0[Y_1 \ne 0] \ge \delta, \tag{7.30}$$

$$\inf_{0<|x|\le L} P_x\big[|Y_1| > L\big] \ge \delta \tag{7.31}$$

and

$$\inf_{|x|>L}\big\{P_x[Y_1 = Y_0 + \hat b] \wedge P_x[Y_1 = Y_0 - \hat b]\big\} \ge \delta. \tag{7.32}$$

Given any initial state x contained in a cube $[-r,r]^d$, there is a sequence of at most $2r$ steps of the types covered by the above estimates that takes the chain Y outside the cube, and this sequence of steps is taken with probability at least δ^{2r}. Thus the exit time from the cube is dominated by $2r$ times a geometric random variable with mean δ^{-2r}.

To prove (7.30)–(7.32) we make use of

$$P_x[Y_1 = z] \ge P_{0,x}\{\beta = \tilde\beta = \infty, \widetilde X_{\tilde\mu_1} = y + z, X_{\mu_1} = y\} \tag{7.33}$$

which is a consequence of the definition of the transition (7.20) and valid for all x, y, z. To this end we construct suitable paths for the X and $\widetilde X$ walks with positive probabilities. We carry out the rest of the proof in Appendix C because this requires a fairly tedious cataloguing of cases. □

Appendix A also requires $0 \le h(x) \le C(1 \vee |x|)^{-p_2}$ for $p_2 > 0$. This we have without further requirements on p_0. Now that the assumptions have been checked, Theorem A.1 gives constants $0 < C < \infty$ and $0 < \eta < 1/2$ such that

$$\sum_{i=1}^{n-1}\sum_y q^{i-1}(x,y)h(y) \le Cn^{1-\eta} \quad \text{for all } x \in \mathbb{V}_d \text{ and } n \ge 1.$$

Going back to (7.28) and collecting the bounds along the way gives the final estimate

$$E_{0,0}|X_{[0,n)} \cap \widetilde X_{[0,n)}| \le C_p n^{1-\eta}$$

for all $n \ge 1$. Taking p large enough, $1-\eta$ can be made as close as desired to $1/2$. This is (7.1) which was earlier shown to imply condition (2.1) required by Theorem 2.1. Previous work in Sections 2 and 5 convert the CLT from Theorem 2.1 into the main result Theorem 1.1. The entire proof is complete, except for the Green function estimate furnished by Appendix A and the remainder of the proof of Lemma 7.13 in Appendix C.

Appendix A. A Green function estimate

This appendix can be read independently of the rest of the paper. Let us write a d-vector in terms of coordinates as $x=(x^1,\dots,x^d)$, and similarly for random vectors $X=(X^1,\dots,X^d)$.

Let $\mathbb{S}$ be some subgroup of $\mathbb{Z}^d$. Let $Y=(Y_k)_{k\ge 0}$ be a Markov chain on $\mathbb{S}$ with transition probability $q(x,y)$, and let $\bar{Y}=(\bar{Y}_k)_{k\ge 0}$ be a symmetric random walk on $\mathbb{S}$ with transition probability $\bar{q}(x,y)=\bar{q}(y,x)=\bar{q}(0,y-x)$. Make the following assumptions.

(A.i) A finite third moment for the random walk: $E_0|\bar{Y}_1|^3<\infty$.

(A.ii) Let $U_r=\inf\{n\ge 0\colon\ Y_n\notin[-r,r]^d\}$ be the exit time from a centered cube of side length $2r+1$ for the Markov chain Y. Then there is a constant $0<K<\infty$ such that

$$\sup_{x\in[-r,r]^d} E_x(U_r)\le K^r \quad \text{for all } r\ge 1. \tag{A.1}$$

(A.iii) For every $i\in\{1,\dots,d\}$, if the one-dimensional random walk $\bar{Y}^i$ is degenerate in the sense that $\bar{q}(0,y)=0$ for $y^i\ne 0$, then so is the process Y^i in the sense that $q(x,y)=0$ whenever $x^i\ne y^i$. In other words, any coordinate that can move in the Y chain somewhere in space can also move in the $\bar{Y}$ walk.

(A.iv) For any initial state $x\ne 0$ the transitions q and $\bar{q}$ can be coupled so that

$$P_{x,x}\{Y_1\ne\bar{Y}_1\}\le C|x|^{-p_1}, \tag{A.2}$$

where $0<C, p_1<\infty$ are constants independent of x and $p_1>15$.

Let h be a function on $\mathbb{S}$ such that $0\le h(x)\le C(|x|\vee 1)^{-p_2}$ for constants $0<C, p_2<\infty$. This section is devoted to proving the following Green function bound on the Markov chain.

Theorem A.1. *There are constants* $0<C,\eta<\infty$ *such that*

$$\sum_{k=0}^{n-1} E_z h(Y_k)=\sum_y h(y)\sum_{k=0}^{n-1} P_z\{Y_k=y\}\le Cn^{1-\eta}$$

for all $n\ge 1$ *and* $z\in\mathbb{S}$. *If* p_1 *and* p_2 *can be taken arbitrarily large, then* $1-\eta$ *can be taken arbitrarily close to* (*but still strictly above*) $1/2$.

Precisely speaking, the bound that emerges is

$$\sum_{k=0}^{n-1} E_z h(Y_k)\le Cn^{\{1-p_2/(2p_1-4)\}\vee\{(1/2)+13/(2p_1-4)\}}. \tag{A.3}$$

The remainder of the section proves the theorem. Throughout C will change value but p_1,p_2 remain the constants in the assumptions above.

For the proof we can assume that each coordinate walk $\bar{Y}^i$ $(1\le i\le d)$ is nondegenerate. For if the random walk has a degenerate coordinate $\bar{Y}^j$ then assumption (A.iii) implies that also for the Markov chain $Y_n^j=Y_0^j$ for all times $n\ge 0$. Then we can project everything onto the remaining $d-1$ coordinates. Given the starting point z of Theorem A.1 write the Markov chain as $Y_n=(z^j,Y_n')$ where Y_n' is the $\mathbb{Z}^{d-1}$-valued Markov chain with transition $q'(x',y')=q((z_j,x'),(z_j,y'))$. Take the $(d-1)$-dimensional random walk $\bar{Y}_n'=(Y_n^1,\dots,Y_n^{j-1},Y_n^{j+1},\dots,Y_n^d)$. Replace h with $h'(x')=h(z_j,x')$. All the assumptions continue to hold with the same constants because $|x'|\le|(z_j,x')|$ and the exit time from a cube only concerns the nondegenerate coordinates. The constants from the assumptions determine the constants of the theorem. Consequently the estimate of the theorem follows with constants that do not depend on the frozen coordinate z^j.

We begin by discarding terms outside a cube of side $r = n^{\varepsilon_1}$ for a small $\varepsilon_1 > 0$ that will be specified at the end of the proof. For convenience, use below the ℓ^1 norm $|\cdot|_1$ on $\mathbb{Z}^d$ because its values are integers.

$$\sum_{|y|_1 > n^{\varepsilon_1}} h(y) \sum_{k=0}^{n-1} P_z\{Y_k = y\} \le \sum_{k=0}^{n-1} \sum_{j \ge [n^{\varepsilon_1}]+1} C j^{-p_2} P_z\{|Y_k|_1 = j\}$$
$$\le \sum_{k=0}^{n-1} C n^{-p_2 \varepsilon_1} \sum_{j \ge [n^{\varepsilon_1}]+1} P_z\{|Y_k|_1 = j\} \le C n^{1-p_2\varepsilon_1}.$$

Let

$$B = \left[-n^{\varepsilon_1}, n^{\varepsilon_1}\right]^d.$$

Since h is bounded, it now remains to show that

$$\sum_{k=0}^{n-1} P_z\{Y_k \in B\} \le C n^{1-\eta}. \tag{A.4}$$

For this we can assume $z \in B$ since accounting for the time to enter B can only improve the estimate.

Bound (A.4) will be achieved in two stages. First we improve the assumed exponential exit time bound (A.1) to a polynomial bound. Second, we show that often enough Y follows the random walk $\bar{Y}$ during its excursions outside B. The random walk excursions are long and thereby we obtain (A.4). Thus our first task is to construct a suitable coupling of Y and $\bar{Y}$.

Lemma A.1. *Let $\zeta = \inf\{n \ge 1\colon \bar{Y}_n \in A\}$ be the first entrance time of the random walk $\bar{Y}$ into some set $A \subseteq \mathbb{S}$. Then we can couple the Markov chain Y and the random walk $\bar{Y}$ so that*

$$P_{x,x}\{Y_k \ne \bar{Y}_k \text{ for some } 1 \le k \le \zeta\} \le C E_x\left[\sum_{k=0}^{\zeta-1} |\bar{Y}_k|^{-p_1}\right].$$

The proof shows that the statement works also if $\zeta = \infty$ is possible, but we will not need this case.

Proof. For each state x create an i.i.d. sequence $(Z_k^x, \bar{Z}_k^x)_{k \ge 1}$ such that Z_k^x has distribution $q(x, x+\cdot)$, $\bar{Z}_k^x$ has distribution $\bar{q}(x, x+\cdot) = \bar{q}(0,\cdot)$, and each pair $(Z_k^x, \bar{Z}_k^x)$ is coupled so that $P(Z_k^x \ne \bar{Z}_k^x) \le C|x|^{-p_1}$. For distinct x these sequences are independent.

Construct the process $(Y_n, \bar{Y}_n)$ as follows: with counting measures

$$L_n(x) = \sum_{k=0}^n \mathbb{1}\{Y_k = x\} \quad \text{and} \quad \bar{L}_n(x) = \sum_{k=0}^n \mathbb{1}\{\bar{Y}_k = x\} \quad (n \ge 0)$$

and with initial point $(Y_0, \bar{Y}_0)$ given, define for $n \ge 1$

$$Y_n = Y_{n-1} + Z_{L_{n-1}(Y_{n-1})}^{Y_{n-1}} \quad \text{and} \quad \bar{Y}_n = \bar{Y}_{n-1} + \bar{Z}_{\bar{L}_{n-1}(\bar{Y}_{n-1})}^{\bar{Y}_{n-1}}.$$

In words, every time the chain Y visits a state x, it reads its next jump from a new variable Z_k^x which is then discarded and never used again. And similarly for $\bar{Y}$. This construction has the property that, if $Y_k = \bar{Y}_k$ for $0 \le k \le n$ with $Y_n = \bar{Y}_n = x$, then the next joint step is $(Z_k^x, \bar{Z}_k^x)$ for $k = L_n(x) = \bar{L}_n(x)$. In other words, given that the processes agree up to the present and reside together at x, the probability that they separate in the next step is bounded by $C|x|^{-p_1}$.

Now follow self-evident steps.

$$
\begin{aligned}
&P_{x,x}\{Y_k \neq \bar{Y}_k \text{ for some } 1 \le k \le \zeta\} \\
&\quad \le \sum_{k=1}^{\infty} P_{x,x}\bigl\{Y_j = \bar{Y}_j \in A^c \text{ for } 1 \le j < k, Y_k \neq \bar{Y}_k\bigr\} \\
&\quad \le \sum_{k=1}^{\infty} E_{x,x}\bigl[\mathbb{1}\bigl\{Y_j = \bar{Y}_j \in A^c \text{ for } 1 \le j < k\bigr\} P_{Y_{k-1},\bar{Y}_{k-1}}\{Y_1 \neq \bar{Y}_1\}\bigr] \\
&\quad \le C\sum_{k=1}^{\infty} E_{x,x}\bigl[\mathbb{1}\bigl\{Y_j = \bar{Y}_j \in A^c \text{ for } 1 \le j < k\bigr\} |\bar{Y}_{k-1}|^{-p_1}\bigr] \\
&\quad \le C E_x \sum_{m=0}^{\zeta-1} |\bar{Y}_m|^{-p_1}.
\end{aligned}
$$

□

For the remainder of this section Y and $\bar{Y}$ are always coupled in the manner that satisfies Lemma A.1.

Lemma A.2. *Fix a coordinate index $j \in \{1,\dots,d\}$. Let r_0 be a positive integer and $\bar{w} = \inf\{n \ge 1\colon \bar{Y}_n^j \le r_0\}$ the first time the random walk $\bar{Y}$ enters the half-space $\mathcal{H} = \{x\colon x^j \le r_0\}$. Couple Y and $\bar{Y}$ starting from a common initial point $x \notin \mathcal{H}$. Then there is a constant C independent of r_0 such that*

$$\sup_{x\notin\mathcal{H}} P_{x,x}\bigl\{Y_k \neq \bar{Y}_k \text{ for some } k \in \{1,\dots,\bar{w}\}\bigr\} \le C r_0^{2-p_1} \text{ for all } r_0 \ge 1.$$

The same result holds for $\mathcal{H} = \{x\colon x^j \ge -r_0\}$.

Proof. By Lemma A.1

$$
\begin{aligned}
&P_{x,x}\bigl\{Y_k \neq \bar{Y}_k \text{ for some } k \in \{1,\dots,\bar{w}\}\bigr\} \\
&\quad \le C E_x\Biggl[\sum_{k=0}^{\bar{w}-1} |\bar{Y}_k|^{-p_1}\Biggr] \le C E_{x^j}\Biggl[\sum_{k=0}^{\bar{w}-1} \bigl|\bar{Y}_k^j\bigr|^{-p_1}\Biggr] = C \sum_{t=r_0+1}^{\infty} t^{-p_1} g\bigl(x^j, t\bigr),
\end{aligned}
$$

where for $s, t \in [r_0+1, \infty)$

$$g(s,t) = \sum_{n=0}^{\infty} P_s\bigl\{\bar{Y}_n^j = t, \bar{w} > n\bigr\}$$

is the Green function of the half-line $(-\infty, r_0]$ for the one-dimensional random walk $\bar{Y}^j$. This is the expected number of visits to t before entering $(-\infty, r_0]$, defined on p. 209 in [17]. The development in Sections 18 and 19 in [17] gives the bound

$$g(s,t) \le C\bigl(1 + (s - r_0 - 1) \wedge (t - r_0 - 1)\bigr) \le C(t - r_0), \quad s, t \in [r_0+1, \infty). \tag{A.5}$$

Here is some more detail. Shift $r_0 + 1$ to the origin to match the setting in [17]. Then P19.3 on p. 209 gives

$$g(x,y) = \sum_{n=0}^{x\wedge y} u(x-n)v(y-n) \quad \text{for } x, y \ge 0,$$

where the functions u and v are defined on p. 201. For a symmetric random walk $u = v$ (E19.3 on p. 204). P18.7 on p. 202 implies that

$$v(m) = \frac{1}{\sqrt{c}} \sum_{k=0}^{\infty} \mathbf{P}\{\mathbf{Z}_1 + \cdots + \mathbf{Z}_k = m\},$$

where c is a certain constant and $\{\mathbf{Z}_i\}$ are i.i.d. strictly positive, integer-valued ladder variables for the underlying random walk. (For $k = 0$ the sum $\mathbf{Z}_1 + \cdots + \mathbf{Z}_k$ is identically zero.) Now $v(m) \le v(0)$ for each m because the $\mathbf{Z}_i$'s are strictly positive. (Either do induction on m, or note that for a particular realization of the sequence $\{\mathbf{Z}_i\}$ a given m can be attained for at most one value of k.) So the quantities $u(m) = v(m)$ are bounded. This justifies (A.5).

Continuing from further above we get the estimate claimed in the statement of the lemma:

$$E_x\left[\sum_{k=0}^{\bar{w}-1} |\bar{Y}_k|^{-p_1}\right] \le C \sum_{t>r_0} (t - r_0) t^{-p_1} \le C r_0^{2-p_1}.$$

□

For the next lemmas abbreviate $B_r = [-r, r]^d$ for d-dimensional centered cubes.

Lemma A.3. *There exist constants* $0 < \alpha_1, A_1 < \infty$ *such that*

$$\inf_{x \in B_r \setminus B_{r_0}} P_x\left\{\text{without entering } B_{r_0} \text{ chain } Y \text{ exits } B_r \text{ by time } A_1 r^3\right\} \ge \frac{\alpha_1}{r} \tag{A.6}$$

for large enough positive integers r_0 *and* r *that satisfy*

$$r^{2/(p_1-2)} \le r_0 < r.$$

Proof. A point $x \in B_r \setminus B_{r_0}$ has a coordinate $x^j \in [-r, -r_0 - 1] \cup [r_0 + 1, r]$. The same argument works for both alternatives, and we treat the case $x^j \in [r_0 + 1, r]$.

One way to realize the event in (A.6) is this: starting at x^j, the $\bar{Y}^j$ walk exits $[r_0 + 1, r]$ by time $A_1 r^3$ through the right boundary into $[r + 1, \infty)$, and Y and $\bar{Y}$ stay coupled together throughout this time. Let $\bar{\zeta}$ be the time $\bar{Y}^j$ exits $[r_0 + 1, r]$ and $\bar{w}$ the time $\bar{Y}^j$ enters $(-\infty, r_0]$. Then $\bar{w} \ge \bar{\zeta}$. Thus the complementary probability of (A.6) is bounded above by

$$P_{x^j}\left\{\bar{Y}^j \text{ exits } [r_0 + 1, r] \text{ into } (-\infty, r_0]\right\} + P_{x^j}\left\{\bar{\zeta} > A_1 r^3\right\} + P_{x,x}\left\{Y_k \ne \bar{Y}_k \text{ for some } k \in \{1, \ldots, \bar{w}\}\right\}. \tag{A.7}$$

We treat the terms one at a time. From the development on pp. 253–255 in [17] we get the bound

$$P_{x^j}\left\{\bar{Y}^j \text{ exits } [r_0 + 1, r] \text{ into } (-\infty, r_0]\right\} \le 1 - \frac{\alpha_2}{r} \tag{A.8}$$

for a constant $\alpha_2 > 0$, uniformly over $0 < r_0 < x^j \le r$. In some more detail: P22.7 on p. 253, the inequality in the third display of p. 255, and the third moment assumption on the steps of $\bar{Y}$ give a lower bound

$$P_{x^j}\left\{\bar{Y}^j \text{ exits } [r_0 + 1, r] \text{ into } [r + 1, \infty)\right\} \ge \frac{x^j - r_0 - 1 - c_1}{r - r_0 - 1} \tag{A.9}$$

for the probability of exiting to the right. Here c_1 is a constant that comes from the term denoted in [17] by $M \sum_{s=0}^{N} (1 + s) a(s)$ whose finiteness follows from the third moment assumption. The text on pp. 254–255 suggests that these steps need the aperiodicity assumption. This need for aperiodicity can be traced back via P22.5 to P22.4 which is used to assert the boundedness of $u(x)$ and $v(x)$. But as we observed above in the derivation of (A.5) boundedness of $u(x)$ and $v(x)$ is true without any additional assumptions.

To go forward from (A.9) fix any $m > c_1$ so that the numerator above is positive for $x^j = r_0 + 1 + m$. The probability in (A.9) is minimized at $x^j = r_0 + 1$, and from $x^j = r_0 + 1$ there is a fixed positive probability θ to take m steps to the right to get past the point $x^j = r_0 + 1 + m$. Thus for all $x^j \in [r_0 + 1, r]$ we get the lower bound

$$P_{x^j}\{\bar{Y}^j \text{ exits } [r_0+1, r] \text{ into } [r+1, \infty)\} \ge \frac{\theta(m - c_1)}{r - r_0 - 1} \ge \frac{\alpha_2}{r},$$

where $\alpha_2 > 0$ is a constant, and (A.8) is verified.

As in (A.5) let $g(s,t)$ be the Green function of the random walk $\bar{Y}^j$ for the half-line $(-\infty, r_0]$, and let $\tilde{g}(s,t)$ be the Green function for the complement of the interval $[r_0+1, r]$. Then $\tilde{g}(s,t) \le g(s,t)$, and by (A.5) we get this moment bound:

$$E_{x^j}[\bar{\zeta}] = \sum_{t=r_0+1}^{r} \tilde{g}(x^j, t) \le \sum_{t=r_0+1}^{r} g(x^j, t) \le Cr^2.$$

Consequently, uniformly over $x^j \in [r_0+1, r]$,

$$P_{x^j}[\bar{\zeta} > A_1 r^3] \le \frac{C}{A_1 r}. \tag{A.10}$$

From Lemma A.2

$$P_x\{Y_k \ne \bar{Y}_k \text{ for some } k \in \{1, \ldots, \bar{w}\}\} \le C r_0^{2-p_1}. \tag{A.11}$$

Putting bounds (A.8), (A.10) and (A.11) together gives an upper bound of

$$1 - \frac{\alpha_2}{r} + \frac{C}{A_1 r} + C r_0^{2-p_1}$$

for the sum in (A.7) which bounds the complement of the probability in (A.6). By assumption $r_0^{2-p_1} \le r^{-2}$. So if A_1 is fixed large enough, then the sum above is not more than $1 - \alpha_1/r$ for a constant $\alpha_1 > 0$, for all large enough r. □

We iterate the last estimate to get down to an iterated logarithmic cube.

Corollary A.1. *Fix a constant $c_1 > 1$ and consider positive integers r_0 and r that satisfy*

$$\log\log r \le r_0 \le c_1 \log\log r < r.$$

Then for large enough r

$$\inf_{x \in B_r \setminus B_{r_0}} P_x\{\textit{without entering } B_{r_0} \textit{ chain } Y \textit{ exits } B_r \textit{ by time } r^4\} \ge r^{-3}. \tag{A.12}$$

Proof. Consider r large enough so that r_0 is also large enough to play the role of r in Lemma A.3. Pick an integer γ such that $3 \le \gamma \le (p_1 - 2)/2$. Put $r_k = r_0^{\gamma^k}$ for $k \ge 0$ (r_0 is still r_0) and $t_n = A_1 \sum_{k=1}^n r_0^{3\gamma^k}$, where A_1 is the constant from Lemma A.3.

We claim that for $n \ge 1$

$$\inf_{x \in B_{r_n} \setminus B_{r_0}} P_x\{\text{without entering } B_{r_0} \text{ chain } Y \text{ exits } B_{r_n} \text{ by time } t_n\} \ge \prod_{k=1}^{n} \left(\frac{\alpha_1}{r_k}\right). \tag{A.13}$$

Here α_1 is the constant coming from (A.6) and we can assume $\alpha_1 \le 1$.

We prove (A.13) by induction. The case $n = 1$ is Lemma A.3 applied to $r_1 = r_0^{\gamma}$ and r_0. The inductive step comes from the Markov property. Assume (A.13) is true for n and consider exiting $B_{r_{n+1}}$ without entering B_{r_0}.

(i) If the initial state x lies in $B_{r_n} \setminus B_{r_0}$ then by induction the chain first takes time t_n to exit B_{r_n} without entering B_{r_0} with probability bounded below by $\prod_{k=1}^{n}(\alpha_1/r_k)$. If the walk landed in $B_{r_{n+1}} \setminus B_{r_n}$ take another time $A_1 r_{n+1}^3 = A_1 r_0^{3\gamma^{n+1}}$ to exit $B_{r_{n+1}}$ without entering B_{r_n} with probability at least α_1/r_{n+1} (Lemma A.3 again). The times taken add up to t_{n+1} and the probabilities multiply to $\prod_{k=1}^{n+1}(\alpha_1/r_k)$.

(ii) If the initial state x lies in $B_{r_{n+1}} \setminus B_{r_n}$ then apply Lemma A.3 to exit $B_{r_{n+1}}$ without entering B_{r_n} in time $A_1 r_{n+1}^3 = A_1 r_0^{3\gamma^{n+1}}$ with probability at least α_1/r_{n+1}.

This completes the inductive proof of (A.13).

Let $N = \min\{k \ge 1\colon\ r_k \ge r\}$. Then $r_0^{\gamma^{N-1}} < r$. If r is large enough, and in particular r_0 is large enough to make $\log\log r_0 > 0$, then also $N < 1 + (\log\log r)/(\log\gamma) < 2\log\log r$.

To prove the corollary take first $n = N-1$ in (A.13). This gets the chain Y out of $B_{r_{N-1}}$ without entering B_{r_0}. If Y landed in $B_r \setminus B_{r_{N-1}}$, apply Lemma A.3 once more to take Y out of B_r without entering $B_{r_{N-1}}$. The probability of achieving this is bounded below by

$$\prod_{k=1}^{N-1}\left(\frac{\alpha_1}{r_k}\right)\cdot\frac{\alpha_1}{r} \ge \alpha_1^N r_0^{-\gamma^N/(\gamma-1)} r^{-1} \ge (\log r)^{2\log\alpha_1} r^{-\gamma/(\gamma-1)-1} \ge r^{-3},$$

where again we required large enough r. For the time elapsed we get the bound

$$t_{N-1} + A_1 r^3 \le A_1(N-1) r_0^{3\gamma^{N-1}} + A_1 r^3 \le r^4$$

for large enough r. □

The reader can see that the exponents in the previous lemmas can be tightened. But in the end the exponents still get rather large so we prefer to keep the statements and proofs simple for readability. We come to one of the main auxiliary lemmas of this development.

Lemma A.4. *Let $U = \inf\{n \ge 0\colon\ Y_n \notin B_r\}$ be the first exit time from $B_r = [-r, r]^d$ for the Markov chain Y. Then there exists a finite positive constant C_1 such that*

$$\sup_{x\in B_r} E_x[U] \le C_1 r^{13} \quad \textit{for all } 1 \le r < \infty.$$

Proof. First observe that $\sup_{x\in B_r} E_x[U] < \infty$ by assumption (A.1). Throughout, let positive integers $r_0 < r$ satisfy $\log\log r \le r_0 \le 2\log\log r$ so that in particular the assumptions of Corollary A.1 are satisfied. Once the statement is proved for large enough r, we obtain it for all $r \ge 1$ by increasing C_1.

Let $0 = T_0 = S_0 \le T_1 \le S_1 \le T_2 \le \cdots$ be the successive exit and entrance times into B_{r_0}. Precisely, for $i \ge 1$ as long as $S_{i-1} < \infty$

$$T_i = \inf\{n \ge S_{i-1}\colon\ Y_n \notin B_{r_0}\} \quad \text{and} \quad S_i = \inf\{n \ge T_i\colon\ Y_n \in B_{r_0}\}.$$

Once $S_i = \infty$ then we set $T_j = S_j = \infty$ for all $j > i$. If $Y_0 \in B_r \setminus B_{r_0}$ then also $T_1 = 0$. From assumption (A.1)

$$\sup_{x\in B_{r_0}} E_x[T_1] \le K^{r_0} \le (\log r)^{2\log K}. \tag{A.14}$$

So a priori T_1 is finite but $S_1 = \infty$ is possible. Since $T_1 \le U < \infty$ we can decompose as follows, for $x \in B_r$:

$$\begin{aligned} E_x[U] &= \sum_{j=1}^{\infty} E_x[U, T_j \le U < S_j] \\ &= \sum_{j=1}^{\infty} E_x[T_j, T_j \le U < S_j] + \sum_{j=1}^{\infty} E_x[U - T_j, T_j \le U < S_j]. \end{aligned} \tag{A.15}$$

We first treat the last sum in (A.15). By an inductive application of Corollary A.1, for any $z \in B_r \setminus B_{r_0}$,

$$\begin{aligned} P_z\{U > jr^4, U < S_1\} &\le P_z\{Y_k \in B_r \setminus B_{r_0} \text{ for } k \le jr^4\} \\ &= E_z\big[\mathbb{1}\{Y_k \in B_r \setminus B_{r_0} \text{ for } k \le (j-1)r^4\} P_{Y_{(j-1)r^4}}\{Y_k \in B_r \setminus B_{r_0} \text{ for } k \le r^4\}\big] \\ &\le \cdots \le (1 - r^{-3})^j. \end{aligned} \tag{A.16}$$

Utilizing this, still for $z \in B_r \setminus B_{r_0}$,

$$\begin{aligned} E_z[U, U < S_1] &= \sum_{m=0}^{\infty} P_z\{U > m, U < S_1\} \\ &\le r^4 \sum_{j=0}^{\infty} P_z\{U > jr^4, U < S_1\} \le r^7. \end{aligned} \tag{A.17}$$

Next we take into consideration the failure to exit B_r during the earlier excursions in $B_r \setminus B_{r_0}$. Let

$$H_i = \{Y_n \in B_r \text{ for } T_i \le n < S_i\}$$

be the event that in between the ith exit from B_{r_0} and entrance back into B_{r_0} the chain Y does not exit B_r. We shall repeatedly use this consequence of Corollary A.1:

$$\text{for } i \ge 1, \text{ on the event } \{T_i < \infty\}, \quad P_x\{H_i | \mathcal{F}_{T_i}\} \le 1 - r^{-3}. \tag{A.18}$$

Here is the first instance.

$$\begin{aligned} E_x[U - T_j, T_j \le U < S_j] &= E_x\left[\prod_{k=1}^{j-1} \mathbb{1}_{H_k} \cdot \mathbb{1}\{T_j < \infty\} \cdot E_{Y_{T_j}}(U, U < S_1)\right] \\ &\le r^7 E_x\left[\prod_{k=1}^{j-1} \mathbb{1}_{H_k} \cdot \mathbb{1}\{T_{j-1} < \infty\}\right] \le r^7 (1 - r^{-3})^{j-1}. \end{aligned}$$

Note that if Y_{T_j} above lies outside B_r then $E_{Y_{T_j}}(U) = 0$. In the other case $Y_{T_j} \in B_r \setminus B_{r_0}$ and (A.17) applies. So for the last sum in (A.15):

$$\sum_{j=1}^{\infty} E_x[U - T_j, T_j \le U < S_j] \le \sum_{j=1}^{\infty} r^7 (1 - r^{-3})^{j-1} \le r^{10}. \tag{A.19}$$

We turn to the second-last sum in (A.15). Separate the $i = 0$ term from the sum below and use (A.14) and (A.18):

$$\begin{aligned} &E_x[T_j, T_j \le U < S_j] \\ &\quad \le \sum_{i=0}^{j-1} E_x\left[\prod_{k=1}^{j-1} \mathbb{1}_{H_k} \cdot \mathbb{1}\{T_j < \infty\} \cdot (T_{i+1} - T_i)\right] \\ &\quad \le (\log r)^{2\log K} (1 - r^{-3})^{j-1} \\ &\qquad + \sum_{i=1}^{j-1} E_x\left[\prod_{k=1}^{i-1} \mathbb{1}_{H_k} \cdot (T_{i+1} - T_i) \mathbb{1}_{H_i} \cdot \mathbb{1}\{T_{i+1} < \infty\}\right] (1 - r^{-3})^{j-1-i}. \end{aligned} \tag{A.20}$$

Split the last expectation as

$$
\begin{aligned}
&E_x\left[\prod_{k=1}^{i-1}\mathbb{1}_{H_k}\cdot(T_{i+1}-T_i)\mathbb{1}_{H_i}\cdot\mathbb{1}\{T_{i+1}<\infty\}\right]\\
&\quad\le E_x\left[\prod_{k=1}^{i-1}\mathbb{1}_{H_k}\cdot(T_{i+1}-S_i)\mathbb{1}_{H_i}\cdot\mathbb{1}\{S_i<\infty\}\right]+E_x\left[\prod_{k=1}^{i-1}\mathbb{1}_{H_k}\cdot(S_i-T_i)\mathbb{1}_{H_i}\cdot\mathbb{1}\{T_i<\infty\}\right]\\
&\quad\le E_x\left[\prod_{k=1}^{i-1}\mathbb{1}_{H_k}\cdot\mathbb{1}\{S_i<\infty\}\cdot E_{Y_{S_i}}(T_1)\right]+E_x\left[\prod_{k=1}^{i-1}\mathbb{1}_{H_k}\cdot\mathbb{1}\{T_i<\infty\}\cdot E_{Y_{T_i}}(S_1\cdot\mathbb{1}_{H_1})\right]\\
&\quad\le E_x\left[\prod_{k=1}^{i-1}\mathbb{1}_{H_k}\cdot\mathbb{1}\{T_{i-1}<\infty\}\right]\bigl((\log r)^{2\log K}+r^7\bigr)\\
&\quad\le\bigl(1-r^{-3}\bigr)^{i-1}\bigl((\log r)^{2\log K}+r^7\bigr).
\end{aligned}\tag{A.21}
$$

In the second-last inequality above, before applying (A.18) to the H_k's, $E_{Y_{S_i}}(T_1)\le(\log r)^{2\log K}$ comes from (A.14). The other expectation is estimated by iterating Corollary A.1 again with $z\in B_r\setminus B_{r_0}$, as was done in calculation (A.16):

$$
\begin{aligned}
E_z[S_1\cdot\mathbb{1}_{H_1}]&=\sum_{m=0}^{\infty}P_z\{S_1>m,H_1\}\\
&\le\sum_{m=0}^{\infty}P_z\{Y_k\in B_r\setminus B_{r_0}\text{ for }k\le m\}\\
&\le r^4\sum_{j=0}^{\infty}P_z\bigl\{Y_k\in B_r\setminus B_{r_0}\text{ for }k\le jr^4\bigr\}\le r^7.
\end{aligned}
$$

Insert the bound from line (A.21) back up into (A.20) to get the bound

$$
E_x[T_j,T_j\le U<S_j]\le\bigl(2(\log r)^{2\log K}+r^7\bigr)j\bigl(1-r^{-3}\bigr)^{j-2}.
$$

Finally, bound the second-last sum in (A.15):

$$
\sum_{j=1}^{\infty}E_x[T_j,T_j\le U<S_j]\le\bigl(2(\log r)^{2\log K}r^6+r^{13}\bigr)\bigl(1-r^{-3}\bigr)^{-1}.
$$

Take r large enough so that $r^{-3}<1/2$. Combine the above bound with (A.15) and (A.19) to get

$$
E_x[U]\le r^{10}+4(\log r)^{2\log K}r^6+2r^{13}\le 4r^{13}
$$

when r is large enough. □

For the remainder of the proof we work with $B=B_r$ for $r=n^{\varepsilon_1}$. The above estimate gives us one part of the argument for (A.4), namely that the Markov chain Y exits $B=[-n^{\varepsilon_1},n^{\varepsilon_1}]^d$ fast enough.

Let $0=V_0<U_1<V_1<U_2<V_2<\cdots$ be the successive entrance times V_i into B and exit times U_i from B for the Markov chain Y, assuming that $Y_0=z\in B$. It is possible that some $V_i=\infty$. But if $V_i<\infty$ then also $U_{i+1}<\infty$ due to assumption (A.1), as already observed. The time intervals spent in B are $[V_i,U_{i+1})$ each of length at least 1.

Thus, by applying Lemma A.4,

$$\begin{aligned}\sum_{k=0}^{n-1} P_z(Y_k \in B) &\le \sum_{i=0}^{n} E_z\big[(U_{i+1} - V_i)\mathbb{1}\{V_i \le n\}\big] \\ &\le \sum_{i=0}^{n} E_z\big[E_{Y_{V_i}}(U_1)\mathbb{1}\{V_i \le n\}\big] \\ &\le Cn^{13\varepsilon_1} E_z\left[\sum_{i=0}^{n} \mathbb{1}\{V_i \le n\}\right].\end{aligned} \tag{A.22}$$

Next we bound the expected number of returns to B by the number of excursions outside B that fit in a time of length n:

$$E_z\left[\sum_{i=0}^{n} \mathbb{1}\{V_i \le n\}\right] = E_z\left[\sum_{i=0}^{n} \mathbb{1}\left\{\sum_{j=1}^{i}(V_j - V_{j-1}) \le n\right\}\right] \le E_z\left[\sum_{i=0}^{n} \mathbb{1}\left\{\sum_{j=1}^{i}(V_j - U_j) \le n\right\}\right]. \tag{A.23}$$

According to the usual notion of stochastic dominance, we say the random vector $(\xi_1, \dots, \xi_n)$ dominates $(\eta_1, \dots, \eta_n)$ if

$$Ef(\xi_1, \dots, \xi_n) \ge Ef(\eta_1, \dots, \eta_n)$$

for any function f that is coordinatewise nondecreasing. If the process $\{\xi_i\colon 1 \le i \le n\}$ is adapted to the filtration $\{\mathcal{G}_i\colon 1 \le i \le n\}$, and $P[\xi_i > a|\mathcal{G}_{i-1}] \ge 1 - F(a)$ for some distribution function F, then the $\{\eta_i\}$ can be taken i.i.d. F-distributed.

Lemma A.5. *There exist positive constants c_1, c_2 such that the following holds: the excursion lengths $\{V_j - U_j\colon 1 \le j \le n\}$ stochastically dominate i.i.d. variables $\{\eta_j\}$ whose common distribution satisfies $\mathbf{P}\{\eta \ge a\} \ge c_1 a^{-1/2}$ for $1 \le a \le c_2 n^{2\varepsilon_1(p_1-2)}$.*

Proof. Since $P_z\{V_j - U_j \ge a|\mathcal{F}_{U_j}\} = P_{Y_{U_j}}\{V \ge a\}$, where V means first entrance time into B, we shall bound $P_x\{V \ge a\}$ below uniformly over $x \notin B$. Fix such an x and an index $1 \le j \le d$ such that $x^j \notin [-r, r]$. As before we work through the case $x^j > r$ because the argument for the other case $x^j < -r$ is the same.

Let $\bar{w} = \inf\{n \ge 1\colon \bar{Y}_n^j \le r\}$ be the first time the one-dimensional random walk $\bar{Y}^j$ enters the half-line $(-\infty, r]$. If both Y and $\bar{Y}$ start at x and stay coupled together until time $\bar{w}$, then $V \ge \bar{w}$. This way we bound V from below. Since the random walk is symmetric and can be translated, we can move the origin to x^j and use classic results about the first entrance time into the left half-line, $\bar{T} = \inf\{n \ge 1\colon \bar{Y}_n^j < 0\}$. Thus

$$P_{x^j}\{\bar{w} \ge a\} \ge P_{r+1}\{\bar{w} \ge a\} = P_0\{\bar{T} \ge a\} \ge \frac{\alpha_5}{\sqrt{a}} \tag{A.24}$$

for a constant α_5. The last inequality follows for one-dimensional symmetric walks from basic random walk theory. For example, combine Eq. (7) on p. 185 of [17] with a Tauberian theorem such as Theorem 5 on p. 447 of [9]. Or see directly Theorem 1a on p. 415 of [9].

Now start both Y and $\bar{Y}$ from x. Apply Lemma A.2 and recall that $r = n^{\varepsilon_1}$.

$$\begin{aligned}P_x\{V \ge a\} &\ge P_{x,x}\{V \ge a, Y_k = \bar{Y}_k \text{ for } k = 1, \dots, \bar{w}\} \\ &\ge P_{x,x}\{\bar{w} \ge a, Y_k = \bar{Y}_k \text{ for } k = 1, \dots, \bar{w}\} \\ &\ge P_{x^j}\{\bar{w} \ge a\} - P_{x,x}\big\{Y_k \ne \bar{Y}_k \text{ for some } k \in \{1, \dots, \bar{w}\}\big\} \\ &\ge \frac{\alpha_5}{\sqrt{a}} - Cn^{\varepsilon_1(2-p_1)} \ge \frac{\alpha_5}{2\sqrt{a}}\end{aligned}$$

if $a \le \alpha_5^2(2C)^{-2}n^{2\varepsilon_1(p_1-2)}$. This lower bound is independent of x. We have proved the lemma. □

We can assume that the random variables η_j given by the lemma satisfy $1 \le \eta_j \le c_2 n^{2\varepsilon_1(p_1-2)}$, and we can assume that $c_2 \le 1$ and ε_1 is small enough to have

$$2\varepsilon_1(p_1-2) \le 1 \tag{A.25}$$

because this merely weakens the conclusion of the lemma. For the renewal process determined by $\{\eta_j\}$ write

$$S_0 = 0, \qquad S_k = \sum_{j=1}^{k} \eta_j \quad \text{and} \quad K(n) = \inf\{k\colon\ S_k > n\}$$

for the renewal times and the number of renewals up to time n (counting the renewal $S_0 = 0$). Since the random variables are bounded, Wald's identity gives

$$\mathbf{E}K(n) \cdot \mathbf{E}\eta = \mathbf{E}S_{K(n)} \le n + c_2 n^{2\varepsilon_1(p_1-2)} \le 2n,$$

while

$$\mathbf{E}\eta \ge \int_1^{c_2 n^{2\varepsilon_1(p_1-2)}} \frac{c_1}{\sqrt{s}}\,\mathrm{d}s \ge c_3 n^{\varepsilon_1(p_1-2)}.$$

Together these give

$$\mathbf{E}K(n) \le \frac{2n}{\mathbf{E}\eta} \le C_2 n^{1-\varepsilon_1(p_1-2)}.$$

Now we pick up the development from line (A.23). Since the negative of the function of $(V_j - U_j)_{1\le i\le n}$ in the expectation on line (A.23) is nondecreasing, the stochastic domination of Lemma A.5 gives an upper bound of (A.23) in terms of the i.i.d. $\{\eta_j\}$. Then we use the renewal bound from above.

$$\begin{aligned} E_z\left[\sum_{i=0}^{n} \mathbb{1}\{V_i \le n\}\right] &\le E_z\left[\sum_{i=0}^{n} \mathbb{1}\left\{\sum_{j=1}^{i}(V_j - U_j) \le n\right\}\right] \\ &\le \mathbf{E}\left[\sum_{i=0}^{n} \mathbb{1}\left\{\sum_{j=1}^{i}\eta_j \le n\right\}\right] = \mathbf{E}K(n) \le C_2 n^{1-\varepsilon_1(p_1-2)}. \end{aligned}$$

Returning back to (A.22) to collect the bounds, we have shown that

$$\sum_{k=0}^{n-1} P_z\{Y_k \in B\} \le C n^{13\varepsilon_1} E_z\left[\sum_{i=0}^{n} \mathbb{1}\{V_i \le n\}\right] \le C n^{1+13\varepsilon_1-\varepsilon_1(p_1-2)} = C n^{1-\eta}.$$

Since $p_1 > 15$ by assumption, $\eta = \varepsilon_1(p_1 - 15) > 0$. We can satisfy (A.25) with $\varepsilon_1 = (1/2)(p_1-2)^{-1}$ in which case the last bound is $Cn^{(1/2)+13/(2p_1-4)}$.

Appendix B. Replacing direction of transience

Hypotheses (1.1) and (M) are made for a specific vector $\hat{u}$. This appendix shows that, at the expense of a further factor in the moment required, the assumption that $\hat{u}$ has integer coordinates entails no loss of generality. This appendix also uses the assumption (S) that the magnitude of a step is bounded by r_0. We learned the proof below from Berger and Zeitouni [1].

Assume some vector $\hat{w} \in \mathbb{R}^d$ satisfies $P_0\{X_n \cdot \hat{w} \to \infty\} = 1$. Let $\{\sigma_k\}_{k\geq 0}$ be the regeneration times in the direction $\hat{w}$. Assume $E_0(\sigma_1^{p_3}) < \infty$ for some $p_3 > 6$. As explained in Section 2, transience and moments on $\hat{w}$ imply the law of large numbers

$$\frac{X_n}{n} \to v = \frac{E_0[X_{\sigma_1} | \hat{\beta} = \infty]}{E_0[\sigma_1 | \hat{\beta} = \infty]} \quad P_0\text{-almost surely,} \tag{B.1}$$

where $\hat{\beta} = \inf\{n \geq 0\colon\ X_n \cdot \hat{w} < X_0 \cdot \hat{w}\}$ is the first backtracking time in the direction $\hat{w}$. The limiting velocity v satisfies $\hat{w} \cdot v > 0$.

Proposition B.1. *Suppose $\hat{u} \in \mathbb{R}^d$ satisfies $\hat{u} \cdot v > 0$. Then*

$$P_0\{X_n \cdot \hat{u} \to \infty\} = 1.$$

For the first regeneration time τ_1 in the direction $\hat{u}$ we have the estimate $E_0(\tau_1^{p_0}) < \infty$ for $1 \leq p_0 < p_3/2 - 2$.

From this lemma we can choose a $\hat{u}$ with rational coordinates and then scale it by a suitable integer to get the integer vector $\hat{u}$ assumed in (1.1) and Hypothesis (M). To get $p_0 > 176d$ as required by (M) of course puts an even larger demand on p_3.

Proof of Proposition B.1. **Step 1.** Transience in direction $\hat{u}$.

Given n choose $k = k(n)$ so that $\sigma_{k-1} < n \leq \sigma_k$. Then by the bounded step Hypothesis (S)

$$\left| \frac{1}{n} X_n \cdot \hat{u} - \frac{1}{n} X_{\sigma_k} \cdot \hat{u} \right| \leq \frac{1}{n} |\hat{u}| r_0 (\sigma_k - \sigma_{k-1}). \tag{B.2}$$

By the moment assumption on σ_1 the right-hand side converges P_0-a.s. to zero while $n^{-1} X_{\sigma_k} \cdot \hat{u} \to v \cdot \hat{u} > 0$, and so in particular $X_n \cdot \hat{u} \to \infty$. From this follows that the regeneration times $\{\tau_k\}$ in direction $\hat{u}$ are finite.

Step 2. Moment bound on the height $X_{\tau_1} \cdot \hat{u}$ of the first $\hat{u}$-regeneration slab.

Let β be the $\hat{u}$-backtracking time as defined in (2.2) and

$$M = \sup_{0 \leq n \leq \beta} X_n \cdot \hat{u}.$$

Lemma 1.2 in [19] shows how the construction of the regeneration time leads to stochastic domination of $X_{\tau_1} \cdot \hat{u}$ under P_0 by a sum of geometrically many i.i.d. terms, each distributed like M plus a fixed constant under the measure $P_0(\cdot | \beta < \infty)$. Hence to prove $E_0[(X_{\tau_1} \cdot \hat{u})^p] < \infty$ it suffices to prove $E_0(M^p | \beta < \infty) < \infty$. We begin with a lemma that helps control the tail probabilities $P\{M > m | \beta < \infty\}$. For the arguments it turns out convenient to multiply m by the constant $|\hat{u}| r_0$.

Lemma B.2. *There exist $\delta_0 > 0$ such that this holds: if $\delta \in (0, \delta_0)$ there exists an $m_0 = m_0(\delta) < \infty$ such that for $m \geq m_0$ the event $\{M > m|\hat{u}|r_0, \beta < \infty\}$ lies in the union of these three events:*

$$\sigma_{[\delta m]} \geq m, \tag{B.3}$$

$$\sigma_k - \sigma_{k-1} \geq \delta k \quad \text{for some } k > [\delta m], \tag{B.4}$$

$$\left| X_{\sigma_k} - E_0(X_{\sigma_k}) \right| \geq \delta k \quad \text{for some } k > [\delta m]. \tag{B.5}$$

Proof of Lemma B.2. Assume that $\beta < \infty$ and $M > m|\hat{u}|r_0$, but conditions (B.3)–(B.5) fail simultaneously. We derive a contradiction from this.

Fix $k \geq 1$ so that

$$\sigma_{k-1} < \beta \leq \sigma_k. \tag{B.6}$$

Since the maximum step size is r_0, at least m steps are needed to realize the event $M > m|\hat{u}|r_0$ and so $\beta > m$. Thus negating (B.3) implies $k > \delta m$. The step bound, (B.6) and the negation of (B.4) imply

$$X_{\sigma_k} \cdot \hat{u} \le X_\beta \cdot \hat{u} + |\hat{u}|r_0(\sigma_k - \sigma_{k-1}) < |\hat{u}|r_0\delta k. \tag{B.7}$$

Introduce the shorthands

$$a = E_0(X_{\sigma_1}) \quad \text{and} \quad b = E_0(\sigma_1|\hat{\beta} = \infty). \tag{B.8}$$

By the i.i.d. property of the regeneration slabs from the second one onwards [recall the discussion around (2.4)] $E_0(\sigma_k - \sigma_{k-1}) = b$ and $E_0(X_{\sigma_k}) = a + b(k-1)v$ for $k \ge 1$. Thus negating (B.5) gives

$$\begin{aligned} X_{\sigma_k} \cdot \hat{u} &= \big(X_{\sigma_k} - a - b(k-1)v\big) \cdot \hat{u} + a \cdot \hat{u} + b(k-1)v \cdot \hat{u} \\ &\ge -\delta k|\hat{u}| - |a| \cdot |\hat{u}| + b(k-1)v \cdot \hat{u}. \end{aligned} \tag{B.9}$$

Since $v \cdot \hat{u} > 0$, comparison of (B.7) and (B.9) reveals that it is possible to first fix $\delta > 0$ small enough and then m_0 large enough so that, if $m \ge m_0$, then $k > \delta m$ forces a contradiction between (B.7) and (B.9). This concludes the proof of Lemma B.2. □

Next we observe that the union of (B.3)–(B.5) has probability $\le Cm^{1-p_3/2}$. The assumptions of $\hat{w}$-directional transience and $E_0(\sigma_1^{p_3}) < \infty$ imply that $P_0(\hat{\beta} = \infty) > 0$ and hence (by the i.i.d. slab property again) for $k \ge 2$,

$$E_0\big[(\sigma_k - \sigma_{k-1})^{p_3}\big] = E_0\big[\sigma_1^{p_3}|\hat{\beta} = \infty\big] \le \frac{E_0(\sigma_1^{p_3})}{P_0(\hat{\beta} = \infty)} < \infty. \tag{B.10}$$

For the next calculation, recall that for i.i.d. mean zero summands and $p \ge 2$ the Burkholder–Davis–Gundy inequality [5] followed by Jensen's inequality gives

$$E\left[\left|\sum_{j=1}^n Z_j\right|^p\right] \le E\left[\left(\sum_{j=1}^n Z_j^2\right)^{p/2}\right] \le n^{p/2} E\big(Z_1^p\big).$$

Recall a and b from (B.8). Shrink δ further (this can be done at the expense of increasing m_0 in Lemma B.2) so that $\delta b < 1/4$.

$$P_0\{\sigma_{[\delta m]} \ge m\} \le P_0\left\{\sigma_1 \ge \frac{m}{2}\right\} + P_0\left\{\sum_{k=2}^{[\delta m]}(\sigma_k - \sigma_{k-1} - b) \ge \frac{m}{4}\right\} \le Cm^{-p_3/2}. \tag{B.11}$$

For the second estimate use (B.10).

$$\sum_{k>[\delta m]} P_0\{\sigma_k - \sigma_{k-1} \ge \delta k\} \le \sum_{k>[\delta m]} C(\delta k)^{-p_3} \le Cm^{1-p_3}.$$

For the third estimate use Chebychev and for the sum of i.i.d. pieces repeat the Burkholder–Davis–Gundy estimate:

$$P_0\big\{\big|X_{\sigma_k} - E_0(X_{\sigma_k})\big| \ge \delta k\big\} \le P_0\left\{\big|X_{\sigma_1} - a\big| \ge \frac{\delta k}{2}\right\} + P_0\left\{\big|X_{\sigma_k} - X_{\sigma_1} - (k-1)bv\big| \ge \frac{\delta k}{2}\right\} \le Ck^{-p_3/2}. \tag{B.12}$$

Summing these bounds over $k > [\delta m]$ gives $Cm^{1-p_3/2}$.

Collecting the above bounds for the events (B.3)–(B.5) and utilizing Lemma B.2 gives the intermediate bound $P_0(M > m|\hat{\beta} < \infty) \le Cm^{1-p_3/2}$ for large enough m. Hence $E_0(M^p|\hat{\beta} < \infty) < \infty$ for $p < p_3/2 - 1$. By the already mentioned appeal to Lemma 1.2 in Sznitman [19] we can conclude Step 2 with the bound

$$E_0\big[(X_{\tau_1} \cdot \hat{u})^p\big] < \infty \quad \text{for } p < \frac{p_3}{2} - 1. \tag{B.13}$$

Step 3. Moment bound for τ_1. We insert one more lemma.

Lemma B.3. *For* $\ell \ge 1$:

$$P_0\{|X_n - nv| \ge \delta n \text{ for some } n \ge \ell\} \le C\ell^{1-p_3/2}.$$

Proof of Lemma B.3. Fix a small $\eta > 0$.

$$\begin{aligned} &P_0\bigl\{|X_n - nv| \ge \delta n \text{ for some } n \ge \ell\bigr\} \\ &\quad\le P_0\{\sigma_{[\eta\ell]} \ge \ell\} + \sum_{j>\eta\ell} P_0\bigl\{|X_n - nv| \ge \delta n \text{ for some } n \in [\sigma_{j-1}, \sigma_j]\bigr\}. \end{aligned}$$

If η is small enough the first probability above is bounded by $C\ell^{-p_3/2}$ as in (B.11). For a term in the sum, note first that if $\sigma_{j-1} \ge \eta j$ then the parameter n in the probability satisfies $n \ge \eta j$. Then replace time n with time σ_j at the expense of an error of a constant times $\sigma_j - \sigma_{j-1}$:

$$\begin{aligned} &P_0\bigl\{|X_n - nv| \ge \delta n \text{ for some } n \in [\sigma_{j-1}, \sigma_j]\bigr\} \\ &\quad\le P_0\{\sigma_{j-1} < \eta j\} + P_0\left\{|X_{\sigma_j} - \sigma_j v| \ge \frac{\delta\eta j}{2}\right\} + P_0\left\{\bigl(|v| + r_0\bigr)(\sigma_j - \sigma_{j-1}) \ge \frac{\delta\eta j}{2}\right\}. \end{aligned}$$

The first probability after the inequality gives again $Cj^{-p_3/2}$ as in (B.11) if η is small enough. In the second one the summands $X_{\sigma_j} - X_{\sigma_{j-1}} - (\sigma_j - \sigma_{j-1})v$ are i.i.d. mean zero for $j \ge 2$ so we can argue in the same spirit as in (B.12) to get $Cj^{-p_3/2}$. The last probability gives Cj^{-p_3} by the moments of $\sigma_j - \sigma_{j-1}$. Adding the bounds gives the conclusion. □

Now we finish the proof of Proposition B.1. To get a contradiction, suppose that $p_0 < p_3/2 - 2$ and $E_0(\tau_1^{p_0}) = \infty$. Pick $\varepsilon \in (0, p_3/2 - 2 - p_0)$. Then there exists a subsequence $\{k_j\}$ such that $P_0(\tau_1 > k_j) \ge k_j^{-p_0-\varepsilon}$. With the above lemma and the choice of ε we have, for large enough k_j

$$P_0\bigl\{\tau_1 > k_j, |X_n - nv| < \delta n \text{ for all } n \ge k_j\bigr\} \ge k_j^{-p_0-\varepsilon} - Ck_j^{1-p_3/2} \ge Ck_j^{-p_0-\varepsilon}.$$

On the event above

$$X_{\tau_1} \cdot \hat{u} \ge \tau_1 v \cdot \hat{u} - \delta\tau_1|\hat{u}| \ge \delta_1 k_j$$

for another small $\delta_1 > 0$ if δ is small enough. Thus we have

$$P_0\{X_{\tau_1} \cdot \hat{u} \ge \delta_1 k_j\} \ge Ck_j^{-p_0-\varepsilon}.$$

From this

$$\delta_1^{-p} E_0\bigl[(X_{\tau_1} \cdot \hat{u})^p\bigr] \ge C\sum_j k_j^{p-1-p_0-\varepsilon}.$$

This sum diverges and contradicts (B.13) if p is chosen to satisfy $p_3/2 - 1 > p \ge 1 + p_0 + \varepsilon$ which can be done by the earlier choice of ε. This contradiction implies that $E_0(\tau_1^{p_0}) < \infty$ and completes the proof of Proposition B.1. □

Appendix C. Completion of a technical proof

In this appendix we finish the proof of Lemma 7.13 by deriving the bounds (7.30)–(7.32).

Proof of (7.30). By Hypothesis (R), there exist two nonzero vectors $w \ne z$ such that $\mathbb{E}\pi_{0,z}\pi_{0,w} > 0$. We will distinguish several cases. In each case we describe two paths that the two walkers can take with positive probability.

The two paths will start at 0, reach a fresh common level at distinct points, will not backtrack below level 0, and will not have a chance of a joint regeneration at any previous positive level. Then the two walks can regenerate with probability $\geq \eta > 0$ (Lemma 7.1). Note that the part of the environment responsible for the two paths and the part responsible for regeneration lie in separate half-spaces. Since $\mathbb{P}$ is product, a positive lower bound for (7.33) is obtained and (7.30) thereby proved.

Case 1. $w \cdot \hat{u} > 0$, $z \cdot \hat{u} > 0$, and they are noncollinear. Let one walk take enough w-steps and the other enough z-steps.

Case 2. $w \cdot \hat{u} > 0$, $z \cdot \hat{u} > 0$, and they are collinear. Since the walk is not confined to a line (Hypothesis (R)), there must exist a vector y that is not collinear with z, w such that $\mathbb{E}\pi_{0y} > 0$.

Subcase 2.a. $y \cdot \hat{u} < 0$. Exchanging w and z, if necessary, we can assume $w \cdot \hat{u} < z \cdot \hat{u}$. Let $n > 0$ and $m > 0$ be such that $nw \cdot \hat{u} + my \cdot \hat{u} = 0$. Let one walk take n w-steps then m y-steps, coming back to level 0, then n w-steps and a z-step. The other walk takes $n - 1$ w-steps (staying with the first walk), a z-step, then a w-step.

Subcase 2.b. $y \cdot \hat{u} \geq 0$. Let $n \geq 0$ and $m > 0$ be such that $nw \cdot \hat{u} = my \cdot \hat{u}$. One walk takes a w-step, m y-steps, then a z-step. The other walk takes a z-step, then $n + 1$ w-steps. Whenever the walks are on a common level, they will be at distinct points.

Case 3. $w \cdot \hat{u} = 0$ while $z \cdot \hat{u} > 0$. The first walk takes a w-step then a z-step. The second walk takes a z-step. The case when $w \cdot \hat{u} > 0$ and $z \cdot \hat{u} = 0$ is similar.

Case 4. $w \cdot \hat{u} = z \cdot \hat{u} = 0$. By $\hat{u}$-transience, there exists a y with $y \cdot \hat{u} > 0$ and $\mathbb{E}\pi_{0y} > 0$. One walk takes a w-step, the other a z-step, then both take a y-step.

The rest of the cases treat the situation when $w \cdot \hat{u} < 0$ or $z \cdot \hat{u} < 0$. Exchanging w and z, if necessary, we can assume that $w \cdot \hat{u} < 0$.

Case 5. $w \cdot \hat{u} < 0$, $z \cdot \hat{u} > 0$, and they are noncollinear. This can be resolved as in the proof of Lemma 7.1 for $x \neq 0$, since now path intersections do not matter. More precisely, let $n > 0$ and $m > 0$ be such that $nw \cdot \hat{u} = mz \cdot \hat{u}$. The first walk takes m z-steps, n w-steps, backtracking all the way back to level 0, then $m + 1$ z-steps. The other walk just takes $m + 1$ z-steps.

Case 6. $w \cdot \hat{u} < 0$, $z \cdot \hat{u} > 0$, and they are collinear. Since the one-dimensional case is excluded, there must exist a vector y noncollinear with them and such that $\mathbb{E}\pi_{0y} > 0$.

Subcase 6.a. $y \cdot \hat{u} > 0$. Let $m > 0$ and $n > 0$ be such that $ny \cdot \hat{u} + mw \cdot \hat{u} = 0$. Let k be a minimal integer such that $kz \cdot \hat{u} + y \cdot \hat{u} + mw \cdot \hat{u} \geq 0$. The first walk takes k z-steps, a y-step, m w-steps, n y-step and a z-step. The other walk takes k z-steps, a y-step, staying so far with the first walk, then splits away and takes a z-step.

Subcase 6.b. $y \cdot \hat{u} = 0$. Let $m > 0$ and $n > 0$ be such that $nz \cdot \hat{u} + mw \cdot \hat{u} = 0$. The first walk takes n z-steps, a y-step, m w-steps, backtracking all the way back to level 0, a y-step, then takes $n + 1$ z-step. The other walk takes n z-steps, a y-step, staying with the first walk, then takes a z-step.

Subcase 6.c. $y \cdot \hat{u} < 0$. Let $k > 0$, $\ell > 0$, $m > 0$, and $n > 0$ be such that $\ell z \cdot \hat{u} = k(w + y) \cdot \hat{u}$ and $mz \cdot \hat{u} = ny \cdot \hat{u}$. The first walk takes $\ell + m$ z-steps, n y-steps, k w-steps, k y-steps, backtracking back to level 0, then $\ell + m + 1$ z-steps. The second walk takes $\ell + m$ z-steps, n y-steps, staying with the other walk, then $m + 1$ z-steps.

Case 7. $w \cdot \hat{u} < 0$, $z \cdot \hat{u} < 0$, and they are collinear. Since the one-dimensional case is excluded, there exists a u noncollinear with them and such that $\mathbb{E}\pi_{0u} > 0$. Furthermore, by $\hat{u}$-transience, there exists a y such that $y \cdot \hat{u} > 0$ and $\mathbb{E}\pi_{0y} > 0$. It could be the case that $y = u$.

Subcase 7.a. y is not collinear with w and z. Let k be the minimal integer such that $w \cdot \hat{u} + ky \cdot \hat{u} > 0$. Let $n > 0$ and $m > 0$ be such that $ny \cdot \hat{u} + mz \cdot \hat{u} = 0$. Let ℓ by the minimal integer such that $\ell y \cdot \hat{u} + w \cdot \hat{u} + mz \cdot \hat{u} \geq 0$. The first walk takes ℓ y-steps, a z-step, a w-step, $m - 1$ z-steps, then $n + k$ y-steps. The other walk takes ℓ y-steps, a w-step, then k y-steps.

Subcase 7.b. y is collinear with w and z and $u \cdot \hat{u} \leq 0$. Let $m > 0$ and $n > 0$ be such that $m(z \cdot \hat{u} + u \cdot \hat{u}) = ny \cdot \hat{u}$. Let k be minimal such that $ky \cdot \hat{u} + w \cdot \hat{u} + 2u \cdot \hat{u} > 0$. Let ℓ be the minimal integer such that $\ell y \cdot \hat{u} + mz \cdot \hat{u} + w \cdot \hat{u} + (m + 2)u \cdot \hat{u} \geq 0$. The first walk takes ℓ y-steps, a u-step, a z-step, a w-step, $m - 1$ z-steps, $m + 1$ u-steps, then $n + k$ y-steps. The other walk takes also ℓ y-steps and a u-step, but then splits from the first walk taking a w-step, a u-step, and k y-steps.

The subcase when y is collinear with w and z and $u \cdot \hat{u} > 0$ is done by using u in place of y in the argument of subcase 7.a.

Case 8. $w \cdot \hat{u} < 0$, $z \cdot \hat{u} \leq 0$, and they are not collinear. By $\hat{u}$-transience, $\exists y : y \cdot \hat{u} > 0$ and $\mathbb{E}\pi_{0y} > 0$.

Subcase 8.a. y is not collinear with w nor with z and $ay + bw + z \neq 0$ for all integers $a, b > 0$. Let $m > 0$ and $n > 0$ be such that $mw \cdot \hat{u} + ny \cdot \hat{u} = 0$. Let ℓ be the minimal integer such that $\ell y \cdot \hat{u} + mw \cdot \hat{u} + z \cdot \hat{u} \geq 0$. Let k be the minimal integer such that $z \cdot \hat{u} + ky \cdot \hat{u} > 0$. The first walk takes ℓ y-steps, m w-steps, a z-step, then $n + k$ y-steps. The other walk takes also ℓ y-steps, a z-step, then k y-steps.

Subcase 8.b. y is not collinear with w nor with z, there exist integers $a, b > 0$ such that $ay + bw + z = 0$ and $z \cdot \hat{u} = 0$. One walk takes a y-steps, one z-step, and one y-step. The other walk takes a y-steps, b w-steps, then $(a + 1)$ y-steps.

Subcase 8.c. y is not collinear with w nor with z, there exist integers $a, b > 0$ such that $ay + bw + z = 0$ and $z \cdot \hat{u} < 0$. Pick $k, n > 0$ such that $kw \cdot \hat{u} = nz \cdot \hat{u}$. Pick $i, j > 0$ so that $iy \cdot \hat{u} + jkw \cdot \hat{u} = 0$. The first walk takes i y-steps, then jk w-steps followed by $(i + 1)$ y-steps. The second walk takes i y-steps, then jn z-steps followed by $(i + 1)$ y-steps.

In subcases 8.b and 8.c there are no self-intersections because the pairs y, w and y, z are not collinear. Also, the two paths cannot intersect because an intersection together with $z = -ay - bw$ would force y and w to be collinear.

Subcase 8.d. y is collinear with z or with w. Exchanging z and w, if necessary, and noting that if $z \cdot \hat{u} = 0$ then y cannot be collinear with z, we can assume that y is collinear with w. Let $k > 0$ and $\ell > 0$ be the minimal integers such that $kw \cdot \hat{u} + \ell y \cdot \hat{u} = 0$. Let $a \geq 0$ and $b > 0$ be the minimal integers such that $ay \cdot \hat{u} + bz \cdot \hat{u} = 0$. Let m be the smallest integer such that $my \cdot \hat{u} + 2z \cdot \hat{u} > 0$. Let n be the smallest integer such that $ny \cdot \hat{u} + (b+2)z \cdot \hat{u} + kw \cdot \hat{u} > 0$. Now, the first walk takes n y-steps, one z-step, k w-steps, $(b + 1)$ z-steps, then $\ell + m + a$ y-steps. The other walk takes n y-steps, two z-steps, and then m y-steps.

Proof of (7.31). We appeal here to the construction done in the proof of Lemma 7.1. For $x \in \mathbb{V}_d \setminus \{0\}$ the paths constructed there gave us a bound

$$P_x\big[|Y_1| > L\big] = P_{0,x}\big\{\beta = \tilde{\beta} = \infty, |\widetilde{X}_{\tilde{\mu}_1} - X_{\mu_1}| > L\big\} \geq \delta(x) > 0$$

for any given L. There was a stage in that proof where x may have been replaced by $-x$, so the above bound is valid for either x or $-x$. But translation shows that

$$P_x\big[|Y_1| = a\big] = P_{-x}\big[|Y_1| = a\big]$$

and so we have the estimate for all $x \in \mathbb{V}_d \setminus \{0\}$. Considering only finitely many x inside a ball gives a uniform lower bound $\delta = \min_{|x| \leq L} \delta(x) > 0$.

Proof of (7.32). The proof of Lemma 7.1 gave us two paths $\sigma_1 = \{0 = x_0, x_1, \ldots, x_{m_1}\}$ and $\sigma_2 = \{0 = y_0, y_1, \ldots, y_{m_2}\}$ with positive probability and these additional properties: the paths do not backtrack below level 0, the final points x_{m_1} and y_{m_2} are distinct but on a common level $\ell = x_{m_1} \cdot \hat{u} = y_{m_2} \cdot \hat{u} > 0$, and no level strictly between 0 and ℓ can serve as a level of joint regeneration for the paths.

To recall more specifically from the proof of Lemma 7.1, these paths were constructed from two nonzero, non-collinear vectors $z, w \in \mathcal{J} = \{x\colon \mathbb{E}\pi_{0,x} > 0\}$ such that $z \cdot \hat{u} > 0$. If also $w \cdot \hat{u} > 0$, then take $\sigma_1 = \{(iz)_{0 \leq i \leq m}\}$ and $\sigma_2 = \{(iw)_{0 \leq i \leq n}\}$ where m, n are the minimal positive integers such that $mz \cdot \hat{u} = nw \cdot \hat{u}$. In the case $z \cdot \hat{u} > 0 \geq w \cdot \hat{u}$ these paths were given by $\sigma_1 = \{(iz)_{0 \leq i \leq m}, (mz + iw)_{1 \leq i \leq n}, (mz + nw + iz)_{1 \leq i \leq m+1}\}$ and $\sigma_2 = \{(iz)_{0 \leq i \leq m+1}\}$, where now $m \geq 0$ and $n > 0$ are minimal for $mz \cdot \hat{u} = -nw \cdot \hat{u}$.

Take L large enough so that $|z_1 - z_2| > L$ guarantees that paths $z_1 + \sigma_1$ and $z_2 + \sigma_2$ cannot intersect. Let $\hat{b} = y_{m_2} - x_{m_1} \in \mathbb{V}_d \setminus \{0\}$. Then by the independence of environments and (7.3), for $|x| > L$,

$$\begin{aligned}
P_x[Y_1 - Y_0 = \hat{b}] &\geq P_{0,x}\{\beta = \tilde{\beta} = \infty, \widetilde{X}_{\tilde{\mu}_1} = x + y_{m_2}, X_{\mu_1} = x_{m_1}\} \\
&\geq P_{0,x}\{X_{0,m_1} = \sigma_1, \widetilde{X}_{0,m_2} = x + \sigma_2, \beta \circ \theta^{m_1} = \tilde{\beta} \circ \theta^{m_2} = \infty\} \\
&\geq \left(\prod_{i=0}^{m_1-1} \mathbb{E}\pi_{x_i, x_{i+1}}\right)\left(\prod_{i=0}^{m_2-1} \mathbb{E}\pi_{y_i, y_{i+1}}\right) \cdot \eta > 0.
\end{aligned}$$

The lower bound is independent of x. The same lower bound for $P_x[Y_1 - Y_0 = -\hat{b}]$ comes by letting X follow σ_2 and $\widetilde{X}$ follow $x + \sigma_1$. This completes the proof of Lemma 7.13.

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