NON-EXISTENCE OF NON-TRIVIAL BI-INFINITE GEODESICS IN GEOMETRIC LAST PASSAGE PERCOLATION

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Abstract. — We show non-existence of non-trivial bi-infinite geodesics in the solvable last-passage percolation model with i.i.d. geometric weights. This gives the first example of a model with discrete weights where non-existence of non-trivial bi-infinite geodesics has been proven. Our proofs rely on the structure of the increment-stationary versions of the model, following the approach recently introduced by Balázs, Busani, and Seppäläinen. Most of our results work for a general weights distribution and we identify the two properties of the stationary distributions which would need to be shown in order to generalize the main result to a non-solvable setting.

1. Introduction

This paper considers directed last-passage percolation (LPP), which is a prototypical example of a lattice interface growth model in 1+1 dimensions. Such lattice growth models have played a central role in the development of modern probability over the last fifty years, with 1+1 dimensional LPP rising in importance over recent decades as a member of the Kardar-Parisi-Zhang (KPZ) universality class. See the recent surveys [3, 14, 15, 28, 42, 43].

Last-passage percolation, along with closely related models like first-passage percolation, directed polymers, and certain stochastic Hamilton-Jacobi equations, have interpretations as a kind of directed analogue of a metric. For this point of view, see for example the discussion in [18, 19] and also [5]. This connection is exact in the case of first-passage percolation, which genuinely describes a random metric on the lattice. In these interpretations, it is often possible to interpret the solution in terms of random paths, which are variously called geodesics, random polymers, or characteristics, among others. The structure of these random paths has been a major focus of research in the field.

This project considers a particular subset of questions related to bi-infinite geodesics, which are bi-infinite paths with the property that the restriction of the path to any finite subpath is a geodesic between its endpoints. The study of such paths traces back at least to a question Furstenburg posed to Kesten on first-passage percolation [37, p. 258], where the existence of such paths is equivalent to the existence of non-trivial ground states in the ferromagnetic Ising model with random impurities [3, p. 105]. It is generally believed that in models of the type we consider, non-trivial bi-infinite geodesics should not exist, for reasons we will discuss shortly. Much of the mathematical progress toward proving this conjecture traces back to the seminal ICM note of Newman [41], which instigated a fruitful line of research on the structure of semi-infinite and bi-infinite geodesics in first-passage percolation [31, 32, 39]. These ideas motivated subsequent work on first [1, 16, 17, 29, 30] and last passage percolation [12, 13, 26, 27, 36], as well as related models [4, 6, 7, 34].

Keywords: Bi-infinite geodesic, Geometric, Last Passage Percolation.
2020 *Mathematics Subject Classification:* 60K35,60K37.
(*) C. Janjigian was partially supported by National Science Foundation grant DMS-2125961.
F. Rassoul-Agha and S. Groathouse were partially supported by NSF grants DMS-1407574 and DMS-1811090.
Last revised: November 30, 2021.

One of the main predictions of the KPZ class concerns the structure of fluctuations of analogues of geodesics, and in particular their characteristic 2/3 scaling exponent in 1+1 dimensions. At an ACM workshop in 2015 [3, Section 4.5.1], Newman gave a heuristic argument that in dimensions for which this transversal fluctuation exponent is greater than 1/2, non-trivial bi-infinite geodesics should not exist. Two different implementations of this heuristic were recently carried out in the last-passage percolation model with i.i.d. exponential weights by Basu, Hoffman, and Sly [**?**] and Balázs, Busani, and Seppäläinen [8]. The former implementation uses integrable methods heavily, while the latter relies on the structure of the increment-stationary distributions for the model. Both approaches rely in essential ways on the exact solvability of the exponential last-passage percolation model. A general version of Newman's argument under strong conditions on the passage time was recently implemented by Alexander in [2]. Perhaps the strongest unconditional result in this direction is the recent [11].

The present paper abstracts the approach of [8]. We consider a novel implementation of the argument to the last-passage percolation model with geometric weights, giving an example of a model with discrete weights for which non-trivial bi-infinite geodesics do not exist. More broadly, we re-cast the approach of [8] without reference to particular weight distributions and identify two properties of increment-stationary distributions, recorded as Assumptions 4.1 and 5.2 below, which would need to be proven in order to realize this program for non-integrable models. After introducing each of these assumptions, we discuss the types of hypotheses on the last-passage percolation model which would need to be proven in order to verify these conditions. It is noteworthy that it is known from [27, 34] that the increment stationary models we discuss in these assumptions have been shown to exist generally.

Our main result, Theorem 5.3, shows that under our abstract hypotheses, non-trivial bi-infinite geodesics do not exist. We verify our conditions in the geometric model using exact solvability. Along the way, we also prove some novel results about geometric last-passage percolation in order to verify our hypotheses. In particular, we prove a new, sharp bound for exit times in increment-stationary geometric last-passage percolation following a strategy recently introduced in [21, 22]. This is recorded as Theorem B.1 below.

2. Setting and the main result

Let $\Omega = \mathbb{R}^{\mathbb{Z}^2}$ and equip it with the product topology and its Borel σ-algebra $\mathcal{F}$. A generic element in Ω is denoted by ω and is sometimes referred to as an *environment*. Let $(\omega_x)_{x\in\mathbb{Z}^2}$ be the coordinates of ω. ω_x is referred to as the *weight* at x. We assume the following throughout the paper: we are given a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that

$$(\omega_x)_{x\in\mathbb{Z}^2} \text{ are i.i.d. under } \mathbb{P}, \quad \exists\epsilon > 0 : \ \mathbb{E}[|\omega_0|^{2+\epsilon}] < \infty, \quad \text{and} \quad \mathbb{V}\mathrm{ar}(\omega_0) > 0. \tag{2.1}$$

Denote by $T = \{T_x : x \in \mathbb{Z}^2\}$ the natural group of shift operators on Ω, which satisfy $(T_y\omega)_x = \omega_{x+y}$ for $x, y \in \mathbb{Z}^2$. Given sites $x, y \in \mathbb{Z}^2$ with $x \leqslant y$ (coordinatewise), an *up-right path* from x to y is a sequence of lattice vertices with increments in the set $\{\mathbf{e}_1, \mathbf{e}_2\}$, the canonical basis of $\mathbb{R}^2$. The collection of up-right paths from x to y is denoted by Π_x^y. The *passage time* (or the weight) of an up-right path $\pi \in \Pi_x^y$ is the sum of the weights of the vertices of the path: $\sum_{v\in\pi}\omega_v$. For $x \leqslant y$ in $\mathbb{Z}^2$, the (bulk) *last-passage time* from x to y is defined to be

$$G_{x,y}(\omega) = \max_{\pi\in\Pi_x^y}\sum_{v\in\pi}\omega_v. \tag{2.2}$$

In particular, $G_{x,x}(\omega) = \omega_x$. As is customary in probability theory we often omit the ω from the argument of $G_{x,y}$.

A path $\pi \in \Pi_x^y$ which realizes the maximum in (2.2) is called a *geodesic*. This terminology is by analogy with the related model of first-passage percolation where $G_{x,y}$ defines a random pseudo-metric on $\mathbb{Z}^2$. Geodesics are unique when $\mathbb{P}(\omega_0 \leqslant t)$ is continuous in t, but when this distribution function is not continuous, multiple geodesics can exit.

Our main interest in the present paper is in the structure of *bi-infinite geodesics*, which we now define.

Definition 2.1. — *A bi-infinite up-right path $\pi_{-\infty:\infty} = (\pi_n)_{n\in\mathbb{Z}}$ is said to be a bi-infinite geodesic if for every $m < n$ in $\mathbb{Z}$, the segment $\pi_{m:n}$ is a geodesic between π_m and π_n.*

For each $x \in \mathbb{Z}^2$ and $k \in \{1, 2\}$, the path $x + \mathbb{Z}\mathbf{e}_k = (x + j\mathbf{e}_k : j \in \mathbb{Z})$ is a *trivial* bi-infinite geodesic. This is because there is only one up-right path between any two sites on such a path. Bi-infinite geodesics which are not of this form are said to be *non-trivial*. Our main theorem says that non-trivial bi-infinite geodesics do not exist when the weights are geometric random variables.

Theorem 2.2. — *Assume ω_0 is a* Geom(r) *random variable for some $r \in (0, 1)$. Then with $\mathbb{P}$-probability one there are no non-trivial bi-infinite geodesics.*

Our main result, Theorem 2.2, follows from Theorem 5.3, which applies to a general weight distribution. It requires two assumptions, which are then verified (in the appendix) to hold when ω_0 is geometrically distributed. These are the only two places where solvability is used. We include the following comments on our use of solvability:

a) The independence property in Theorem A.2(c) and the explicit knowledge of marginal distributions in Theorem A.2(d) are used in the proof of the tail bound in Theorem B.1, which verifies our Assumption 4.1. These methods seem unlikely to generalize, as they rely on a certain structure of Radon-Nikodym derivatives of the marginal distributions which is satisfied for solvable poylmer and percolation models, but not general distributions. See [21–23]. Using these methods, the bound we prove is sharp, with cubic exponential decay. This is stronger than is necessary for the rest of our arguments: Assumption 4.1 only asks for a polynomial bound with exponent strictly greater than two.

b) The independence property in Theorem A.2(b) is used when verifying Assumption 5.2. This is an assumption concerning certain random walks which, in a general setting, would be built out of using the Busemann process constructed in [27, 34]. In the setting with geometric weights, this independence allows us to turn the probability of an intersection in (5.5) to a product of probabilities. Moreover, it is used to deduce that the random walks in (5.2) have independent increments. Our key random walk estimate, Lemma C.1, assumes that the random walk increments are independent for this reason. For a general weight distribution, we expect that the increments of the associated random walks in question are mixing, but not independent. A version of Lemma C.1 can be expected to hold for such random variables, subject to some extra moment hypotheses.

Organization of the paper: Section 3 introduces boundary models. In Section 4 we derive geodesic fluctuation bounds under Assumption 4.1. Section 5 has the proof of the nonexistence of bi-infinte geodesics, under Assumptions 4.1 and 5.2. Appendix A.1 recalls results that provide the boundary weights for the boundary models needed for the proofs. Sections 3-5 and Appendix A.1 are for general weights and can be read independently. The rest of the appendixes deal with the case of geometric weights and can each be read independently. Theorem B.1 in Section B verifies that Assumption 4.1 holds in the case of geometric weights. Lemma C.2 uses the extra independence structure in Theorem A.2 and the random walk estimates in Lemma C.1 to verify that Assumption 5.2 holds in the case of geometric weights.

2.1. Notation

$\mathbb{N}$ denotes the natural numbers $\{1, 2, \dots\}$, $\mathbb{Z}$ is the set of integers $\{0, \pm 1, \pm 2, \dots, \}$, and $\mathbb{R}$ is the set of real numbers. For $a \in \mathbb{R}$, $\mathbb{R}_{\geqslant a} = [a, \infty)$, $\mathbb{R}_{>a} = (a, \infty)$, $\mathbb{Z}_{\geqslant a} = [a, \infty) \cap \mathbb{Z}$, and $\mathbb{Z}_{>a} = (a, \infty) \cap \mathbb{Z}$. $\mathbb{R}_{\leqslant a}$, $\mathbb{R}_{<a}$, $\mathbb{Z}_{\leqslant a}$, and $\mathbb{Z}_{<a}$ are defined analogously. For $a, b \in \mathbb{R}$ with $a \leqslant b$ we write $[\![a, b]\!]$ to denote the integers that are in $[a, b]$ and we abbreviate $[\![n]\!] = [\![1, n]\!]$. For points $u, v \in \mathbb{R}^2$, $u \leqslant v$ and $v \geqslant u$ mean $u_1 \leqslant v_1$ and $u_2 \leqslant v_2$. For such u and v, let $[u, v] = \{x \in \mathbb{R}^2 : u \leqslant x \leqslant v\}$ and $[\![u, v]\!] = \{x \in \mathbb{Z}^2 : u \leqslant x \leqslant v\}$.

We denote the canonical basis vectors of $\mathbb{R}^2$ by $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. Set $\mathbf{0} = (0, 0)$. An up-right path $\pi_{m:n} = (\pi_m, \pi_{m+1}, \dots, \pi_n)$ is a collection of vertices $\pi_i \in \mathbb{Z}^2$ which satisfies $\pi_i - \pi_{i-1} \in \{\mathbf{e}_1, \mathbf{e}_2\}$ for $i \in [\![m + 1, n]\!]$. For $x \leqslant y$, the set of up-right paths which start at x and end at y is denoted by Π_x^y.

Let $\mathcal{U} = [\mathbf{e}_2, \mathbf{e}_1] = \{t\mathbf{e}_1 + (1-t)\mathbf{e}_2 : 0 \leqslant t \leqslant 1\}$. Its relative interior is denoted by $\operatorname{ri}\mathcal{U} = (\mathbf{e}_2, \mathbf{e}_1) = \{t\mathbf{e}_1 + (1-t)\mathbf{e}_2 : 0 < t < 1\}$. We will use the notation $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$.

For $r \in (0,1)$, a $\operatorname{Geom}(r)$ random variable X satisfies $\mathbb{P}(X = n) = r^n(1-r)$ for $n \in \mathbb{Z}_{\geqslant 0}$. For $p \in [0,1]$, a $\operatorname{Ber}(p)$ random variable X satisfies $\mathbb{P}(X = 1) = 1 - \mathbb{P}(X = 0) = p$.

3. Models with boundary

The main player in the proof of Theorem 2.2 is a coupling of the bulk passage times and a collection of passage times in models with boundary conditions. Given weights $\omega \in \Omega$ and numbers $\{I_x, J_x : x \in \mathbb{Z}^2\}$, referred to as *boundary weights*, the *boundary passage time* $G^{\mathrm{SW}}_{x,y}(\omega, I, J)$ from x to y, with $x \leqslant y$, is the maximum weight of up-right paths from x to y, where each path collects 0 weight at the site x, I weights at each vertex on the horizontal boundary $x + \mathbb{Z}_{\geqslant 0}\mathbf{e}_1$, J weights at each vertex on the vertical boundary $x + \mathbb{Z}_{\geqslant 0}\mathbf{e}_2$, and bulk weights ω at each vertex in the bulk $x + \mathbb{N}^2$. See Figure 3.1. Rigorously, for $x = (x_1, x_2) \in \mathbb{Z}^2$ and $k \in \mathbb{N}$ we set $G^{\mathrm{SW}}_{x,x} = 0$,

$$G^{\mathrm{SW}}_{x,x+k\mathbf{e}_1} = \sum_{i=1}^{k} I_{x+i\mathbf{e}_1}, \quad \text{and} \quad G^{\mathrm{SW}}_{x,x+k\mathbf{e}_2} = \sum_{i=1}^{k} J_{x+i\mathbf{e}_2}. \tag{3.1}$$

Then for $y \in x + \mathbb{N}^2$ we let

$$G^{\mathrm{SW}}_{x,y} = \max_{1 \leqslant k \leqslant y_1 - x_1} \Big\{ \sum_{i=1}^{k} I_{x+i\mathbf{e}_1} + G_{x+k\mathbf{e}_1+\mathbf{e}_2,y} \Big\} \bigvee \max_{1 \leqslant \ell \leqslant y_2 - x_2} \Big\{ \sum_{j=1}^{\ell} J_{x+j\mathbf{e}_2} + G_{x+\mathbf{e}_1+\ell\mathbf{e}_2,y} \Big\}. \tag{3.2}$$

Note that $G^{\mathrm{SW}}_{x,y}$ is a function of $\{I_{x+i\mathbf{e}_1}, I_{x+j\mathbf{e}_2}, \omega_z : i,j \in \mathbb{N}, z \in x + \mathbb{N}^2\}$. Hence the superscript SW which stands for southwest as this is where the ω-weights are switched to I and J.

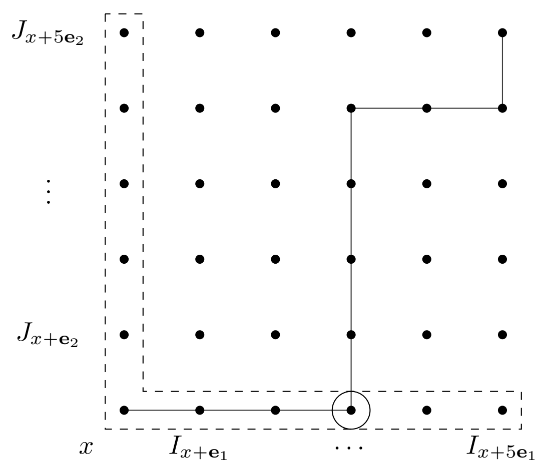

Figure 3.1. An illustration of paths in the model with boundary conditions. The boundary is contained between the dashed lines, the geodesic is solid, and the exit point of the geodesic from the boundary is circled.

As in the bulk model, a geodesic in the model with boundary conditions is an up-right path that achieves the maximum in (3.1-3.2). Recall that geodesics are not necessarily unique if the weights do not have a continuous distribution.

Each geodesic path must exit the boundary at some point. See the circled vertex in Figure 3.1 for an illustration of such an exit point. We denote by $\operatorname{Exit}^{\mathrm{SW}}_{x,y}(\omega, I, J)$ the set of locations of exit points of the geodesics from x to y, relative to the starting point x. That is, with the convention that we index exit points from the vertical boundary with negative numbers,

$$\mathrm{Exit}^{\mathrm{SW}}_{x,y} = \Big\{k \in [\![1, y_1 - x_1]\!] : \sum_{i=1}^{k} I_{x+i\mathbf{e}_1} + G_{x+k\mathbf{e}_1+\mathbf{e}_2,y} = G^{\mathrm{SW}}_{x,y}\Big\}$$
$$\bigcup \Big\{-\ell : \ell \in [\![1, y_2 - x_2]\!] \text{ and } \sum_{j=1}^{\ell} J_{x+j\mathbf{e}_2} + G_{x+\ell\mathbf{e}_2+\mathbf{e}_1,y} = G^{\mathrm{SW}}_{x,y}\Big\}.$$

The furthest exit point in the $\mathbf{e}_1$ direction is then given by

$$Z^{\mathrm{SW},\mathbf{e}_1}_{x,y} = \max \mathrm{Exit}^{\mathrm{SW}}_{x,y}$$

and the furthest exit point in the $\mathbf{e}_2$ direction is given by

$$Z^{\mathrm{SW},\mathbf{e}_2}_{x,y} = \min \mathrm{Exit}^{\mathrm{SW}}_{x,y}\,.$$

Note that if $I_{x+k\mathbf{e}_1} \leqslant \bar{I}_{x+k\mathbf{e}_2}$ for integers $1 \leqslant k \leqslant (y-x)\cdot\mathbf{e}_1$, then $Z^{\mathrm{SW},\mathbf{e}_1}_{x,y}(\omega, I, J) \leqslant Z^{\mathrm{SW},\mathbf{e}_1}_{x,y}(\omega, \bar{I}, J)$, with a similar statement if the J-weights are increased.

The boundary weights that we will use in this paper are the random variables $\{I^\xi_x, J^\xi_y : x, y \in \mathbb{Z}^2\}$, supplied by Theorem A.1 for each fixed $\xi \in \mathrm{ri}\,\mathcal{U}$. We then use the notation

$$G^\xi_{x,y}(\omega) = G^{\mathrm{SW}}_{x,y}(\omega, I^\xi(\omega), J^\xi(\omega)). \tag{3.3}$$

$\mathrm{Exit}^\xi_{x,y}$ and $Z^{\xi,\mathbf{e}_k}_{x,y}$ are defined similarly.

When the starting point is the origin $\mathbf{0}$ we will omit it from the index and abbreviate quantities of the form $A_{\mathbf{0},x}$ by writing A_x or $A(x)$. We will also sometimes write $A(m,n) = A_{m\mathbf{e}_1+n\mathbf{e}_2}$.

The significance of the particular choice of boundary weights is that while the bulk passage times are subadditive:

$$G_{x,y} - \omega_y + G_{y,z} - \omega_z \leqslant G_{x,z} - \omega_z, \quad \forall x \leqslant y \leqslant z, \tag{3.4}$$

the boundary passage times are additive:

$$G^\xi_{x,y} + G^\xi_{y,z} = G^\xi_{x,z}. \tag{3.5}$$

In particular, for any $x = (m,n) \in \mathbb{Z}^2_{\geqslant 0}$

$$\mathbb{E}[G^\xi_{\mathbf{0},x}] = \mathbb{E}[G^\xi_{\mathbf{0},\mathbf{e}_1}]m + \mathbb{E}[G^\xi_{\mathbf{0},\mathbf{e}_2}]n = \mathbb{E}[I^\xi_{\mathbf{e}_1}]m + \mathbb{E}[J^\xi_{\mathbf{e}_2}]n.$$

The above leads to a variational characterization of the limiting shape of the bulk model. Indeed, [40, Theorem 2.3] and a standard coarse-graining argument (see, for example, [33]) imply that if $\mathbb{E}[|\omega_0|^{2+\varepsilon}] < \infty$ for some $\varepsilon > 0$, then for $\mathbb{P}$-almost every ω,

$$\varlimsup_{n\to\infty} n^{-1} \max_{\substack{x\in\mathbb{Z}^2_{\geqslant 0}\\ |x|_1=n}} |G_{\mathbf{0},x} - \gamma(x)| = 0, \tag{3.6}$$

where

$$\gamma(x) = \gamma(x_1,x_2) = \inf_{\xi\in\mathrm{ri}\,\mathcal{U}} \big\{\mathbb{E}[I^\xi_{\mathbf{e}_1}]x_1 + \mathbb{E}[J^\xi_{\mathbf{e}_2}]x_2\big\} \quad \text{for } x \in \mathbb{R}^2_{\geqslant 0}. \tag{3.7}$$

This expression for γ is an immediate consequence of the construction in [34, Theorem 4.7], which defines our I^ξ and J^ξ (see (4.3) and Lemma 4.12 there), and the variational characterization of a homogeneous concave function in terms of its superdifferential.

Another property of the (I^ξ, J^ξ) process is that it is stationary and, as a consequence,

$$\{G^\xi_{z+x,z+y} : y \geqslant x \text{ in } \mathbb{Z}^2\} \quad \text{has the same distribution as} \quad \{G^\xi_{x,y} : y \geqslant x \text{ in } \mathbb{Z}^2\} \quad \forall z \in \mathbb{Z}^2. \tag{3.8}$$

This explains why the last-passage percolation model with these boundary weights is called the *stationary model* or the *stationary LPP*.

The significance of specializing to the case where the weights are geometric random variables, i.e. $\omega_0 \sim \mathrm{Geom}(r)$ for some $r \in (0,1)$, is that because of the memoryless property of the geometric distribution, many explicit computations are possible. For this reason, this case is said to be *solvable*. For example, for any $x \leqslant y$ in $\mathbb{Z}^2$ and any $\xi \in \operatorname{ri}\mathcal{U}$,

$$\big\{\omega_z, G^{\xi}_{x,y+(n+1)\mathbf{e}_i} - G^{\xi}_{x,y+n\mathbf{e}_i} : n \in \mathbb{Z}_{\geqslant 0}, i \in \{1,2\}, z - x \in \mathbb{N}^2\big\} \quad \text{are independent} \tag{3.9}$$

and, marginally, $G^{\xi}_{x,y+\mathbf{e}_1} - G^{\xi}_{x,y} \sim \mathrm{Geom}(p)$ and $G^{\xi}_{x,y+\mathbf{e}_2} - G^{\xi}_{x,y} \sim \mathrm{Geom}(r/p)$, with $p = \overline{p}(\xi)$ given by

$$\overline{p}(\xi) = \overline{p}(\xi_1, \xi_2) = \frac{r(\xi_1+\xi_2) + (r+1)\sqrt{r\xi_1\xi_2}}{\xi_1 + r\xi_2 + 2\sqrt{r\xi_1\xi_2}} \in (r,1) \quad \text{for } \xi \in \mathbb{R}^2_{>0}. \tag{3.10}$$

These are some of the properties contained in Theorem A.2.

For a fixed $r \in (0,1)$, (3.10) gives a bijection from $\operatorname{ri}\mathcal{U}$ to $(r,1)$ with the inverse function given by

$$\overline{\xi}(p) = \left(\frac{r(1-p)^2}{p^2(r+1) - 4pr + r(r+1)}, \frac{(p-r)^2}{p^2(r+1) - 4pr + r(r+1)}\right) \in \operatorname{ri}\mathcal{U}. \tag{3.11}$$

Switching from ξ to p in the variational formula (3.7) and then using the explicit distributions allows to solve (3.7) explictly and get

$$\gamma(x) = \gamma(x_1,x_2) = \inf_{p\in(r,1)} M^p(x) = \frac{r}{1-r}(x_1+x_2) + \frac{2\sqrt{r}}{1-r}\sqrt{x_1x_2}\,, \tag{3.12}$$

where

$$M^p(x) = M^p(x_1,x_2) = \frac{px_1}{1-p} + \frac{\frac{r}{p}x_2}{1-\frac{r}{p}} = \frac{px_1}{1-p} + \frac{rx_2}{p-r}\,. \tag{3.13}$$

4. Geodesic Fluctuation Bounds

Theorem A.1 produces random variables $\{I^{\xi}_x, J^{\xi}_y : x,y \in \mathbb{Z}^2, \xi \in \operatorname{ri}\mathcal{U}\}$ and the passage times that use these variables as boundary weights, which we denote by $G^{\xi}_{x,y}(\omega) = G^{\mathrm{SW}}_{x,y}(\omega, I^{\xi}(\omega), J^{\xi}(\omega))$.

In this section, we give bounds on the size of the fluctuations of the point-to-point geodesics under the following assumption on the tails of $Z^{\xi,\mathbf{e}_k}_{0,x}$, when $|x|_1$ is large and $x/|x|_1$ is close to ξ. For $\delta \in (0,1)$, define the cone

$$S_\delta = \{x \in \mathbb{R}^2_{>0} : x\cdot\mathbf{e}_1 \geqslant \delta x\cdot\mathbf{e}_2 \text{ and } x\cdot\mathbf{e}_2 \geqslant \delta x\cdot\mathbf{e}_1\}.$$

Assumption 4.1. — *There exist a $\nu > 2$ and a $\delta_0 \in (0,1)$ such that for any $\delta \in (0,\delta_0)$ and $\kappa \geqslant 0$, there exist positive finite constants $C_0(\delta)$, $N_0(\delta,\kappa)$, and $s_0(\delta,\kappa)$ such that*

$$\mathbb{P}\big\{|Z^{\xi,\mathbf{e}_1}(m,n)| \vee |Z^{\xi,\mathbf{e}_2}(m,n)| \geqslant s(m+n)^{2/3}\big\} \leqslant C_0 s^{-\nu}, \tag{4.1}$$

for all $(m,n) \in S_\delta \cap \mathbb{Z}^2_{\geqslant N_0}$, $s \geqslant s_0$, and $\xi \in \operatorname{ri}\mathcal{U}$ such that $\xi_1 \in (\delta, 1-\delta)$ and $|\xi_1 - \frac{m}{m+n}| \leqslant \kappa(m+n)^{-1/3}$.

By Theorem B.1, this assumption is satisfied for any $\nu > 2$ when ω_0 is geometrically distributed. This assumption is verified in the case of exponential weights in [8, Corollary 4.3], with a sharp bound appearing in [22, Corollary 3.2].

We begin with some preliminary observations about the structure of last-passage percolation. Given points $x \leqslant y$ in $\mathbb{Z}^2$ and weights ω, define the boundary weights

$$\begin{aligned} I^{[x]}_y(\omega) &= G_{x,y}(\omega) - G_{x,y-\mathbf{e}_1}(\omega), \quad \text{when } x \leqslant y - \mathbf{e}_1, \text{ and} \\ J^{[x]}_y(\omega) &= G_{x,y}(\omega) - G_{x,y-\mathbf{e}_2}(\omega), \quad \text{when } x \leqslant y - \mathbf{e}_2. \end{aligned} \tag{4.2}$$

Then for $z \in y + \mathbb{Z}^2_{\geqslant 0}$, let

$$G^{[x]}_{y,z}(\omega) = G^{\mathrm{SW}}_{y,z}(\omega, I^{[x]}(\omega), J^{[x]}(\omega)) \quad \text{and} \quad Z^{[x],\mathbf{e}_2}_{y,z}(\omega) = Z^{\mathrm{SW},\mathbf{e}_2}_{y,z}(\omega, I^{[x]}(\omega), J^{[x]}(\omega)).$$

The following is immediate from the definitions. See, for example, [**?** , Lemma A.1].

Lemma 4.2. — *Let $x \leqslant y \leqslant z$ in $\mathbb{Z}^2$. Fix a configuration of weights $\omega \in \Omega$. Then $G_{x,z}(\omega) = G_{x,y}(\omega) + G^{[x]}_{y,z}(\omega)$. Furthermore, if an upright path is a geodesic of $G_{x,z}(\omega)$, then its restriction to $y + \mathbb{Z}^2_{\geqslant 0}$ is part of a geodesic of $G^{[x]}_{y,z}(\omega)$. Likewise, if an upright path is part of a geodesic of $G^{[x]}_{y,z}(\omega)$, then it can be extended to a geodesic of $G_{x,z}(\omega)$.*

The next lemma is a direct consequence of the one above.

Lemma 4.3. — *Let ℓ, m be positive integers. Let $x \geqslant y$ be in $\mathbb{Z}^2$. Take $i \in \{1,2\}$. Fix a configuration of weights $\omega \in \Omega$. Then $Z^{\mathrm{SW},\mathbf{e}_i}_{x,y}(\omega) = \ell + m$ if and only if $Z^{[x],\mathbf{e}_i}_{x+\ell\mathbf{e}_1,y}(\omega) = m$. Similarly, $Z^{\mathrm{SW},\mathbf{e}_i}_{x,y}(\omega) = -\ell - m$ if and only if $Z^{[x],\mathbf{e}_i}_{x+\ell\mathbf{e}_2,y}(\omega) = -m$.*

The above definitions and lemmas are deterministic statements and work for every fixed choice of the environment ω. Therefore, by considering passage times with boundary weights I^ξ and J^ξ and recalling (3.5), we see that both lemmas hold if we replace $G_{x,z}$, $G_{x,y}$, $G^{[x]}_{y,z}$, $Z^{\mathrm{SW},\mathbf{e}_i}_{x,y}$, and $Z^{[x],\mathbf{e}_i}_{x+\ell\mathbf{e}_i,y}$, $i \in \{1,2\}$, with, respectively, $G^\xi_{x,z}$, $G^\xi_{x,y}$, $G^\xi_{y,z}$, $Z^{\xi,\mathbf{e}_i}_{x,y}$, and $Z^{\xi,\mathbf{e}_i}_{x+\ell\mathbf{e}_i,y}$, $i \in \{1,2\}$.

Corollary 4.4. — *Suppose Assumption 4.1 holds. Then for any $\delta \in (0,\delta_0)$, $A > 0$, and $\kappa \geqslant 0$ there exist positive finite constants $C_1(\delta,\delta_0,\nu,A)$, $N_1(\delta,\delta_0,\kappa) \geqslant 1$, and $s_1(\delta,\delta_0,\kappa)$ such that*

$$\mathbb{P}\{Z^{\xi,\mathbf{e}_2}(m, n - \lfloor s(m+n)^{2/3}\rfloor) < 0\} \leqslant C_1 s^{-\nu} \tag{4.3}$$

and

$$\mathbb{P}\{Z^{\xi,\mathbf{e}_1}(m, n + \lfloor s(m+n)^{2/3}\rfloor) > 0\} \leqslant C_1 s^{-\nu} \tag{4.4}$$

for all $(m,n) \in S_\delta \cap \mathbb{Z}^2_{\geqslant N_1}$, $s \geqslant s_1$, $\xi \in \mathrm{ri}\,\mathcal{U}$ such that $\xi_1 \in (\delta, 1-\delta)$ and $|\xi_1 - \frac{m}{m+n}| \leqslant \kappa(m+n)^{-1/3}$, and with $n - \lfloor s(m+n)^{2/3}\rfloor \geqslant 1$, in the case of (4.3), and $s \leqslant A(m+n)^{1/3}$, in the case of (4.4).

Proof. — Fix δ and κ as in the claim. Recall the constants N_0 and s_0 in Assumption 4.1. Take $(m,n) \in S_\delta \cap \mathbb{Z}^2_{\geqslant N_0}$ and $s \geqslant \max(2s_0, 2^{1/3}N_0^{-2/3})$ such that $n - \lfloor s(m+n)^{2/3}\rfloor \geqslant 1$. Take $\xi \in \mathrm{ri}\,\mathcal{U}$ such that $\xi_1 \in (\delta, 1-\delta)$ and $|\xi_1 - \frac{m}{m+n}| \leqslant \kappa(m+n)^{-1/3}$. Apply shift-invariance, Lemma 4.3, and (4.1) to obtain

$$\begin{aligned}\mathbb{P}\big\{Z^{\xi,\mathbf{e}_2}_{(0,0),(m,n-\lfloor s(m+n)^{2/3}\rfloor)} < 0\big\} &= \mathbb{P}\big\{Z^{\xi,\mathbf{e}_2}_{(0,\lfloor s(m+n)^{2/3}\rfloor),(m,n)} < 0\big\}\\ &= \mathbb{P}\big\{Z^{\xi,\mathbf{e}_2}_{(0,0),(m,n)} < -\lfloor s(m+n)^{2/3}\rfloor\big\} \leqslant \mathbb{P}\big\{Z^{\xi,\mathbf{e}_2}_{(0,0),(m,n)} < -s(m+n)^{2/3}/2\big\}\\ &\leqslant 2^\nu C_0 s^{-\nu}.\end{aligned}$$

For (4.4), let $\bar N_0 = N_0(\delta/2, \kappa+1)$ and

$$\bar s_0 = \max\big(s_0(\delta/2,\kappa+1), 2(1+\delta)\delta^{-1}\bar N_0^{-2/3}, (1+\delta)^{-1}\delta \bar N_0^{1/3}\big).$$

Take $(m,n) \in S_\delta \cap \mathbb{Z}^2_{\geqslant \bar N_0}$ and $s \geqslant \bar s_0$. Let $d = \lfloor s(m+n)^{2/3}\rfloor$, $\widetilde n = n + d$ and $\widetilde m = m + \lfloor \frac{dm}{n}\rfloor$. Then

$$\frac{\delta}{2} \leqslant \frac{\delta n + \delta s(m+n)^{2/3} - 1 - \delta}{n + s(m+n)^{2/3}} \leqslant \frac{\widetilde m}{\widetilde n} \leqslant \frac{1}{\delta} \leqslant \frac{2}{\delta}\,.$$

Take $\xi \in \mathrm{ri}\,\mathcal{U}$ such that $\xi_1 \in (\delta, 1-\delta)$ and $|\xi_1 - \frac{m}{m+n}| \leqslant \kappa(m+n)^{-1/3}$. Then

$$\Big|\xi_1 - \frac{\widetilde m}{\widetilde m + \widetilde n}\Big| \leqslant \Big|\xi_1 - \frac{m}{m+n}\Big| + \frac{md - n\lfloor dm/n\rfloor}{(m+n)^2} \leqslant \kappa(m+n)^{-1/3} + (m+n)^{-1} \leqslant (\kappa+1)(m+n)^{-1/3}.$$

Furthermore, if we take $s \leqslant A(m+n)^{1/3}$, then we get

$$\frac{\widetilde m + \widetilde n}{m+n} \leqslant 1 + \frac{(\delta^{-1}+1)d}{m+n} \leqslant 1 + (\delta^{-1}+1)s(m+n)^{-1/3} \leqslant 1 + (\delta^{-1}+1)A.$$

Now, similar to the above computation, we have

$$\begin{aligned}\mathbb{P}\big\{Z^{\xi,\mathbf{e}_1}(m,n+\lfloor s(m+n)^{2/3}\rfloor)>0\big\} &= \mathbb{P}\Big\{Z^{\xi,\mathbf{e}_1}(\widetilde m,\widetilde n)>\Big\lfloor\frac{dm}{n}\Big\rfloor\Big\}\\ &\leqslant \mathbb{P}\big\{Z^{\xi,\mathbf{e}_1}(\widetilde m,\widetilde n)>\tfrac12\delta s(m+n)^{2/3}\big\}\\ &\leqslant \mathbb{P}\big\{Z^{\xi,\mathbf{e}_1}(\widetilde m,\widetilde n)>\tfrac12\delta\big(1+(\delta^{-1}+1)A\big)^{-2/3}s(\widetilde m+\widetilde n)^{2/3}\big\}\\ &\leqslant (2/\delta)^{\nu}\big(1+(\delta^{-1}+1)A\big)^{2\nu/3}C_0(\delta/2)s^{-\nu}.\end{aligned}$$ □

For $(m,n)\in\mathbb{N}^2$, $\alpha\in(0,1)$, and $s>0$, define

$$\mathcal{C}^{(m,n)}_{\alpha,s}=[\![(\lfloor\alpha m\rfloor,\lfloor\alpha n\rfloor)-s(m+n)^{2/3}\mathbf{e}_2,(\lfloor\alpha m\rfloor,\lfloor\alpha n\rfloor)+s(m+n)^{2/3}\mathbf{e}_2]\!].$$

$\mathcal{C}^{(m,n)}_{\alpha,s}$ is the symmetric vertical line segment centered at $(\lfloor\alpha m\rfloor,\lfloor\alpha n\rfloor)$ with length $2s(m+n)^{2/3}$. For $x\in\mathbb{R}^2_+\backslash\{\mathbf{0}\}$, let

$$\zeta(x)=\frac{x}{|x|_1}\in\mathcal{U}.$$

Let $\pi^{(m,n),\mathbf{e}_1}$ and $\pi^{(m,n),\mathbf{e}_2}$ denote, respectively, the rightmost and the upmost geodesics of $G(m,n)$.

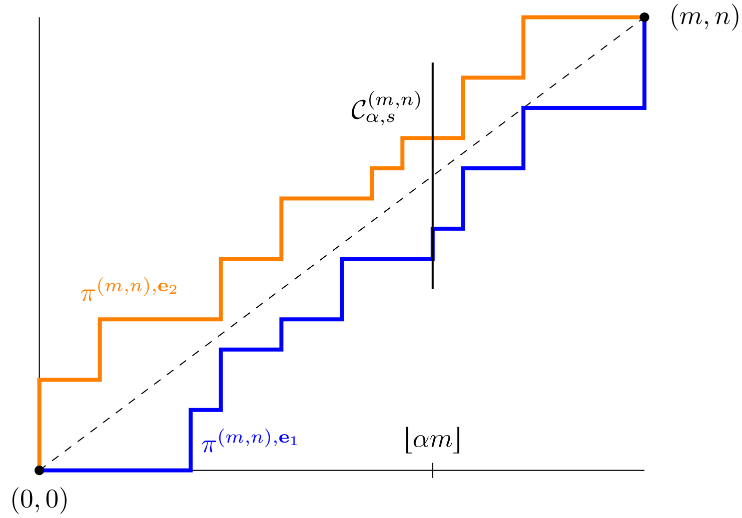

Figure 4.1. An illustration of the high probability event in Lemma 4.5. The upmost and rightmost geodesics from $(0,0)$ to (m,n) will intersect the vertical line segment $\mathcal{C}^{(m,n)}_{\alpha,s}$.

Lemma 4.5. — *Suppose Assumption 4.1 holds. For $0<\delta<\delta_0$ and $0<\varepsilon<\frac12$, there exist finite positive constants $C_2(\delta,\delta_0,\nu,\varepsilon)$, $N_2(\delta,\delta_0)\geqslant1$, and $s_2(\delta,\delta_0)$ such that the following holds: for all $(m,n)\in S_\delta\cap\mathbb{Z}^2_{\geqslant N_2}$, $\alpha\in(\varepsilon,1-\varepsilon)$, and $s\in\big[s_2,\frac{\varepsilon\delta}{3(\delta+1)}(m+n)^{1/3}\big]$,*

$$\mathbb{P}\Big\{(\pi^{(m,n),\mathbf{e}_1}\cap\mathcal{C}^{(m,n)}_{\alpha,s}=\varnothing)\cup(\pi^{(m,n),\mathbf{e}_2}\cap\mathcal{C}^{(m,n)}_{\alpha,s}=\varnothing)\Big\}\leqslant C_2s^{-\nu}. \tag{4.5}$$

Proof. — Fix δ,ε as in the claim. Take

$$\begin{aligned}N_2=\max\Big\{&\varepsilon^{-1}N_1(\delta/3,\delta_0,1),(3-\delta)\delta^{-1}\varepsilon^{-1}\big(N_1(\delta/3,\delta_0,1)+1\big),3(3-\delta)\delta^{-2}\varepsilon^{-1},\\ &2(1-\delta/3)\varepsilon^{-1},(2-2\delta/3)^{3/2}(1+3/\delta)^{1/2}\varepsilon^{-1},\frac12\big(2\delta^{-1}(3-\delta)\big)^{3/2},(1-\varepsilon)^{-1},\\ &\frac65\varepsilon^{-1}N_0(\delta/3,1),\varepsilon^{-1},\frac12(3\varepsilon^{-1})^{3/2}\big(1-\frac{\varepsilon\delta}{3(\delta+1)}\big)^{-3}\Big\}.\end{aligned} \tag{4.6}$$

Take

$$s_2=\max\big\{2,8\delta^{-2/3}s_1(\delta/3,\delta_0,1),2s_0(\delta/3,1)\big\}.$$

Take $(m,n) \in S_\delta \cap \mathbb{Z}^2_{\geqslant N_2}$ and

$$s \in \big[s_2, \frac{\varepsilon\delta}{3(\delta+1)}(m+n)^{1/3}\big].$$

This ensures that $n - s(m+n)^{2/3} \geqslant (1-\varepsilon/3)n > 0$. Let $\xi^\star = \zeta(m, n - s(m+n)^{2/3})$ and $\xi_\star = \zeta(m, n + s(m+n)^{2/3})$. Let $\pi^\star$ denote the upmost geodesic of $G^{\xi^\star}(m,n)$ and $\pi_\star$ be the rightmost geodesic of $G^{\xi_\star}(m,n)$.

Theorem A.1 gives a coupling of the weights $\{\omega_x, I^{\xi^\star}_x, J^{\xi^\star}_x, I^{\xi_\star}_x, J^{\xi_\star}_x : x \in \mathbb{Z}^2\}$ that is stationary and such that almost surely, for all $x \in \mathbb{Z}^2$,

$$\omega_x \leqslant I^{\xi^\star}_x \leqslant I^{\xi_\star}_x \quad \text{and} \quad \omega_x \leqslant J^{\xi_\star}_x \leqslant J^{\xi^\star}_x. \tag{4.7}$$

Take $\alpha \in (\varepsilon, 1-\varepsilon)$ and define $\widetilde{o} = (\lfloor \alpha m \rfloor, \lfloor \alpha(n - s(m+n)^{2/3}) \rfloor)$. By Lemma 4.2, the point where $\pi^\star$ crosses the southwest boundary of the rectangle $[\widetilde{o}, (m,n)]$ is the same as the exit point of the upmost geodesic of $G^{\xi^\star}_{\widetilde{o},(m,n)}$ from the same boundary. Furthermore, we clearly have

$$\lfloor \alpha n \rfloor + 2s(m+n)^{2/3} \geqslant \lfloor \alpha(n - s(m+n)^{2/3}) \rfloor + 2s(m+n)^{2/3},$$

and the fact that $m+n \geqslant 1$ and $s \geqslant 1$ implies $(2-\alpha)s(m+n)^{2/3} \geqslant 1+\varepsilon \geqslant 1$, which implies

$$\lfloor \alpha n \rfloor - 2s(m+n)^{2/3} \leqslant \lfloor \alpha(n - s(m+n)^{2/3}) \rfloor.$$

Therefore,

$$\big\{\pi^\star \cap \mathcal{C}^{(m,n)}_{\alpha,2s} = \varnothing\big\} \subset \big\{\pi^\star \cap [\widetilde{o}, \widetilde{o} + 2s(m+n)^{2/3}\mathbf{e}_2] = \varnothing\big\} \subset \big\{Z^{\xi^\star,\mathbf{e}_2}_{\widetilde{o},(m,n)} \notin [-2s(m+n)^{2/3}, -1]\big\}.$$

Consequently,

$$\begin{aligned} \mathbb{P}(\pi^\star \cap \mathcal{C}^{(m,n)}_{\alpha,2s} = \varnothing) &\leqslant \mathbb{P}(Z^{\xi^\star,\mathbf{e}_2}_{\widetilde{o},(m,n)} \notin [\![-2s(m+n)^{2/3}, -1]\!]) \\ &= \mathbb{P}(Z^{\xi^\star,\mathbf{e}_2}_{\widetilde{o},(m,n)} > 0) + \mathbb{P}(Z^{\xi^\star,\mathbf{e}_2}_{\widetilde{o},(m,n)} < -2s(m+n)^{2/3}). \end{aligned} \tag{4.8}$$

To bound the first of these two probabilities, let

$$\widetilde{m} = m - \lfloor \alpha m \rfloor, \quad \widetilde{n} = \lfloor \xi^\star_2 \widetilde{m}/\xi^\star_1 \rfloor, \quad \text{and} \quad \widetilde{s} = \frac{n - \lfloor \alpha(n - s(n+m)^{2/3}) \rfloor - \widetilde{n}}{(\widetilde{m}+\widetilde{n})^{2/3}}.$$

We next check that we can apply Corollary 4.4 with these parameters.

Note that

$$\xi^\star_1 = \frac{m}{m+n-s(m+n)^{2/3}} \geqslant \frac{m}{m+n} \geqslant \frac{\delta}{\delta+1} \geqslant \frac{\delta}{3}$$

and

$$\xi^\star_1 \leqslant \frac{m}{m+n-\frac{\varepsilon\delta}{3(\delta+1)} \cdot (m+n)} = \frac{m}{m+n} \cdot \frac{\delta+1}{\delta+1-\varepsilon\delta/3} \leqslant \frac{1}{1+5\delta/6} \leqslant 1 - \frac{\delta}{3}.$$

Next, we use the choice of N_2 in (4.6) repeatedly. First, we have

$$\widetilde{m} \geqslant (1-\alpha)m \geqslant \varepsilon N_2 \geqslant N_1(\delta/3, \delta_0, 1)$$

and

$$\widetilde{n} \geqslant \xi^\star_2 \widetilde{m}/\xi^\star_1 - 1 \geqslant \frac{\delta/3}{1-\delta/3} \cdot \varepsilon N_2 - 1 \geqslant N_1(\delta/3, \delta_0, 1).$$

We also have

$$\frac{\widetilde{m}}{\widetilde{n}} \geqslant \frac{\widetilde{m}}{\xi^\star_2 \widetilde{m}/\xi^\star_1} \geqslant \frac{\delta/3}{1-\delta/3} \geqslant \frac{\delta}{3}$$

and

$$\frac{\widetilde{m}}{\widetilde{n}} \leqslant \frac{\widetilde{m}}{\xi^\star_2 \widetilde{m}/\xi^\star_1 - 1} \leqslant \frac{1}{\frac{\delta}{3-\delta} - \frac{1}{\varepsilon N_2}} \leqslant \frac{3}{\delta}.$$

In other words, $(\widetilde{m}, \widetilde{n}) \in S_{\delta/3}$. Furthermore,

$$\Big|\xi^\star_1 - \frac{\widetilde{m}}{\widetilde{m}+\widetilde{n}}\Big| = \Big|\widetilde{n} - \frac{\widetilde{m}\xi^\star_2}{\xi^\star_1}\Big| (\widetilde{m}+\widetilde{n})^{-1}\xi^\star_1 \leqslant (\widetilde{m}+\widetilde{n})^{-1}.$$

Lastly, we derive bounds on $\widetilde{s}$. An upper bound is given by

$$\begin{aligned}\widetilde{s} = \frac{n - \lfloor\alpha(n - s(n+m)^{2/3})\rfloor - \widetilde{n}}{(\widetilde{m}+\widetilde{n})^{2/3}} &\leqslant \frac{n}{\big(\widetilde{m} + \xi_2^\star \widetilde{m}/\xi_1^\star - 1\big)^{2/3}} \leqslant \frac{m/\delta}{\big(\widetilde{m}/(1-\delta/3) - 1\big)^{2/3}} \\ &\leqslant \frac{\varepsilon^{-1}(1-\alpha)m/\delta}{\big(\widetilde{m}/(1-\delta/3)-1\big)^{2/3}} \leqslant \big(2(1-\delta/3)\big)^{2/3}\delta^{-1}\varepsilon^{-1}\widetilde{m}^{1/3} \leqslant A(\widetilde{m}+\widetilde{n})^{1/3},\end{aligned}$$

where A is the constant in front of $\widetilde{m}^{1/3}$ in the middle of the second line. We similarly have the lower bound

$$\begin{aligned}\widetilde{s} &\geqslant \frac{n - \alpha n + \alpha s(n+m)^{2/3} - \xi_2^\star(m - \alpha m + 1)/\xi_1^\star}{\big((m-\alpha m+1)/\xi_1^\star\big)^{2/3}} = \frac{(1-\alpha)(n - \xi_2^\star m/\xi_1^\star) + \alpha s(n+m)^{2/3} - \xi_2^\star/\xi_1^\star}{\big((m-\alpha m+1)/\xi_1^\star\big)^{2/3}} \\ &= \frac{s(m+n)^{2/3} - \xi_2^\star/\xi_1^\star}{\big((m-\alpha m+1)/\xi_1^\star\big)^{2/3}} \geqslant \frac{s(m+n)^{2/3} - (3-\delta)/\delta}{\big(3\delta^{-1}((1-\varepsilon)m+1)\big)^{2/3}} \geqslant \frac{s(m+n)^{2/3}}{2\big(6\delta^{-1}(1-\varepsilon)m\big)^{2/3}} \geqslant \delta^{2/3}s/8 \geqslant s_1(\delta/3,\delta_0,1).\end{aligned}$$

In the last inequality we used $s \geqslant s_2$ and the choice of s_2.

Now, apply Corollary 4.4 to get

$$\begin{aligned}\mathbb{P}(Z^{\xi^\star,\mathbf{e}_2}_{\widetilde{o},(m,n)} > 0) = \mathbb{P}(Z^{\xi^\star,\mathbf{e}_2}(\widetilde{m}, \widetilde{n} + \lfloor\widetilde{s}(\widetilde{m}+\widetilde{n})^{2/3}\rfloor) > 0) &\leqslant C_1(\delta/3,\delta_0,\varepsilon,A)\widetilde{s}^{-\nu} \\ &\leqslant C_1(\delta/3,\delta_0,\varepsilon,A)(\delta^{2/3}/8)^{-\nu}s^{-\nu}\,.\end{aligned}$$

To bound the second probability in (4.8) start by using Lemma 4.3 to write

$$\begin{aligned}\mathbb{P}\big(Z^{\xi^\star,\mathbf{e}_2}_{\widetilde{o},(m,n)} < -2s(m+n)^{2/3}\big) &\leqslant \mathbb{P}\big(Z^{\xi^\star,\mathbf{e}_2}_{\widetilde{o},(m,n)} < -2\lfloor s(m+n)^{2/3}\rfloor\big) \\ &= \mathbb{P}\big(Z^{\xi^\star,\mathbf{e}_2}_{\widetilde{o}+\lfloor s(m+n)^{2/3}\rfloor\mathbf{e}_2,(m,n)} < -\lfloor s(m+n)^{2/3}\rfloor\big) \\ &= \mathbb{P}\big(Z^{\xi^\star,\mathbf{e}_2}(m',n') < -\lfloor s(m+n)^{2/3}\rfloor\big),\end{aligned}$$

where

$$m' = m - \lfloor\alpha m\rfloor \quad \text{and} \quad n' = n - \lfloor\alpha(n - s(n+m)^{2/3})\rfloor - \lfloor s(m+n)^{2/3}\rfloor.$$

Now we check that we can use Assumption 4.1. We have

$$m' \geqslant \varepsilon N_2 \geqslant N_0(\delta/3,1)$$

and

$$n' \geqslant (1-\alpha)\big(n - s(m+n)^{2/3}\big) \geqslant (1-\alpha)(1-\varepsilon/3)n \geqslant 5(1-\alpha)n/6 \geqslant 5\varepsilon N_2/6 \geqslant N_0(\delta/3,1).$$

Also,

$$\frac{\delta}{3} \leqslant \frac{\delta}{1 + 1/3 + \varepsilon^{-1}/N_2} \leqslant \frac{(1-\alpha)m}{(1-\alpha)n + \frac{\varepsilon\delta}{3(\delta+1)}(n+m) + 1} \leqslant \frac{m'}{n'} \leqslant \frac{(1-\alpha)m+1}{5(1-\alpha)n/6} \leqslant \frac{6}{5\delta} + \frac{6}{5\varepsilon N_2} \leqslant \frac{3}{\delta}\,.$$

We already checked that $\xi_1^\star \in (\delta/3, 1-\delta/3)$, and we have

$$\begin{aligned}\Big|\xi_1^\star - \frac{m'}{m'+n'}\Big| &\leqslant \frac{\big|s(m+n)^{2/3}(m - \lfloor\alpha m\rfloor) - m\lfloor s(n+m)^{2/3}\rfloor + n\lfloor\alpha m\rfloor - m\lfloor\alpha(n - s(n+m)^{2/3})\rfloor\big|}{(1-\alpha)\big(1 - \frac{\varepsilon\delta}{3(\delta+1)}\big)^2(m+n)^2} \\ &\leqslant \frac{s(m+n)^{2/3} + 2m}{(1-\alpha)\big(1 - \frac{\varepsilon\delta}{3(\delta+1)}\big)^2(m+n)^2} \leqslant \frac{\frac{\varepsilon\delta}{3(\delta+1)} + 2}{(1-\alpha)\big(1 - \frac{\varepsilon\delta}{3(\delta+1)}\big)^2(m+n)} \\ &\leqslant \frac{3}{\varepsilon\big(1 - \frac{\varepsilon\delta}{3(\delta+1)}\big)^2(2N_2)^{2/3}} \cdot (m+n)^{-1/3} \leqslant (m'+n')^{-1/3}.\end{aligned}$$

We can now use (4.1) to write

$$\begin{aligned}\mathbb{P}\big(Z^{\xi^\star,\mathbf{e}_2}(m',n') < -\lfloor s(m+n)^{2/3}\rfloor\big) &\leqslant \mathbb{P}\big(Z^{\xi^\star,\mathbf{e}_2}(m',n') < -\lfloor s\rfloor(m'+n')^{2/3}\big) \\ &\leqslant C_0(\delta/3)\lfloor s\rfloor^{-\nu} \leqslant 2^\nu C_0 s^{-\nu}.\end{aligned}$$

Using the above bounds in (4.8), there exists a finite constant $C(\delta,\delta_0,\nu,\varepsilon) > 0$ such that

$$\mathbb{P}(\pi^\star \cap \mathcal{C}^{(m,n)}_{\alpha,2s} = \varnothing) \leqslant Cs^{-\nu}. \tag{4.9}$$

Next, write

$$\begin{aligned}\mathbb{P}\Big\{Z^{\xi^\star,\mathbf{e}_1}(m,n) > 0\Big\} &= \mathbb{P}\Big\{Z^{\xi^\star,\mathbf{e}_1}\big(m, n - \lfloor s(m+n)^{2/3}\rfloor + \lfloor s(m+n)^{2/3}\rfloor\big) > 0\Big\}\\ &= \mathbb{P}\Big\{Z^{\xi^\star,\mathbf{e}_1}\big(m'', n'' + \lfloor s''(m''+n'')^{2/3}\rfloor\big) > 0\Big\},\end{aligned}$$

where

$$m'' = m, \quad n'' = n - \lfloor s(m+n)^{2/3}\rfloor, \quad \text{and} \quad s'' = \frac{s(m+n)^{2/3}}{(m''+n'')^{2/3}}.$$

Similarly to the above, we can check that the conditions of Corollary 4.4 are satisfied, with $\delta/3$ in place of δ and with $\kappa = 1$, provided we choose N_2 large enough. Therefore, (4.4) gives the upper bound

$$\mathbb{P}\Big\{Z^{\xi^\star,\mathbf{e}_1}(m,n) > 0\Big\} \leqslant Cs^{-\nu}, \tag{4.10}$$

where C is a (possibly different larger) finite positive constant depending only on δ, δ_0, ν, and ε.

An identical reasoning gives the bounds

$$\mathbb{P}(\pi_\star \cap \mathcal{C}^{(m,n)}_{\alpha,2s} = \varnothing) \leqslant Cs^{-\nu} \quad \text{and} \quad \mathbb{P}\Big\{Z^{\xi_\star,\mathbf{e}_2}(m,n) < 0\Big\} \leqslant Cs^{-\nu}. \tag{4.11}$$

Next, we argue that $Z^{\xi^\star,\mathbf{e}_2}(m,n) < 0$ implies that $\pi^{(m,n),\mathbf{e}_2}$ never goes strictly above $\pi^\star$. To argue by contradiction, suppose there existed a positive integer k and $x \in \mathbb{Z}^2_{\geqslant 0}$ such that $\pi^\star_k = \pi^{(m,n),\mathbf{e}_2}_k = x$, $\pi^\star_{k+1} = x+\mathbf{e}_1$, and $\pi^{(m,n),\mathbf{e}_2}_{k+1} = x+\mathbf{e}_2$. Since $Z^{\xi^\star,\mathbf{e}_2}(m,n) < 0$, the upmost geodesic $\pi^\star$ goes from $\mathbf{0}$ to $\mathbf{e}_2$ and therefore $k \geqslant 1$ and $x+\mathbf{e}_1$ lies in the bulk $\mathbb{N}^2$. Consequently, $\pi^\star_{k+1:m+n}$ is a geodesic for $G_{x+\mathbf{e}_1,(m,n)}$. Since $\pi^{(m,n),\mathbf{e}_2}_{k:m+n}$ is the upmost geodesic of $G_{x,(m,n)}$, it must be that the passage time of $\pi^{(m,n),\mathbf{e}_2}_{k+1:m+n}$ is at least as large as the passage time of $\pi^\star_{k+1:m+n}$ and the former path never goes strictly below the latter one. Now, the bounds in (4.7) say that the edge weights on the boundary $\mathbb{N}\mathbf{e}_2$ are at least as large as the bulk weights there. Therefore, the passage time of $\pi^\star_{k+1:m+n}$ (which only uses bulk weights) is no larger than the passage time of $\pi^{(m,n),\mathbf{e}_2}_{k+1:m+n}$, even when the latter uses boundary weights on $\mathbb{N}\mathbf{e}_2$ (which is possible if x is on that boundary). But this means that replacing $\pi^\star_{k+1:m+n}$ by $\pi^{(m,n),\mathbf{e}_2}_{k+1:m+n}$ in $\pi^\star$ gives a geodesic for $G^{\xi^\star}(m,n)$ that at some point goes strictly above $\pi^\star$. This contradicts the definition of $\pi^\star$ as the upmost geodesic. Consequently, $\pi^{(m,n),\mathbf{e}_2}$ can never go strictly above $\pi^\star$.

Similarly, if $Z^{\xi_\star,\mathbf{e}_1}(m,n) > 0$, then $\pi^{(m,n),\mathbf{e}_1}$ never goes strictly right of $\pi_\star$. Consequently, if we have both $Z^{\xi^\star,\mathbf{e}_2}(m,n) < 0$ and $Z^{\xi_\star,\mathbf{e}_1}(m,n) > 0$, then all the geodesics of $G(m,n)$ are sandwiched between $\pi^\star$ and $\pi_\star$. If, furthermore, $\pi^\star$ and $\pi_\star$ both intersect $\mathcal{C}^{(m,n)}_{\alpha,2s}$, then both $\pi^{(m,n),\mathbf{e}_k}$, $k \in \{1,2\}$, are forced to intersect it as well. We have thus shown that

$$\begin{aligned}&\Big\{(\pi^{(m,n),\mathbf{e}_1} \cap \mathcal{C}^{(m,n)}_{\alpha,s} = \varnothing) \cup (\pi^{(m,n),\mathbf{e}_2} \cap \mathcal{C}^{(m,n)}_{\alpha,s} = \varnothing)\Big\}\\ &\quad\subset \Big\{Z^{\xi^\star,\mathbf{e}_1}(m,n) > 0\Big\} \cup \Big\{Z^{\xi_\star,\mathbf{e}_2}(m,n) < 0\Big\} \cup \Big\{\pi^\star \cap \mathcal{C}^{(m,n)}_{\alpha,2s} = \varnothing\Big\} \cup \Big\{\pi_\star \cap \mathcal{C}^{(m,n)}_{\alpha,2s} = \varnothing\Big\}.\end{aligned}$$

This, together with (4.9-4.11) complete the proof of the lemma. ☐

5. Non-existence of bi-infinite geodesics

We begin by proving non-existence of non-trivial axis-directed bi-infinite geodesics, which is essentially an immediate consequence of the uniqueness of axis-directed semi-infinite geodesics.

LEMMA 5.1. — *With probability one, for each $x \in \mathbb{Z}^2_{\geqslant 0}$ and $\ell \in \{1,2\}$, the only semi-infinite geodesic starting at x satisfying $\underline{\lim}_{k\to\infty} k^{-1}x_k \cdot \mathbf{e}_{3-\ell} = 0$ is the trivial geodesic $\{x + k\mathbf{e}_\ell\}_{k=0}^\infty$.*

Proof. — The proof of this result is essentially the same as that of [36, Lemma A.6], where there is an additional assumption that the weight distribution is continuous. We include the proof for completeness. It suffices to prove the result for semi-infinite geodesics starting at the origin which are $\mathbf{e}_1$-directed. Fix a strictly decreasing sequence of directions $\xi_i \in \operatorname{ri}\mathcal{U}$ such that $\xi_i \searrow \mathbf{e}_1$ and γ is differentiable at ξ_i for each i. By Lemma 4.1(b) in [26], each B^{ξ_i} given by Theorem A.1 produces an upmost semi-infinite geodesic $x^i_{0:\infty}$ that starts at the origin and follows the minimal increments of B^{ξ_i}, taking an $\mathbf{e}_2$ increment in case of a tie. By [26, Theorem 4.3], for each i, the limit points of x^i_n/n, as $n \to \infty$, lie on the same (possibly degenerate) linear segment of γ that contains ξ_i.

Due to the uniqueness of the upmost geodesic between any pair of points $x \leqslant y$, any $\mathbf{e}_1$-directed semi-infinite geodesics starting from the origin must stay weakly to the right of all of the geodesics $x^i_{0:\infty}$.

The result now follows if we show that for any $m \in \mathbb{Z}_{\geqslant 0}$ and any i large enough, $x^i_{0:m} = [\![\mathbf{0}, m\mathbf{e}_1]\!]$. We prove this by induction. This claim is trivial for $m = 0$. Suppose the claim is true for some $m \in \mathbb{Z}_{\geqslant 0}$. Lemma 5.1 in [26] says that $B^{\xi_i}(m\mathbf{e}_1, m\mathbf{e}_1 + \mathbf{e}_2) \to \infty$ as $i \to \infty$. This implies that for i large enough $B^{\xi_i}(m\mathbf{e}_1, m\mathbf{e}_1 + \mathbf{e}_2) > \omega_{m\mathbf{e}_1} = B^{\xi_i}(m\mathbf{e}_1, (m+1)\mathbf{e}_1)$, which implies that $x^i_{m+1} = (m+1)\mathbf{e}_1$. □

Next, we turn to interior-directed bi-infinite geodesics. Recall that the passage times $G^{\xi}_{x,y}$ that use the boundary weights $\{I^{\xi}_{x+k\mathbf{e}_1}, J^{\xi}_{x+k\mathbf{e}_2} : k \in \mathbb{N}\}$ on the southwest boundary of $x + \mathbb{Z}^2_+$ give a stationary LPP process satisfying (3.5). We will now need to consider the stationary LPP process that corresponds to putting appropriate weights on the northeast boundary. To this end, define the reflected weights $\widehat{\omega} = (\widehat{\omega}_x)_{x\in\mathbb{Z}^2}$ with $\widehat{\omega}_x = \omega_{-x}$. Define the boundary weights

$$\widehat{I}^{\xi}_x(\omega) = I^{\xi}_{-x}(\widehat{\omega}) \quad \text{and} \quad \widehat{J}^{\xi}_x(\omega) = J^{\xi}_{-x}(\widehat{\omega}). \tag{5.1}$$

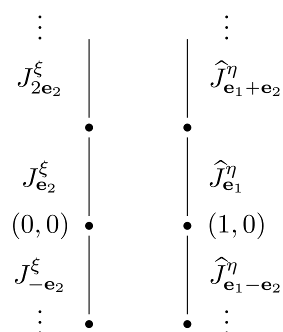

Figure 5.1. The edges involved in $S^{\xi,\eta}_n$.

Given $\xi, \eta \in \operatorname{ri}\mathcal{U}$ let

$$S^{\xi,\eta}_n = \begin{cases} \sum_{j=1}^{n} (J^{\xi}_{j\mathbf{e}_2} - \widehat{J}^{\eta}_{\mathbf{e}_1+(j-1)\mathbf{e}_2}), & n \in \mathbb{Z}_{\geqslant 1}, \\ 0, & n = 0, \\ -\sum_{j=n+1}^{0} (J^{\xi}_{j\mathbf{e}_2} - \widehat{J}^{\eta}_{\mathbf{e}_1+(j-1)\mathbf{e}_2}), & n \in \mathbb{Z}_{\leqslant -1}. \end{cases} \tag{5.2}$$

Given $\xi^\star, \xi_\star, \eta^\star, \eta_\star \in \operatorname{ri}\mathcal{U}$, we can use Theorem A.1 to couple the weights

$$\{\omega_x, I^{\xi^\star}_x, J^{\xi^\star}_x, I^{\xi_\star}_x, J^{\xi_\star}_x, I^{\eta^\star}_x, J^{\eta^\star}_x, I^{\eta_\star}_x, J^{\eta_\star}_x : x \in \mathbb{Z}^2\}. \tag{5.3}$$

This produces a coupling of $\{\omega_x, J^{\xi^\star}_x, J^{\xi_\star}_x, \widehat{J}^{\eta^\star}_x, \widehat{J}^{\eta_\star}_x : x \in \mathbb{Z}^2\}$ and of $\{\omega_x, I^{\xi^\star}_x, I^{\xi_\star}_x, \widehat{I}^{\eta^\star}_x, \widehat{I}^{\eta_\star}_x : x \in \mathbb{Z}^2\}$.

Assumption 5.2. — *There exist an $a_0 \in (1/3, 2/3)$ and a $\delta_0 \in (0,1)$ such that for any $\delta \in (0, \delta_0)$, there exist positive finite constants $C_3(\delta)$ and $N_3(\delta)$ such that for all $N \geqslant N_3$, and all $\eta_\star, \eta^\star, \xi_\star, \xi^\star \in \operatorname{ri}\mathcal{U}$ with $\mathbf{e}_1$-coordinate in $(\delta, 1-\delta)$ and such that*

$$-N^{-a_0/2} \leqslant \xi_\star \cdot \mathbf{e}_1 - \eta^\star \cdot \mathbf{e}_1 < 0 \quad \textit{and} \quad -N^{-a_0/2} \leqslant \eta_\star \cdot \mathbf{e}_1 - \xi^\star \cdot \mathbf{e}_1 < 0, \tag{5.4}$$

we have

$$\mathbb{P}\Big\{\sup_{0<k\leqslant N^{2/3}} S_k^{\xi_\star,\eta^\star}\leqslant 0 \quad \text{and} \sup_{-N^{2/3}+1\leqslant k<0} S_k^{\xi^\star,\eta_\star}\leqslant 0\Big\}\leqslant C_3N^{-a_0}. \tag{5.5}$$

By Lemma C.2, this assumption is satisfied for any $a_0\in(1/3,2/3)$ when ω_0 is geometrically distributed. This assumption is verified in the exponential model in [8, Lemma C.1] with $a_0=2/5$.

Theorem 2.2 now follows from Theorem B.1, Lemma C.2, and the following, more general result.

THEOREM 5.3. — *Suppose Assumptions 4.1 and 5.2 hold with*

$$a_0<\frac{2(\nu-1)}{3\nu}\,. \tag{5.6}$$

Then with $\mathbb{P}$-probability one there are no non-trivial bi-infinite geodesics.

Remark 5.4. — By exchanging the roles of the two axes, one sees that the above theorem also holds if Assumption 5.2 holds with $I^\bullet_{j\mathbf{e}_1}$-increments instead of $J^\bullet_{j\mathbf{e}_2}$. We expect that if the assumption holds with one set of increments, then it holds with the other set as well.

Given $\delta\in(0,1)$ and a positive integer N, define the southwest boundary,

$$\partial^{N,\delta}=(\{-N\}\times[\![-N,-\delta N]\!])\cup([\![-N,-\delta N]\!]\times\{-N\}), \tag{5.7}$$

and the northeast boundary,

$$\widehat{\partial}^{N,\delta}=(\{N\}\times[\![\delta N,N]\!])\cup([\![\delta N,N]\!]\times\{N\}). \tag{5.8}$$

By Lemma 5.1 a nontrivial bi-infinite geodesic must eventually take an $\mathbf{e}_1$ step. Then by the shift-invariance of $\mathbb{P}$, to prove Theorem 5.3 it suffices to show that almost surely there are no nontrivial bi-infinite geodesics that take the edge $(\mathbf{0},\mathbf{e}_1)$. Thus, this theorem follows from Lemma 5.1 and the next result.

For $u\leqslant v$ in $\mathbb{Z}^2$ define the event

$$U^{u,v}=\{\text{at least one geodesic of } G_{u,v} \text{ goes through both } \mathbf{0} \text{ and } \mathbf{e}_1\}\,. \tag{5.9}$$

THEOREM 5.5. — *Suppose Assumptions 4.1 and 5.2 hold with* (5.6) *satisfied. Let*

$$a_1=\min\Big(a_0,\big(\frac{1}{3}-\frac{a_0}{2}\big)\nu\Big)\in(1/3,2/3). \tag{5.10}$$

For each $\delta\in(0,\delta_0)$ there exist positive finite constants $N_4(\delta,\delta_0,\nu,a_0)$ and $C_4(\delta,\delta_0,\nu,a_0)$ such that for all $N\geqslant N_4$

$$\mathbb{P}\Big(\bigcup_{u\in\partial^{N,\delta},v\in\widehat{\partial}^{N,\delta}}U^{u,v}\Big)\leqslant C_4N^{-(a_1-1/3)}.$$

The reason behind the relation (5.6) is that if ν is close to 2, then this affects the bound (4.1) and, as a consequence, we do not have good control over the geodesic fluctuations in (4.5). Then, when using (5.5) in the argument against the existence of bi-infinite geodesics, we need to allow for a larger interval in (5.4), which means using a smaller a_0. That said, it should be the case that if the i.i.d. environment has finite exponential moments, then Assumption 4.1 holds for all $\nu>2$ and Assumption 5.2 holds for $a_0\in(1/3,2/3)$.

The rest of the section builds up towards the proof of the above theorem. Define the vertical segment

$$\mathcal{I}=\{0\}\times[\![-N^{2/3},N^{2/3}]\!]. \tag{5.11}$$

For $N\geqslant 1$, $1\leqslant s\leqslant\frac{\delta}{4}N^{1/3}$, and $o\in\mathbb{Z}^2_{>0}\cup\mathbb{Z}^2_{<0}$, define the directions

$$\begin{aligned}\zeta(o)&=\frac{o}{o_1+o_2}\,,\quad \zeta_\star(o)=\zeta(o)+(-sN^{-1/3},sN^{-1/3})\,,\\ &\text{and}\quad \zeta^\star(o)=\zeta(o)+(sN^{-1/3},-sN^{-1/3})\,.\end{aligned} \tag{5.12}$$

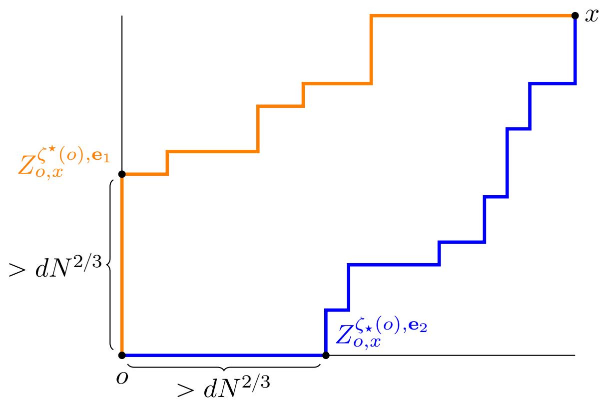

Figure 5.2. An illustration of the high probability event in Lemma 5.6. The upmost geodesic of $G_{o,x}^{\zeta_\star(o)}$ exits at least $dN^{2/3}$ to the right of o. The rightmost geodesic of $G_{o,x}^{\zeta^\star(o)}$ exits at least $dN^{2/3}$ above o.

Lemma 5.6. — *Suppose Assumption 4.1 holds. For any $\delta \in (0,\delta_0)$ there exist finite positive constants $C_5(\delta,\delta_0,\nu)$, $N_5(\delta,\delta_0) \geqslant 8\delta^{-3}$, and $s_5(\delta,\delta_0)$ such that for all $N \geqslant N_5$, if*

$$1 \leqslant d \leqslant \frac{\delta}{64}N^{1/3} \qquad \text{and} \qquad \max(s_5, 8d) \leqslant s \leqslant \frac{\delta}{4}N^{1/3}, \tag{5.13}$$

then for all $x \in \mathcal{I}$, and $o \in \partial^{N,\delta}$,

$$\mathbb{P}(Z_{o,x}^{\zeta_\star(o),\mathbf{e}_2} \leqslant dN^{2/3}) \leqslant C_5 s^{-\nu} \tag{5.14}$$

$$\mathbb{P}(Z_{o,x}^{\zeta^\star(o),\mathbf{e}_1} \geqslant -dN^{2/3}) \leqslant C_5 s^{-\nu}. \tag{5.15}$$

Proof. — The condition that $N \geqslant 8/\delta^3$ guarantees that $-\delta N + 1 \leqslant -N^{2/3}$, which implies that $-\lfloor \delta N \rfloor \leqslant -\lfloor N^{2/3} \rfloor$ and hence $\partial^{N,\delta}$ is entirely below $\mathcal{I}$. We prove (5.14) and the second bound follows analogously. Let $o = -(aN, bN)$ where $a \vee b = 1$ and $a \wedge b \in [\delta, 1]$. Abbreviate $\xi_\star = \zeta_\star(o)$. The upmost geodesic from o to $\lfloor N^{2/3} \rfloor \mathbf{e}_2$ must stay above any geodesic from o to $x \in \mathcal{I}$. This, Lemma 4.3, and shift-invariance give

$$\begin{aligned}\mathbb{P}\Big\{Z_{o,x}^{\xi_\star,\mathbf{e}_2} \leqslant dN^{2/3}\Big\} &\leqslant \mathbb{P}\Big\{Z_{o,\lfloor N^{2/3}\rfloor \mathbf{e}_2}^{\xi_\star,\mathbf{e}_2} \leqslant \lfloor dN^{2/3} \rfloor\Big\} = \mathbb{P}\Big\{Z_{o+\lfloor dN^{2/3}\rfloor \mathbf{e}_1,\lfloor N^{2/3}\rfloor \mathbf{e}_2}^{\xi_\star,\mathbf{e}_2} < 0\Big\} \\ &= \mathbb{P}\Big\{Z_{o,\lfloor N^{2/3}\rfloor \mathbf{e}_2 - \lfloor dN^{2/3}\rfloor \mathbf{e}_1}^{\xi_\star,\mathbf{e}_2} < 0\Big\} = \mathbb{P}\Big\{Z^{\xi_\star,\mathbf{e}_2}(aN - \lfloor dN^{2/3}\rfloor, bN + \lfloor N^{2/3}\rfloor) < 0\Big\}.\end{aligned} \tag{5.16}$$

Next, we check that we can apply Corollary 4.4 with

$$m = aN - \lfloor dN^{2/3} \rfloor, \quad n = \Big\lfloor \frac{m\xi_\star \cdot \mathbf{e}_2}{\xi_\star \cdot \mathbf{e}_1} \Big\rfloor = \Big\lfloor \frac{b + (a+b)sN^{-1/3}}{a - (a+b)sN^{-1/3}} \cdot m \Big\rfloor, \quad \text{and} \quad s' = \frac{n - bN - \lfloor N^{2/3} \rfloor}{(m+n)^{2/3}},$$

if we take N large enough and s as in (5.13). Here are the details. Take

$$N_5 = \max\big(8\delta^{-3} + 1, 64(N_1(\delta/4,\delta_0,1)+1)/(63\delta^2)\big)$$

and

$$s_5 = \max\big(4, 8/(3\delta), 2^{11/3}s_1(\delta/4,\delta_0,1)/(3\delta), 4/(1+\delta), 2^{10/3}s_1(\delta/4,\delta_0,1)/\delta^{2/3}\big).$$

Take $N \geqslant N_5$ and d and s as in (5.13). Then $m \geqslant 63\delta N/64 \geqslant N_1$, $n \geqslant bm/a - 1 \geqslant 63\delta^2 N/64 - 1 \geqslant N_1$, and

$$\frac{\delta}{4} \leqslant \frac{a - (a+b)\delta/4}{b + (a+b)\delta/4} \leqslant \frac{m}{n} \leqslant \frac{m}{bm/a - 1} \leqslant \frac{a}{b - 64a/(63\delta N_5)} \leqslant \frac{2}{\delta}.$$

Also

$$(n+m)^{1/3} \geqslant s' \geqslant \frac{(a+b)^2 s - bd - d(a+b)\delta/4 - 2}{a^{2/3}\Big(1 + \frac{b+(a+b)\delta/4}{a-(a+b)\delta/4}\Big)^{2/3}} \geqslant \frac{s/2 - 2d}{\big(1 + \frac{2+\delta}{\delta}\big)^{2/3}} \geqslant \frac{\delta^{2/3}s}{2^{10/3}} \geqslant s_1.$$

And

$$\frac{\delta}{4} \leqslant \frac{1}{1+1/\delta} - \frac{\delta}{4} \leqslant \frac{1}{1+o_2/o_1} - \frac{\delta}{4} \leqslant \xi_\star \cdot \mathbf{e}_1 \leqslant \frac{1}{1+o_2/o_1} \leqslant \frac{1}{1+\delta} \leqslant 1 - \frac{\delta}{2}\,. \tag{5.17}$$

And lastly,

$$\left|\xi_\star \cdot \mathbf{e}_1 - \frac{m}{m+n}\right| = \left|n - \frac{m\xi_\star \cdot \mathbf{e}_2}{\xi_\star \cdot \mathbf{e}_1}\right| (m+n)^{-1}\xi_\star \cdot \mathbf{e}_1 \leqslant (m+n)^{-1}.$$

Thus, (4.3) gives

$$\mathbb{P}\Big\{Z^{\xi_\star,\mathbf{e}_2}_{o,x} \leqslant dN^{2/3}\Big\} \leqslant \mathbb{P}\Big\{Z^{\xi_\star,\mathbf{e}_2}(m, n - s'(m+n)^{2/3}) < 0\Big\} \leqslant Cs^{-\nu}$$

for some positive finite constant $C(\delta, \delta_0, \nu)$. □

To control coarse graining on the scale $N^{2/3}$, we use the parameters d_1 for the southwest boundary and d_2 for the northeast boundary of the square $[\![-N, N]\!]^2$. Let $d = (d_1, d_2)$. For $o \in \partial^{N,\delta}$ define

$$\mathcal{I}_{o,d} = \Big\{u \in \partial^{N,\delta} : |u - o|_1 \leqslant \frac{d_1 N^{2/3} - 1}{2}\Big\}.$$

Because $\mathcal{I}_{o,d}$ is a connected portion of the boundary of a square, it contains a unique point o_c such that $o_c \leqslant u$ coordinate-wise for each point $u \in \mathcal{I}_{o,d}$.

For $s \leqslant \frac{\delta}{4}N^{1/3}$, define the directions

$$\xi_\star = \zeta_\star(o_c) \quad \text{and} \quad \xi^\star = \zeta^\star(o_c) \tag{5.18}$$

as in (5.12). Use Theorem A.1 to couple the weights $\{\omega_x, I^{\xi^\star}_x, J^{\xi^\star}_x, I^{\xi_\star}_x, J^{\xi_\star}_x : x \in \mathbb{Z}^2\}$ so that (4.7) holds almost surely and for all $x \in \mathbb{Z}^2$. Define the event

$$A_{o,d} = \Big\{Z^{\xi^\star,\mathbf{e}_1}_{o_c,-\lfloor N^{2/3}\rfloor\mathbf{e}_2} < -d_1 N^{2/3} \quad \text{and} \quad Z^{\xi_\star,\mathbf{e}_2}_{o_c,\lfloor N^{2/3}\rfloor\mathbf{e}_2} > d_1 N^{2/3}\Big\}. \tag{5.19}$$

Recall the boundary weights defined in (4.2).

Lemma 5.7. — *Suppose Assumption 4.1 holds. Then for any $\delta \in (0, \delta_0)$, $N \geqslant N_5(\delta, \delta_0)$, $o \in \partial^{N,\delta}$, and (d_1, s) satisfying* (5.13),

$$\mathbb{P}(A^c_{o,d}) \leqslant 2C_5(\delta, \delta_0, \nu)s^{-\nu}. \tag{5.20}$$

On the event $A_{o,d}$, the following inequalities hold for all $x \in \mathcal{I}$ and $u \in \mathcal{I}_{o,d}$:

$$J^{\xi_\star}_{x+\mathbf{e}_2} \leqslant J^{[u]}_{x+\mathbf{e}_2} \leqslant J^{\xi^\star}_{x+\mathbf{e}_2}. \tag{5.21}$$

Proof. — Lemma 5.6 implies (5.20). We prove the second inequality of (5.21) and the first inequality follows similarly. Let $\widetilde{G}_{x,y}$ be the LPP process on the quadrant $o_c + \mathbb{Z}^2_{\geqslant 0}$ with weights $\widetilde{\omega}_{o_c} = 0$, $\widetilde{\omega}_{o_c + j\mathbf{e}_2} = J^{\xi^\star}_{o_c + j\mathbf{e}_2}$ for each $j \geqslant 1$, and $\widetilde{\omega}_{o_c+x} = \omega_{o_c+x}$ whenever $x \cdot \mathbf{e}_1 > 0$.

First consider the case that $u = o_c + j\mathbf{e}_2$ for some $j \geqslant 0$. On the event $A_{o,d}$, we have that the rightmost geodesic of $G^{\xi^\star}_{o_c,-\lfloor N^{2/3}\rfloor\mathbf{e}_2}$ exits the boundary above $o_c + d_1 N^{2/3}\mathbf{e}_2$. Therefore, for any $x \in (-\lfloor N^{2/3}\rfloor + \mathbb{Z}_{\geqslant 0})\mathbf{e}_2$, every geodesic of $G^{\xi^\star}_{o_c,x}$ must exit the boundary above $o_c + d_1 N^{2/3}\mathbf{e}_2$, i.e.,

$$Z^{\xi^\star,\mathbf{e}_1}_{o_c,x} < -d_1 N^{2/3}.$$

Thus, every geodesic of $G^{\xi^\star}_{o_c,x}$ includes u and $u + \mathbf{e}_2$. Since the weights used by $\widetilde{G}$ and $G^{\xi^\star}$ are the same away from the horizontal boundary, and on that boundary the weights used by the former are smaller than the ones used by the latter, we get that

$$G^{\xi^\star}_{o_c,x+\mathbf{e}_2} - G^{\xi^\star}_{o_c,x} = \widetilde{G}_{u,x+\mathbf{e}_2} - \widetilde{G}_{u,x}.$$

By [8, Lemma B.1], $\widetilde{G}_{u,x+\mathbf{e}_2} - \widetilde{G}_{u,x} \geqslant G_{u,x+\mathbf{e}_2} - G_{u,x}$. Putting these together gives $J^{\xi^\star}_{x+\mathbf{e}_2} \geqslant J^{[u]}_{x+\mathbf{e}_2}$.

If instead $u = o_c + i\mathbf{e}_1$ for some $i \geqslant 1$, then as before $G^{\xi^\star}_{o_c,x} = \widetilde{G}_{o_c,x}$. Furthermore, since $\widetilde{G}_{u,x}$ does not use the vertical weights above o_c, then $\widetilde{G}_{u,x} = G_{u,x}$. By [8, Lemma B.2],

$$G_{o_c+\mathbf{e}_1,x+\mathbf{e}_2} - G_{o_c+\mathbf{e}_1,x} \leqslant G_{o_c,x+\mathbf{e}_2} - G_{o_c,x}.$$

Inductively, this gives $G_{o_c,x+\mathbf{e}_2}-G_{o_c,x}\geqslant G_{u,x+\mathbf{e}_2}-G_{u,x}$ and applying [8, Lemma B.1] for the first inequality we get

$$G^{\xi^\star}_{o_c,x+\mathbf{e}_2}-G^{\xi^\star}_{o_c,x}=\widetilde G_{o_c,x+\mathbf{e}_2}-\widetilde G_{o_c,x}\geqslant G_{o_c,x+\mathbf{e}_2}-G_{o_c,x}\geqslant G_{u,x+\mathbf{e}_2}-G_{u,x}.\qquad\square$$

Now we do an analogous construction for the stationary process with a northeast boundary. Recall the northeast boundary (5.8). We continue to drop the δ from the notation. Recall also the weights (5.1). For $x=(x_1,x_2)$ and $y=(y_1,y_2)$ in $\mathbb{Z}^2$ set $\widehat G^\xi_{x,y}=0$ if $x\nleqslant y$, while if $x\leqslant y$ then let

$$\widehat G^\xi_{y,x}=\max_{1\leqslant k\leqslant y_1-x_1}\Big\{G_{x,y-k\mathbf{e}_1-\mathbf{e}_2}+\sum_{i=1}^k \widehat I^\xi_{y-i\mathbf{e}_1}+\Big\}\bigvee\max_{1\leqslant \ell\leqslant y_2-x_2}\Big\{G_{x,y-\mathbf{e}_1-\ell\mathbf{e}_2}+\sum_{j=1}^\ell \widehat J^\xi_{y-j\mathbf{e}_2}\Big\},\tag{5.22}$$

with the convention that $\max_\varnothing=0$. In particular, $\widehat G^\xi_{x,x}=0$. Then

$$\widehat G^\xi_{y,x}(\omega)=G^\xi_{-x,-y}(\widehat\omega).$$

The additivity (3.5) becomes

$$\widehat G^\xi_{z,y}+\widehat G^\xi_{y,x}=\widehat G^\xi_{z,x},\tag{5.23}$$

for $x\leqslant y\leqslant z$ in $\mathbb{Z}^2$. The quantities $\widehat{\mathrm{Exit}}^\xi_{y,x}$ and $\widehat Z^{\xi,-\mathbf{e}_k}_{y,x}$ are defined analogously to $\mathrm{Exit}^\xi_{y,x}$ and $Z^{\xi,\mathbf{e}_k}_{y,x}$. Precisely,

$$\widehat{\mathrm{Exit}}^\xi_{y,x}=\Big\{k\in[\![1,y_1-x_1]\!]:\sum_{i=1}^k\widehat I^\xi_{y-i\mathbf{e}_1}+G_{x,y-k\mathbf{e}_1-\mathbf{e}_2}=\widehat G^\xi_{y,x}\Big\}$$
$$\bigcup\Big\{-\ell:\ell\in[\![1,y_2-x_2]\!]\text{ and }\sum_{j=1}^\ell\widehat J^\xi_{y-j\mathbf{e}_2}+G_{x,y-\ell\mathbf{e}_2-\mathbf{e}_1}=\widehat G^\xi_{y,x}\Big\},$$

$\widehat Z^{\xi,-\mathbf{e}_1}_{y,x}=\max\widehat{\mathrm{Exit}}^\xi_{y,x}$, and $\widehat Z^{\xi,-\mathbf{e}_2}_{y,x}=\min\widehat{\mathrm{Exit}}^\xi_{y,x}$.

For $\widehat o\in\widehat\partial^{N,\delta}$ let

$$\widehat{\mathcal I}_{\widehat o,d}=\Big\{v\in\widehat\partial^{N,\delta}:|v-\widehat o|_1\leqslant\frac{d_2N^{2/3}-1}{2}\Big\}$$

and let $\widehat o_c$ be the unique point of $\widehat{\mathcal I}_{\widehat o,d}$ such that $\widehat o_c\geqslant v$ for each point $v\in\widehat{\mathcal I}_{\widehat o,d}$. For $1\leqslant s\leqslant\frac{\delta}{4}N^{1/3}$ define

$$\eta_\star=\zeta_\star(\widehat o_c)\quad\text{and}\quad\eta^\star=\zeta^\star(\widehat o_c)\tag{5.24}$$

as in (5.12). Couple the weights $\{\omega_x,I^{\eta^\star}_x,J^{\eta^\star}_x,I^{\eta_\star}_x,J^{\eta_\star}_x\}$ using Theorem A.1. This produces a coupling of $\{\omega_x,\widehat I^{\eta^\star}_x,\widehat J^{\eta^\star}_x,\widehat I^{\eta_\star}_x,\widehat J^{\eta_\star}_x\}$ such that

$$\omega_x\leqslant\widehat I^{\eta^\star}_x\leqslant\widehat I^{\eta_\star}_x\quad\text{and}\quad\omega_x\leqslant\widehat J^{\eta_\star}_x\leqslant\widehat J^{\eta^\star}_x\ ,$$

the analogue of (4.7), holds almost surely and for all $x\in\mathbb{Z}^2$.

Define the increment variables analogously to (4.2):

$$\widehat I^{[y]}_x=G_{x,y}-G_{x+\mathbf{e}_1,y},\quad\text{when }x+\mathbf{e}_1\leqslant y,\text{ and}$$
$$\widehat J^{[y]}_x=G_{x,y}-G_{x+\mathbf{e}_2,y},\quad\text{when }x+\mathbf{e}_2\leqslant y.$$

Define the event

$$B_{\widehat o,d}=\Big\{\widehat Z^{\eta^\star,-\mathbf{e}_1}_{\widehat o_c,\lfloor N^{2/3}\rfloor\mathbf{e}_2+\mathbf{e}_1}<-d_2N^{2/3},\quad\widehat Z^{\eta_\star,-\mathbf{e}_2}_{\widehat o_c,-\lfloor N^{2/3}\rfloor\mathbf{e}_2+\mathbf{e}_1}>d_2N^{2/3}\Big\}\tag{5.25}$$

The next result follows from Lemma 5.7.

Lemma 5.8. — *Suppose Assumption 4.1 holds. Then for any $\delta\in(0,\delta_0)$, $N\geqslant N_5(\delta,\delta_0)$, $\widehat o\in\widehat\partial^{N,\delta}$, and (d_2,s) satisfying* (5.13),

$$\mathbb{P}(B^c_{\widehat o,d})\leqslant 2C_5(\delta,\delta_0,\nu)s^{-\nu}.\tag{5.26}$$

On the event $B_{\widehat o,d}$, the following inequalities hold for all $x\in\mathcal I$ and $v\in\widehat{\mathcal I}_{\widehat o,d}$:

$$\widehat J^{\eta_\star}_{x+\mathbf{e}_1+\mathbf{e}_2}\leqslant\widehat J^{[v]}_{x+\mathbf{e}_1+\mathbf{e}_2}\leqslant\widehat J^{\eta^\star}_{x+\mathbf{e}_1+\mathbf{e}_2}.\tag{5.27}$$

Let $o \in \partial^{N,\delta}$, $\widehat{o} \in \widehat{\partial}^{N,\delta}$ and consider the LPP process from points $u \in \mathcal{I}_{o,d}$ to the interval $\mathcal{I}$ and the reverse LPP process from points $v \in \widehat{\mathcal{I}}_{\widehat{o},d}$ to the shifted interval $\mathbf{e}_1 + \mathcal{I}$. Recall (5.18) and (5.24). Use Theorem A.1 again to couple the weights in (5.3) and thus produce a coupling of the weights

$$\{\omega_x, J_x^{\xi^\star}, J_x^{\xi_\star}, \widehat{J}_x^{\eta^\star}, \widehat{J}_x^{\eta_\star} : x \in \mathbb{Z}^2\}.$$

Recall the random walks $S^{\xi_\star,\eta^\star}$ and $S^{\xi^\star,\eta_\star}$, as defined in (5.2). Define also

$$S_n^{u,v} = \begin{cases} \sum_{j=1}^{n}(J_{j\mathbf{e}_2}^{[u]} - \widehat{J}_{\mathbf{e}_1+(j-1)\mathbf{e}_2}^{[v]}), & n \geqslant 1, \\ 0, & n = 0, \\ -\sum_{j=n+1}^{0}(J_{j\mathbf{e}_2}^{[u]} - \widehat{J}_{\mathbf{e}_1+(j-1)\mathbf{e}_2}^{[v]}), & n \leqslant -1. \end{cases}$$

The following is immediate from (5.21) and (5.27).

Lemma 5.9. — *On the event $A_{o,d} \cap B_{\widehat{o},d}$, for all $u \in \mathcal{I}_{o,d}$ and $v \in \widehat{\mathcal{I}}_{\widehat{o},d}$,*

$$\begin{aligned} &S_n^{\xi_\star,\eta^\star} \leqslant S_n^{u,v} \leqslant S_n^{\xi^\star,\eta_\star} \text{ for } n \in [\![0, N^{2/3}]\!] \quad \text{and} \\ &S_n^{\xi^\star,\eta_\star} \leqslant S_n^{u,v} \leqslant S_n^{\xi_\star,\eta^\star} \text{ for } n \in [\![-N^{2/3}+1, 0]\!]. \end{aligned} \tag{5.28}$$

Recall the event $U^{u,v}$ defined in (5.9).

Lemma 5.10. — *Suppose Assumptions 4.1 and 5.2 hold with* (5.6) *satisfied. For any $\delta \in (0, \delta_0)$ there exist finite positive constants $C_6(\delta, \delta_0, \nu, a_0)$ and $N_6(\delta, \delta_0) \geqslant 8\delta^{-3}$ such that for all $N \geqslant N_6$ and $o \in \partial^{N,\delta}$, if $\widehat{o} = -o \in \widehat{\partial}^{N,\delta}$, $d_1 = 1$, $d_2 = N^{\frac{1}{3}-\frac{a_0}{2}}/18$, and $s = 8d_2$, then*

$$\mathbb{P}\Big(\bigcup_{u\in\mathcal{I}_{o,d}, v\in\widehat{\mathcal{I}}_{\widehat{o},d}} U_{u,v}\Big) \leqslant C_6 N^{-a_1}, \tag{5.29}$$

where a_1 is defined in (5.10).

Proof. — Let $o \in \partial^{N,\delta}$, $\widehat{o} = -o$, $u \in \mathcal{I}_{o,d}$, and $v \in \widehat{\mathcal{I}}_{\widehat{o},d}$. The walk $S^{u,v}$ determines where the geodesics of $G_{u,v}$ leave the vertical axis, since

$$\begin{aligned} G_{u,v} &= \max_{u_2 \leqslant n \leqslant v_2} \Big\{ G_{u,(0,n)} + \widehat{G}_{v,(1,n)} \Big\} \\ &= \max_{u_2 \leqslant n \leqslant v_2} \Big\{ \big[G_{u,(0,n)} - G_{u,(0,0)}\big] + G_{u,(0,0)} + \widehat{G}_{v,(1,0)} - \Big[\widehat{G}_{v,(1,0)} - \widehat{G}_{v,(1,n)}\Big] \Big\} \\ &= \max_{u_2 \leqslant n \leqslant v_2} \Big\{ G_{u,(0,0)} + \widehat{G}_{v,(1,0)} + S_n^{u,v} \Big\}. \end{aligned}$$

Therefore, a geodesic of $G_{u,v}$ takes the edge $(j\mathbf{e}_2, \mathbf{e}_1 + j\mathbf{e}_2)$ if and only if $j \in [\![u_2, v_2]\!]$ is such that $S_j^{u,v} = \max_{u_2 \leqslant n \leqslant v_2} S_n^{u,v}$. Consequently,

$$U^{u,v} \subset \Big\{ \sup_{0<k\leqslant N^{2/3}} S_k^{u,v} \leqslant 0 \Big\} \cap \Big\{ \sup_{-N^{2/3}+1\leqslant k<0} S_k^{u,v} \leqslant 0 \Big\}.$$

This and (5.28) imply that on the event $A_{o,d} \cap B_{\widehat{o},d}$

$$\bigcup_{u\in\mathcal{I}_{o,d}, v\in\widehat{\mathcal{I}}_{\widehat{o},d}} U^{u,v} \subset \Big\{ \sup_{0<k\leqslant N^{2/3}} S_k^{\xi_\star,\eta^\star} \leqslant 0 \Big\} \cap \Big\{ \sup_{-N^{2/3}+1\leqslant k<0} S_k^{\xi^\star,\eta_\star} \leqslant 0 \Big\},$$

where $\xi^\star, \xi_\star, \eta^\star, \eta_\star$ were defined in (5.18) and (5.24). As a result, we have

$$\mathbb{P}\Big(\bigcup_{u\in\mathcal{I}_{o,d}, v\in\widehat{\mathcal{I}}_{\widehat{o},d}} U^{u,v}\Big) \leqslant \mathbb{P}\Big(\Big\{ \sup_{0<k\leqslant N^{2/3}} S_k^{\xi_\star,\eta^\star} \leqslant 0 \Big\} \cap \Big\{ \sup_{-N^{2/3}+1\leqslant k<0} S_k^{\xi^\star,\eta_\star} \leqslant 0 \Big\}\Big) + \mathbb{P}(A_{o,d}^c \cup B_{\widehat{o},d}^c). \tag{5.30}$$

Take $N \geqslant N_3(\delta) \vee N_5(\delta, \delta_0)$ and such that $s \geqslant s_5(\delta, \delta_0)$, $d \leqslant \delta N^{1/3}/64$, and hence $s \leqslant \delta N^{1/3}/4$. Since $o = -\widehat{o}$

$$|o_c + \widehat{o}_c|_1 \leqslant |o_c - o|_1 + |\widehat{o}_c - \widehat{o}|_1 \leqslant (d_1 N^{2/3} + d_2 N^{2/3})/2 \leqslant d_2 N^{2/3}$$

and thus

$$|\zeta(o_c)-\zeta(\widehat{o}_c)|_1 \leqslant \Big|\frac{o_c}{|o_c|_1}+\frac{\widehat{o}_c}{|\widehat{o}_c|_1}\Big|_1 \leqslant \frac{|o_c+\widehat{o}_c|_1}{|o_c|_1}+\frac{\big||o_c|_1-|\widehat{o}_c|_1\big|}{|o_c|_1} \leqslant \frac{2|o_c+\widehat{o}_c|_1}{|o_c|_1} \leqslant 2d_2N^{-1/3}.$$

Therefore

$$-N^{-a_0/2} = -2d_2N^{-1/3}-2sN^{-1/3} \leqslant \xi_\star\cdot\mathbf{e}_1-\eta^\star\cdot\mathbf{e}_1 \leqslant 2d_2N^{-1/3}-2sN^{-1/3}<0$$

and similarly

$$-N^{-a_0/2} \leqslant \eta_\star\cdot\mathbf{e}_1-\xi^\star\cdot\mathbf{e}_1<0.$$

Furthermore, the inequalities in (5.17) verify that the $\mathbf{e}_1$-coordinates of $\xi^\star,\xi_\star,\eta^\star,\eta_\star$ are all in $(\delta/4,1-\delta/4)$. We can now apply (5.5), (5.20), and (5.26), which together with (5.30) give

$$\mathbb{P}\Big(\bigcup_{u\in\mathcal{I}_{o,d},v\in\widehat{\mathcal{I}}_{\widehat{o},d}} U^{u,v}\Big) \leqslant C_3(\delta)N^{-a_0}+4C_5(\delta,\delta_0,\nu)s^{-\nu} \leqslant C_6N^{-a_1}. \qquad \square$$

Just as above, for $o\in\partial^{N,\delta}$, let $\widehat{o}=-o$ and set

$$\widehat{\mathcal{F}}_{\widehat{o},d}=\Big\{v\in\widehat{\partial}^{N,\delta}:|\widehat{o}-v|_1>\frac{d_2N^{2/3}-1}{2}\Big\}.$$

Lemma 5.11. — *Suppose Assumption 4.1 holds. For any $a\in(0,2/3)$ and $\delta\in(0,\delta_0)$, there exist positive finite constants $C_7(\delta,\delta_0,\nu)$ and $N_7(\delta,\delta_0,a)\geqslant 8\delta^{-3}$ such that for any $N\geqslant N_7$ and $o\in\partial^{N,\delta}$, if $d_1=1$ and $d_2=N^{\frac{1}{3}-\frac{a}{2}}/18$, then*

$$\mathbb{P}\Big(\bigcup_{u\in\mathcal{I}_{o,d},v\in\widehat{\mathcal{F}}_{\widehat{o},d}} U^{u,v}\Big) \leqslant C_7N^{-(\frac{1}{3}-\frac{a}{2})\nu}. \tag{5.31}$$

Proof. — Define the boundaries

$$\partial\widehat{\mathcal{F}}_{\widehat{o},d}=\Big\{v\in\widehat{\mathcal{F}}_{\widehat{o},d}:\exists u\in\widehat{\mathcal{I}}_{\widehat{o},d}\text{ such that }|v-u|_1=1\Big\}\quad\text{and}$$
$$\partial\mathcal{I}_{o,d}=\big\{v\in\mathcal{I}_{o,d}:\exists u\in\partial^{N,\delta}\backslash\mathcal{I}_{o,d}\text{ such that }|v-u|_1=1\big\}.$$

Their cardinalities are either 1 or 2, since it may happen that $\widehat{\mathcal{I}}_{\widehat{o},d}$ contains an endpoint such as $(N,\lfloor\delta N\rfloor)$. Additionally, $1\leqslant|\partial\widehat{\mathcal{F}}_{\widehat{o},d}|\leqslant|\partial\mathcal{I}_{o,d}|\leqslant 2$ because $d_1<d_2$, so $\widehat{\mathcal{I}}_{\widehat{o},d}$ would include an endpoint of the boundary whenever $\mathcal{I}_{o,d}$ does. Label the points in $\partial\mathcal{I}_{o,d}$ as h^1 and h^2 and label those of $\partial\widehat{\mathcal{F}}_{\widehat{o},d}$ as f^1 and f^2 so that

$$h_1^1\geqslant o_1\geqslant h_1^2,\quad h_2^1\leqslant o_2\leqslant h_2^2,\quad f_1^1\leqslant\widehat{o}_1\leqslant f_1^2,\quad\text{and}\quad f_2^1\geqslant\widehat{o}_2\geqslant f_2^2.$$

Traveling clockwise around the boundary of the square $[\![-N,N]\!]^2$ starting at $(0,N)$, the points that exist come in this order: $f^1,\widehat{o},f^2,h^1,o,h^2$.

We will show that if some geodesic from $u\in\mathcal{I}_{o,d}$ to $v\in\widehat{\mathcal{F}}_{\widehat{o},d}$ uses the edge $(\mathbf{0},\mathbf{e}_1)$ then, for some $i\in\{1,2\}$, $\pi^{u,v,\mathbf{e}_i}$, the $\mathbf{e}_i$-most geodesic of G_{h^i,f^i}, deviates by at least $\delta d_2N^{2/3}/16$ from the straight line segment from h^i to f^i. To this end, define

$$\mathcal{P}_m^{u,v,\mathbf{e}_i}=\pi^{u,v,\mathbf{e}_i}\cap\big\{x\in\mathbb{Z}^2:x_1=m\big\}.$$

This is the intersection of the $\mathbf{e}_i$-most geodesic of $G_{u,v}$ with the vertical line $x_1=m$. For $t>0$ let

$$D_{m,t}^{u,v}=\bigcup_{i=1}^{2}\Big\{\inf_{p=(p_1,p_2)\in\mathcal{P}_m^{u,v,\mathbf{e}_i}}\Big|u_2+\frac{v_2-u_2}{v_1-u_1}(m-u_1)-p_2\Big|>t\Big\} \tag{5.32}$$

be the event that at the vertical line $x_1=m$, some geodesic from u to v deviates from the straight line segment from u to v by more than t.

For $u\in\mathcal{I}_{o,d}$ and $v\in\widehat{\mathcal{F}}_{\widehat{o},d}$, let $e^u=u-o$ and $e^v=v-\widehat{o}$. Then $\widehat{\mathcal{F}}_{\widehat{o},d}$ is the union of two disjoint pieces

$$\widehat{\mathcal{F}}_{\widehat{o},d}^1=\Big\{v\in\widehat{\mathcal{F}}_{\widehat{o},d}:e_1^v\leqslant 0\leqslant e_2^v\Big\}\quad\text{and}\quad\widehat{\mathcal{F}}_{\widehat{o},d}^2=\Big\{v\in\widehat{\mathcal{F}}_{\widehat{o},d}:e_2^v\leqslant 0\leqslant e_1^v\Big\},$$

separated by $\widehat{\mathcal{I}}_{\hat{o},d}$, one of which can be empty. $\widehat{\mathcal{F}}^1_{\hat{o},d}$ is to the left and above $\widehat{\mathcal{I}}_{\hat{o},d}$, and if it is not empty, then it is separated from $\widehat{\mathcal{I}}_{\hat{o},d}$ by the point f^1. $\widehat{\mathcal{F}}^2_{\hat{o},d}$ is to the right and below $\widehat{\mathcal{I}}_{\hat{o},d}$, and if it is not empty, then it is separated from $\widehat{\mathcal{I}}_{\hat{o},d}$ by the point f^2.

Take $N \geqslant \max\big(\sqrt{8}, (1+\delta)/\delta^2, N_2(\delta,\delta_0)/(2\delta)\big)$ and large enough so that

$$\frac{2d_1}{\delta} \leqslant \frac{\delta d_2}{16}, \quad d_2 \geqslant 16 s_2(\delta,\delta_0), \quad \text{and} \quad \frac{d_2}{16} \leqslant \frac{\delta}{4(1+\delta)} \cdot \frac{\delta(4N)^{1/3}}{3(1+\delta)}.$$

If $u \in \mathcal{I}_{o,d}$ and $v \in \widehat{\mathcal{F}}^1_{\hat{o},d}$, then

$$|e^u|_1 \leqslant \frac{d_1 N^{2/3}-1}{2}, \quad |e^v|_1 > \frac{d_2 N^{2/3}-1}{2} \geqslant \frac{d_2 N^{2/3}}{4}, \quad \text{and} \quad e_1^v \leqslant 0 \leqslant e_2^v.$$

Using this, together with $v_i - u_i = \hat{o}_i + e_i^v - (o_i + e_i^u) = -2o_i + e_i^v - e_i^u$, $-N \leqslant o_i \leqslant -\delta N$, and $\delta \leqslant (v_2-u_2)/(v_1-u_1) \leqslant 1/\delta$, we get

$$\begin{aligned} u_2 + \frac{v_2-u_2}{v_1-u_1}(-u_1) &= \frac{o_2 e_1^v - o_1 e_2^v}{v_1-u_1} - \frac{o_2 e_1^u - o_1 e_2^u}{v_1-u_1} + e_2^u + \frac{v_2-u_2}{v_1-u_1}(-e_1^u) \\ &\geqslant \frac{\delta N\,|e^v|_1}{2N} - \Big(\frac{N}{2N\delta} + 1 + \delta^{-1}\Big)|e^u|_1 \\ &\geqslant \frac{1}{8}\delta d_2 N^{2/3} - 2\delta^{-1} d_1 N^{2/3} \geqslant \frac{1}{16}\delta d_2 N^{2/3}. \end{aligned} \tag{5.33}$$

Similarly, for $u \in \mathcal{I}_{o,d}$ and $v \in \widehat{\mathcal{F}}^2_{\hat{o},d}$, we have

$$\begin{aligned} u_2 + \frac{v_2-u_2}{v_1-u_1}(1-u_1) &= \frac{o_2 e_1^v - o_1 e_2^v}{v_1-u_1} - \frac{o_2 e_1^u - o_1 e_2^u}{v_1-u_1} + e_2^u + \frac{v_2-u_2}{v_1-u_1}(1-e_1^u) \\ &\leqslant -\frac{\delta N\,|e^v|_1}{2N} + \left(\frac{N}{2N\delta} + 1 + \delta^{-1}\right)|e^u|_1 + \delta^{-1} \\ &\leqslant -\frac{1}{8}\delta d_2 N^{2/3} + 2\delta^{-1} d_1 N^{2/3} \leqslant -\frac{1}{16}\delta d_2 N^{2/3}. \end{aligned} \tag{5.34}$$

Now suppose that for some $u \in \mathcal{I}_{o,d}$ and $v \in \widehat{\mathcal{F}}_{\hat{o},d}$ some geodesic of $G_{u,v}$ goes through the edge $(\mathbf{0},\mathbf{e}_1)$. We have these two cases:

(i) If $v \in \widehat{\mathcal{F}}^1_{\hat{o},d}$, then the rightmost geodesic $\pi^{h^1,f^1,\mathbf{e}_1}$ stays to the right of all the geodesics from u to v. Consequently, this geodesic crosses the axis $\mathbb{R}\mathbf{e}_2$ at or below $\mathbf{0}$. Then (5.33) with $u = h^1$ and $v = f^1$ shows that $\pi^{h^1,f^1,\mathbf{e}_1}$ avoids the vertical interval of radius $\frac{1}{16}\delta d_2 N^{2/3}$, centered around the point on the line segment from h^1 to f^1 with $\mathbf{e}_1$-coordinate $x_1 = 0$.

(ii) If $v \in \widehat{\mathcal{F}}^2_{\hat{o},d}$, then the upmost geodesic $\pi^{h^2,f^2,\mathbf{e}_2}$ stays above all the geodesics from u to v and therefore crosses $\mathbb{R}\mathbf{e}_2$ at or above $\mathbf{0}$. Then (5.34) with $u = h^2$ and $v = f^2$ shows that $\pi^{h^2,f^2,\mathbf{e}_2}$ avoids the vertical interval of radius $\frac{1}{16}\delta d_2 N^{2/3}$, centered around the point on the line segment from h^2 to f^2 with $\mathbf{e}_1$-coordinate $x_1 = 0$.

We can now apply Lemma 4.5 with $\varepsilon = \frac{\delta}{4(1+\delta)}$ because we took N large enough so that $f^i - h^i \in S_\delta \cap \mathbb{Z}_{\geqslant N_2(\delta,\delta_0)}$,

$$s = \frac{d_2 N^{2/3}}{16|f^i - h^i|_1^{2/3}} \in \Big[s_2(\delta,\delta_0), \frac{\varepsilon\delta}{3(\delta+1)}|f_i - h_i|_1^{1/3}\Big]$$

and

$$\alpha = \begin{cases} \frac{-h_1^1}{f_1^1 - h_1^1} \in \big[\frac{\delta}{1+\delta}, \frac{1}{1+\delta}\big] \subset (\varepsilon, 1-\varepsilon) & \text{in case (i)}, \\ \frac{1-h_1^2}{f_1^2 - h_1^2} \in \big[\frac{\delta}{1+\delta}, \frac{1}{1+\delta} + \frac{1}{2\delta N}\big] \subset (\varepsilon, 1-\varepsilon) & \text{in case (ii)}. \end{cases}$$

Combining these results, we conclude that

$$\mathbb{P}\Big(\bigcup_{u\in\mathcal{I}_{o,d},v\in\widehat{\mathcal{F}}_{\hat{o},d}} U^{u,v}\Big) \leqslant \mathbb{P}\left(D^{h^1,f^1}_{0,\delta d_2 N^{2/3}/16} \cup D^{h^2,f^2}_{1,\delta d_2 N^{2/3}/16}\right)$$

$$\leqslant 2C_2(\delta,\delta_0,\nu,\varepsilon)s^{-\nu} \leqslant C_2\Big(\frac{d_2}{16\cdot 2^{2/3}}\Big)^{-\nu} = C_7 N^{-(\frac{1}{3}-\frac{a}{2})\nu}.$$

The lemma is proved. □

Using a union bound and Lemmas 5.10 and 5.11 we get the following.

Lemma 5.12. — *Suppose Assumptions 4.1 and 5.2 hold with* (5.6) *satisfied. For any* $\delta \in (0,\delta_0)$ *there exist positive finite constants* $C_8(\delta,\delta_0,\nu,a_0)$ *and* $N_8(\delta,\delta_0,a_0) \geqslant 8\delta^{-3}$ *such that for all* $N \geqslant N_8$ *and* $o \in \partial^{N,\delta}$, *if* $d_1 = 1$ *and* $d_2 = N^{\frac{1}{3}-\frac{a_0}{2}}/18$, *then*

$$\mathbb{P}\Big(\bigcup_{u\in\mathcal{I}_{o,d},v\in\widehat{\partial}^{N,\delta}} U^{u,v}\Big) \leqslant C_8 N^{-a_1}.$$

where a_1 *is given in* (5.10).

We are now ready to prove Theorem 5.5.

Proof of Theorem 5.5. — Take $d_1 = 1$ and $d_2 = N^{\frac{1}{3}-\frac{a_0}{2}}/18$. Let

$$\mathcal{O}^N = \partial^{N,\delta} \cap \left(\Big\{\Big(-N + id_1(\lfloor N^{2/3}\rfloor - 1), -N\Big)\Big\}_{i\in\mathbb{Z}\geqslant 0} \bigcup \Big\{\Big(-N, -N + jd_1(\lfloor N^{2/3}\rfloor - 1)\Big)\Big\}_{j\in\mathbb{Z}_{\geqslant 0}}\right).$$

Then we can decompose

$$\bigcup_{u\in\partial^{N,\delta},v\in\widehat{\partial}^{N,\delta}} U^{u,v} \subset \bigcup_{o\in\mathcal{O}^N}\ \bigcup_{u\in\mathcal{I}_{o,d},v\in\widehat{\partial}^{N,\delta}} U^{u,v}.$$

Since $|\mathcal{O}^N| \leqslant Cd_1^{-1}N^{1-2/3} = CN^{1/3}$, for some positive finite constant C, a union bound and Lemma 5.12 give

$$\mathbb{P}\Big(\bigcup_{u\in\partial^{N,\delta},v\in\widehat{\partial}^{N,\delta}} U^{u,v}\Big) \leqslant \sum_{o\in\mathcal{O}^N} \mathbb{P}\Big(\bigcup_{u\in\mathcal{I}_{o,d},v\in\widehat{\partial}^{N,\delta}} U^{u,v}\Big) \leqslant C_4 N^{-(a_1-1/3)}.$$

The theorem is proved. □

Appendix A. Stationary boundary

A.1. General weight distribution

The next theorem provides the boundary weights I^ξ_x and J^ξ_x that are used throughout our proofs. It follows directly from Theorem 4.7 of [34]. Note that when the weights are geometric, random variables, Theorem A.2 below gives an alternate construction of these boundary weights, with some additional independence properties. The purpose of the theorem in this section is to give a construction that works for a general weight distribution. If the reader is only interested in the geometric weights setting, then Theorem A.1 can be bypassed and Theorem A.2 can be used instead.

Recall the shape function γ defined in (3.7). The subadditivity (3.4) and the limit (3.6) imply that γ is a convex positively homogeneous function on $\mathbb{R}^2_{\geqslant 0}$. As such, we can define the right-gradient $\gamma(\xi+)$ via the limits

$$\mathbf{e}_1\cdot\nabla\gamma(\xi+) = \lim_{\varepsilon\searrow 0}\frac{\gamma(\xi+\varepsilon\mathbf{e}_1)-\gamma(\xi)}{\varepsilon} \quad\text{and}\quad \mathbf{e}_2\cdot\nabla\gamma(\xi+) = \lim_{\varepsilon\searrow 0}\frac{\gamma(\xi)-\gamma(\xi-\varepsilon\mathbf{e}_2)}{\varepsilon}.$$

Let $\mathcal{U}_0$ be a countable dense subset of ri$\mathcal{U}$. Let $\mathcal{H}_0 = \{-\nabla\gamma(\xi+) : \xi\in\mathcal{U}_0\}$. Let $\widehat{\Omega} = \Omega\times\mathbb{R}^{\mathbb{Z}^2\times\{1,2\}\times\mathcal{H}_0}$ and equip it with the product topology and the Borel σ-algebra $\widehat{\mathcal{G}}$. Let $\widehat{T} = (\widehat{T}_x)_{x\in\mathbb{Z}^2}$ be the natural group of shifts on $\widehat{\Omega}$. For $A\subset\mathbb{Z}^2$ let $A^{\leqslant} = \{x\in\mathbb{Z}^2 : \exists y\in A \text{ with } x\leqslant y\}$ and $A^{>} = \mathbb{Z}^2\backslash A^{\leqslant}$.

Theorem A.1. — *Assume* (2.1). *There exist a $\widehat{T}$-invariant probability measure $\widehat{\mathbb{P}}$ on $(\widehat{\Omega}, \widehat{\mathcal{G}})$ and random variables $(x, \xi, \widehat{\omega}) \in \mathbb{Z}^2 \times \mathrm{ri}\,\mathcal{U} \times \widehat{\Omega} \mapsto (I^\xi_x, J^\xi_x) \in \mathbb{R}^2$ such that the following properties hold.*

(a) *$\mathbb{P}$ is the restriction of $\widehat{\mathbb{P}}$ onto Ω.*

(b) *For any $A \subset \mathbb{Z}^2$, the process $\{\omega_x, I^\xi_x, J^\xi_x : x \in A, \xi \in \mathrm{ri}\,\mathcal{U}\}$ is independent of $\{\omega_x : x \in A^>\}$.*

(c) *For each $\xi \in \mathrm{ri}\,\mathcal{U}$ and $x \in \mathbb{Z}^2$, I^ξ_x and J^ξ_x are integrable and $\widehat{\mathbb{E}}[(I^\xi_x, J^\xi_x)] = \nabla\gamma(\xi+)$.*

(d) *There exists an event $\widehat{\Omega}_0$ such that $\widehat{\mathbb{P}}(\widehat{\Omega}_0) = 1$ and the following all hold for $\widehat{\omega} \in \widehat{\Omega}_0$:*

(d.1) *For each $x, y \in \mathbb{Z}^2$ and $\xi \in \mathrm{ri}\,\mathcal{U}$, $I^\xi_x(\widehat{T}_y\widehat{\omega}) = I^\xi_{x+y}(\widehat{\omega})$ and $J^\xi_x(\widehat{T}_y\widehat{\omega}) = J^\xi_{x+y}(\widehat{\omega})$.*

(d.2) *For each $x \in \mathbb{Z}^2$ and $\xi, \zeta \in \mathrm{ri}\,\mathcal{U}$ with $\xi_1 \leqslant \zeta_1$ we have $\omega_x = I^\xi_x \wedge J^\xi_x$,*

$$\omega_x \leqslant I^\zeta_x \leqslant I^\xi_x, \quad \text{and} \quad \omega_x \leqslant J^\xi_x \leqslant J^\zeta_x.$$

(d.3) *For $\xi \in \mathrm{ri}\,\mathcal{U}$, $x = (x_1, x_2) \in \mathbb{Z}^2$, and $k \in \mathbb{N}$ set $G^\xi_{x,x} = 0$,*

$$G^\xi_{x,x+k\mathbf{e}_1} = \sum_{i=1}^k I^\xi_{x+i\mathbf{e}_1} \quad \text{and} \quad G^\xi_{x,x+k\mathbf{e}_2} = \sum_{i=1}^k J^\xi_{x+i\mathbf{e}_2}. \tag{A.1}$$

For $y \in x + \mathbb{N}^2$ let

$$G^\xi_{x,y} = \max_{1\leqslant k\leqslant y_1-x_1} \Big\{\sum_{i=1}^k I^\xi_{x+i\mathbf{e}_1} + G_{x+k\mathbf{e}_1+\mathbf{e}_2,y}\Big\} \bigvee \max_{1\leqslant \ell\leqslant y_2-x_2} \Big\{\sum_{j=1}^{\ell} J^\xi_{x+j\mathbf{e}_2} + G_{x+\mathbf{e}_1+\ell\mathbf{e}_2,y}\Big\}. \tag{A.2}$$

Then for all $x \leqslant y \leqslant z$ in $\mathbb{Z}^2$ and $\xi \in \mathrm{ri}\,\mathcal{U}$ we have

$$G^\xi_{x,y} + G^\xi_{y,z} = G^\xi_{x,z}. \tag{A.3}$$

In particular, for any $\xi \in \mathrm{ri}\,\mathcal{U}$ and $x \in \mathbb{Z}^2$

$$I^\xi_{x+\mathbf{e}_1} + J^\xi_{x+\mathbf{e}_1+\mathbf{e}_2} = J^\xi_{x+\mathbf{e}_2} + I^\xi_{x+\mathbf{e}_1+\mathbf{e}_2}. \tag{A.4}$$

(e) *For each $u \geqslant v$ in $\mathbb{Z}^2_{\geqslant 0}$*

$$\{(G^\xi_{u,v+x} - G^\xi_{u,v} : x \in \mathbb{Z}^2_{\geqslant 0}, \xi \in \mathrm{ri}\,\mathcal{U}\} \overset{\mathrm{d}}{=} \{(G^\xi_{u,u+x} : x \in \mathbb{Z}^2_{\geqslant 0}, \xi \in \mathrm{ri}\,\mathcal{U}\}.$$

Proof. — Taking $\beta = \infty$ in Theorem 4.7 of [34] we get a process $B^{\infty,h(\xi)+}(x, y, \widehat{\omega})$, $x, y \in \mathbb{Z}^2$, $\widehat{\omega} \in \widehat{\Omega}$, and $\xi \in \mathrm{ri}\,\mathcal{U}$. For $\widehat{\omega} \in \widehat{\Omega}$ let $\overline{\omega} \in \widehat{\Omega}$ be such that $\overline{\omega}_x = \widehat{\omega}_{-x}$, for all $x \in \mathbb{Z}^2$. Set $I^\xi_x(\widehat{\omega}) = B^{\infty,h(\xi)+}(-x, -x + \mathbf{e}_1, \overline{\omega})$ and $J^\xi_x(\widehat{\omega}) = B^{\infty,h(\xi)+}(-x, -x + \mathbf{e}_2, \overline{\omega})$.

Properties (a-c) follow from [34, Theorem 4.7(a-c)]. (d.1) comes from [34, (4.4)] and (d.2) comes from [34, (4.7-4.8)].

It is immediate from the cocycle property (4.4) and the recovery property (4.7) in [34, Theorem 4.7] that for any $x \leqslant y$ in $\mathbb{Z}^2$, we have $\widehat{\mathbb{P}}$-almost surely, for any $\xi \in \mathrm{ri}\,\mathcal{U}$, $G^\xi_{x,y}(\omega) = B^{\infty,h(\xi)+}(-y, -x, \overline{\omega})$. Then the additivity (A.3) (and (A.4)) is exactly the cocycle property [34, (4.4)].

Next, note that (A.3) implies $G^\xi_{u,v+x} - G^\xi_{u,v} = G^\xi_{v,v+x}$. Then property (e) follows from the $\widehat{T}$-invariance of $\widehat{\mathbb{P}}$ and the shift-covariance property in part (d.1). □

A.2. Geometric weights

When the weights are geometric the process in Theorem A.1 has some independence features and explicit one-dimensional marginals. Recall the bijection (3.11). Let $\overline{\Omega} = \Omega \times \mathbb{R}^{\mathbb{Z}^2\times\{1,2\}} \times \mathbb{R}^{\mathbb{Z}^2\times\{1,2\}}$ and equip it with the product topology and the Borel σ-algebra $\overline{\mathcal{G}}$. Let $(\omega(\overline{\omega}), I^1_x(\overline{\omega}), J^1_x(\overline{\omega}), I^2_x(\overline{\omega}), J^2_x(\overline{\omega}))$, $x \in \mathbb{Z}^2$, denote the coordinate projections of an element $\overline{\omega} \in \overline{\Omega}$. Let $\overline{T} = (\overline{T}_x)_{x\in\mathbb{Z}^2}$ be the natural group of shifts on $\overline{\Omega}$.

Theorem A.2. — *Fix $0 < r < 1$ and let the bulk weights $\{\omega_x : x \in \mathbb{Z}^2\}$ be i.i.d.* Geom(r) *random variables. Then* (2.1) *is satisfied and for each $r < q_1 < q_2 < 1$ there exist a $\overline{T}$-invariant probability measure $\overline{\mathbb{P}}_{q_1,q_2}$ on $(\overline{\Omega}, \overline{\mathcal{G}})$ such that the following properties hold.*

(a) *The properties in Theorem A.1*(a-e) *all hold:*

(a.i) *$\mathbb{P}$ is the restriction of $\overline{\mathbb{P}}_{q_1,q_2}$ onto Ω.*

(a.ii) *For any $A \subset \mathbb{Z}^2$, the process $\{\omega_x, I_x^1, J_x^1, I_x^2, J_x^2 : x \in A\}$ is independent of $\{\omega_x : x \in A^>\}$.*

(a.iii) *For each $\ell \in \{1,2\}$ and $x \in \mathbb{Z}^2$, I_x^ℓ and J_x^ℓ are integrable and*

$$\widehat{\mathbb{E}}[(I_x^\ell, J_x^\ell)] = \nabla\gamma(\overline{\xi}(q_\ell)) = \Big(\frac{q_\ell}{1-q_\ell}, \frac{r}{q_\ell - r}\Big). \tag{A.5}$$

(a.iv) *There exists an event $\overline{\Omega}_0$ such that $\overline{\mathbb{P}}_{q_1,q_2}(\overline{\Omega}_0) = 1$ and the following all hold for $\overline{\omega} \in \overline{\Omega}_0$:*

(a.iv.1) *For each $x, y \in \mathbb{Z}^2$ and $\ell \in \{1,2\}$, $I_x^\ell(\overline{T}_y\overline{\omega}) = I_{x+y}^\ell(\overline{\omega})$ and $J_x^\ell(\overline{T}_y\overline{\omega}) = J_{x+y}^\ell(\overline{\omega})$.*

(a.iv.2) *For each $x \in \mathbb{Z}^2$ and $\ell \in \{1,2\}$ we have $\omega_x = I_x^\ell \wedge J_x^\ell$,*

$$\omega_x \leqslant I_x^2 \leqslant I_x^1, \quad \text{and} \quad \omega_x \leqslant J_x^1 \leqslant J_x^2.$$

(a.iv.3) *For $\ell \in \{1,2\}$, if we define $G_{x,y}^\ell$ as in (A.1-A.2), with ξ replaced by ℓ, then for all $x \leqslant y \leqslant z$ in $\mathbb{Z}^2$ we have*

$$G_{x,y}^\ell + G_{y,z}^\ell = G_{x,z}^\ell. \tag{A.6}$$

In particular, for any $x \in \mathbb{Z}^2$

$$I_{x+\mathbf{e}_1}^\ell + J_{x+\mathbf{e}_1+\mathbf{e}_2}^\ell = J_{x+\mathbf{e}_2}^\ell + I_{x+\mathbf{e}_1+\mathbf{e}_2}^\ell. \tag{A.7}$$

(a.v) *For each $u \geqslant v$ in $\mathbb{Z}^2_{\geqslant 0}$*

$$\big\{(G_{u,v+x}^\ell - G_{u,v}^\ell : x \in \mathbb{Z}^2_{\geqslant 0}, \ell \in \{1,2\}\big\} \overset{\mathrm{d}}{=} \big\{(G_{u,u+x}^\ell : x \in \mathbb{Z}^2_{\geqslant 0}, \ell \in \{1,2\}\big\}.$$

In addition, we have the following independence properties.

(b) *The vertical increments $\{J_{u+j\mathbf{e}_2}^1 : j \leqslant 0\}$ and $\{J_{u+j\mathbf{e}_2}^2 : j \geqslant 1\}$ are mutually independent. Similarly, the horizontal increments $\{I_{u+i\mathbf{e}_1}^2 : i \leqslant 0\}$ and $\{I_{u+i\mathbf{e}_1}^1 : i \geqslant 1\}$ are mutually independent.*

(c) *For each $\ell \in \{1,2\}$, the increment variables $\{I_{u+i\mathbf{e}_1}^\ell, J_{u+j\mathbf{e}_2}^\ell : i, j \geqslant 1\}$ are mutually independent. Also the increment variables $\{I_{u-i\mathbf{e}_1}^\ell, J_{u-j\mathbf{e}_2}^\ell : i \geqslant 0, j \geqslant 0\}$ are mutually independent.*

(d) *For each $i \geqslant 1$, $j \geqslant 1$, and $\ell \in \{1,2\}$, the increments have marginal distributions: $I_{u+i\mathbf{e}_1}^\ell \sim \mathrm{Geom}(q_\ell)$ and $J_{u+j\mathbf{e}_2}^\ell \sim \mathrm{Geom}(r/q_\ell)$.*

Remark A.3. — The edge weights needed to define the stationary boundary models we used in Sections 4 and 5 came from the process produced by Theorem A.1. Theorem A.2 can be used just the same to produce these edge weights, since by Theorem A.2(a), the process $\{\omega_x, I_x^1, J_x^1, I_x^2, J_x^2 : x \in \mathbb{Z}^2\}$, under $\overline{\mathbb{P}}_{q_1,q_2}$, satisfies all the properties of the process $\{\omega_x, I_x^{\overline{\xi}(q_1)}, J_x^{\overline{\xi}(q_1)}, I_x^{\overline{\xi}(q_2)}, J_x^{\overline{\xi}(q_2)} : x \in \mathbb{Z}^2\}$, under $\widehat{\mathbb{P}}$. The advantage of using the process from Theorem A.2 is that in the case of geometric weights, one has the additional independence properties in Theorem A.2(b-d). These properties are used to verify that Assumptions 4.1 and 5.2 hold when the weights are geometric random variables. In the rest of this appendix (specifically, in Corollary A.4, Theorem B.1, and Lemma C.2 below), although we continue using the notation from Theorem A.1, we mean to use the process from Theorem A.2. We also remark that the proof of Theorem 5.6 in [**?**] implies that the two processes actually have the same distribution, but we do not need this fact.

Corollary A.4. — *Fix $0 < r < 1$ and let the bulk weights $\{\omega_x : x \in \mathbb{Z}^2\}$ be i.i.d.* $\mathrm{Geom}(r)$ *random variables. Let $\xi_\star, \xi^\star, \eta_\star, \eta^\star \in \mathrm{ri}\,\mathcal{U}$ be such that $\xi_\star \cdot \mathbf{e}_1 < \xi^\star \cdot \mathbf{e}_1$ and $\eta_\star \cdot \mathbf{e}_1 > \eta^\star \cdot \mathbf{e}_1$. The processes $\big\{S_m^{\xi^\star,\eta_\star} : m \in [\![-N^{2/3}, -1]\!]\big\}$ and $\big\{S_n^{\xi_\star,\eta^\star} : n \in [\![1, N^{2/3}]\!]\big\}$, as defined in* (5.2)*, are independent.*

Proof. — Examining the construction in the proof of the previous theorem one sees that the processes $\{J_x^{\xi_\star}, J_x^{\xi^\star}\}_{x\in\mathcal{I}}$ and $\{\widehat{J}_{x+\mathbf{e}_1}^{\eta_\star}, \widehat{J}_{x+\mathbf{e}_1}^{\eta^\star}\}_{x\in\mathcal{I}}$ can be constructed simultaneously. Then the independence of $\{\omega_x : x \in \mathbb{Z}_{\geqslant 1} \times \mathbb{Z}\}$ and $\{\omega_x : x \in \mathbb{Z}_{\leqslant 0} \times \mathbb{Z}\}$ implies that the joint distribution of the two process (that are now defined on a larger, product space) is in fact a product measure and the two processes are independent.

Next, Theorem A.2 says that $\{J_{j\mathbf{e}_2}^{\xi^\star}\}_{j\leqslant 0}$ is independent of $\{J_{j\mathbf{e}_2}^{\xi_\star}\}_{j\geqslant 1}$ and that $\{\widehat{J}_{\mathbf{e}_1+(j-1)\mathbf{e}_2}^{\eta_\star}\}_{j\leqslant 0}$ is independent of $\{\widehat{J}_{\mathbf{e}_1+(j-1)\mathbf{e}_2}^{\eta^\star}\}_{j\geqslant 1}$. The claim follows from these independence properties. □

The proof of Theorem A.2 follows closely that of [8, Theorem 3.1]. It is based on a few results from queuing theory. The queuing-theoretic interpretation is not important for this paper; however, it gives some intuition behind the algebra that follows. To this end, consider a *queue* or *service station* with a single *server* and unbounded room for customers waiting to be served. Index the bi-infinite sequence of customers by j. The server serves one customer at a time. Once the service of customer j is complete, they leave the queue and customer $j+1$ enters service if they were already waiting in the queue. If the queue is empty after the departure of customer j, then the server remains idle until customer $j+1$ arrives. Let $\mathbf{s} = (s_j)_{j\in\mathbb{Z}}$ denote the service process, i.e. s_j is the time it takes to service customer j. Let $\mathbf{a} = (a_j)_{j\in\mathbb{Z}}$ be the inter-arrival process, i.e. a_j is the time elapsed between the arrivals of customers $j-1$ and j. Assume that

$$\lim_{n\to-\infty}\sum_{i=n}^{0}(s_i - a_{i+1}) = -\infty. \tag{A.8}$$

Let $\mathbf{G} = (G_j)_{j\in\mathbb{Z}}$ be a sequence of customer arrival times such that $a_j = G_j - G_{j-1}$. Define the sequence $\widetilde{\mathbf{G}} = (\widetilde{G}_j)_{j\in\mathbb{Z}}$ by

$$\widetilde{G}_j = \sup_{k\leqslant j}\Big\{G_k + \sum_{i=k}^{j} s_i\Big\}. \tag{A.9}$$

Condition (A.8) guaranties that the supremum is achieved and that $\widetilde{G}_j$ is a finite real number. The recurrence relation

$$\widetilde{G}_j = (\widetilde{G}_{j-1} + s_j) \vee (G_j + s_j) \tag{A.10}$$

provides a natural interpretation of $\widetilde{G}_j$ as the time customer j leaves the service station.

It is noteworthy that (A.9) is not the only solution to (A.10). For example, the sequence that is identically equal to ∞ is another solution. However, adapting the proof of [35, Lemma 4.3] to the current setting shows that under the assumptions that (s_j) is i.i.d., (a_j) is ergodic and independent of (s_j), and the mean of a_j is strictly larger than the mean of s_j, (A.9) is the unique stationary almost surely finite solution to (A.10).

Define the inter-departure process $\mathbf{d} = (d_j)_{j\in\mathbb{Z}} = D(\mathbf{a},\mathbf{s})$ by $d_j = \widetilde{G}_j - \widetilde{G}_{j-1}$. Define the sojourn process $\mathbf{t} = (t_j)_{j\in\mathbb{Z}} = S(\mathbf{a},\mathbf{s})$ by $t_j = \widetilde{G}_j - G_j$. Define the dual service times $\check{\mathbf{s}} = (\check{s}_j)_{j\in\mathbb{Z}} = R(\mathbf{a},\mathbf{s})$ by $\check{s}_j = a_j \wedge t_{j-1}$. These definitions do not depend on the particular sequence $\mathbf{G}$ which was selected.

Note that

$$t_j + a_j = \widetilde{G}_j - G_{j-1} = t_{j-1} + d_j. \tag{A.11}$$

Also, (A.10) implies

$$s_j \leqslant d_j \quad \text{for all } j\in\mathbb{Z}. \tag{A.12}$$

Subtracting G_j from both sides in (A.9) and expanding $G_k - G_j = -\sum_{i=k+1}^{j} a_i$ shows that

(A.13) the sojourn times t_j are non-increasing functions of the inter-arrival times a_j.

The following is Lemma A.1 from [8].

Lemma A.5. — *The following holds for any $\mathbf{a},\mathbf{b},\mathbf{s}$ for which all the involved departure times are defined:*

$$D(D(\mathbf{b},\mathbf{a}),\mathbf{s}) = D(D(\mathbf{b},R(\mathbf{a},\mathbf{s})),D(\mathbf{a},\mathbf{s})).$$

For horizontal edge weight I, vertical edge weight J, and vertex weight ω, define

$$I' = \omega + (I-J)^+,\quad J' = \omega + (I-J)^-,\quad \text{and}\quad \omega' = I\wedge J.$$

The next lemma can be proved for example using Laplace transforms. It is essentially a consequence of the memoryless property of the Geometric distribution.

Lemma A.6. — *Fix $0<r<1$ and $r\leqslant q\leqslant 1$. Let $\omega\sim\mathrm{Geom}(r)$, $I\sim\mathrm{Geom}(q)$, and $J\sim\mathrm{Geom}\left(\frac{r}{q}\right)$ be independent. Then the following hold.*

(a) *$I-J$ and $I\wedge J$ are independent.*

(b) *The distribution of* $(I-J)^+$ *is the same as that of the product of a* $\mathrm{Ber}\big(\frac{q-r}{q(1-r)}\big)$ *and an independent* $\mathrm{Geom}(q)$ *random variables.*

(c) *The triple* (I', J', ω') *has the same distribution as* (I, J, ω).

Take $0 < \sigma < \alpha_1 < \alpha_2$ in $(0,1)$. Let $\mathbf{b}^i$ be an i.i.d. sequence of $\mathrm{Geom}(\alpha_i)$ random variables for $i \in \{1,2\}$ and let $\mathbf{s}$ be an i.i.d. sequence of $\mathrm{Geom}(\sigma)$ random variables, which are all mutually independent. Define the arrival sequences $(\mathbf{a}^1, \mathbf{a}^2) = (\mathbf{b}^1, D(\mathbf{b}^2, \mathbf{b}^1))$. Define $\mathbf{d}^k = D(\mathbf{a}^k, \mathbf{s})$, $\mathbf{t}^k = S(\mathbf{a}^k, \mathbf{s})$, and $\check{\mathbf{s}}^k = R(\mathbf{a}^k, \mathbf{s})$ for $k \in \{1,2\}$.

Lemma A.7. — *The following statements are true.*

(a) *Marginally,* $\mathbf{a}^2$ *is a sequence of i.i.d.* $\mathrm{Geom}(\alpha_2)$ *random variables.*

(b) *For each* $k \in \{1,2\}$ *and* $m \in \mathbb{Z}$*, the random variables* $\{d^k_j\}_{j \leqslant m}$, t^k_m, *and* $\{\check{s}^k_j\}_{j \leqslant m}$ *are mutually independent. Their marginal distributions are* $d^k_j \sim \mathrm{Geom}(\alpha_k)$, $t^k_m \sim \mathrm{Geom}\big(\frac{\sigma}{\alpha_k}\big)$, *and* $\check{s}^k_j \sim \mathrm{Geom}(\sigma)$.

(c) *For each* $k \in \{1,2\}$*, the sequences* $\mathbf{d}^k$ *and* $\check{\mathbf{s}}^k$ *are mutually independent. Their marginal distributions are* $d^k_j \sim \mathrm{Geom}(\alpha_k)$ *and* $\check{s}^k_j \sim \mathrm{Geom}(\sigma)$.

(d) $(\mathbf{d}^1, \mathbf{d}^2) \overset{\mathrm{d}}{=} (\mathbf{a}^1, \mathbf{a}^2)$.

(e) *For each* $m \in \mathbb{Z}$*, the random variables* $\{a^2_i\}_{i \leqslant m}$ *and* $\{a^1_j\}_{j \geqslant m+1}$ *are mutually independent.*

The proof of the first three claims follows from Lemma B.2 of [**?**] by replacing the exponential version of the induction with the geometric version in Lemma A.6. The proof of the last two claims follows from Lemma A.2 of [8] with the same replacements. Note that (A.12) and (A.13) imply

$$a^1_j \leqslant a^2_j \quad \text{and} \quad t^1_j \geqslant t^2_j \quad \text{for all } j \in \mathbb{Z}. \tag{A.14}$$

Proof of Theorem A.2. — Fix $u \in \mathbb{Z}^2$. We start by constructing a joint LPP process $(L^1_x, L^2_x)_{x \in u+\mathbb{Z}_{\geqslant 0}\times\mathbb{Z}}$. In the bulk, we have the i.i.d. $\mathrm{Geom}(r)$ weights $\{\omega_x : x_1 > u_1\}$. For $\ell \in \{1,2\}$, let $\mathbf{Y}^\ell = \{Y^\ell_j\}_{j\in\mathbb{Z}}$ be a sequence of i.i.d. $\mathrm{Geom}(r/q_\ell)$ random variables such that $\{\mathbf{Y}^1, \mathbf{Y}^2, \omega\}$ are mutually independent. Note that $q_1 < q_2$ implies that (A.8) holds almost surely with $\mathbf{s} = \mathbf{Y}^1$ and $\mathbf{a} = \mathbf{Y}^2$. For $\ell \in \{1,2\}$, define $\mathbf{J}^\ell = \{J^\ell_{u+j\mathbf{e}_2}\}_{j\in\mathbb{Z}}$ by $(\mathbf{J}^1, \mathbf{J}^2) = (\mathbf{Y}^1, D(\mathbf{Y}^2, \mathbf{Y}^1))$. By Lemma A.7(a), marginally $\{J^\ell_{u+j\mathbf{e}_2}\}_{j\in\mathbb{Z}}$ are i.i.d. $\mathrm{Geom}(r/q_\ell)$.

For $\ell \in \{1,2\}$, define the LPP values on this vertical axis by

$$L^\ell_u = 0 \quad \text{and} \quad L^\ell_{u+j\mathbf{e}_2} - L^\ell_{u+(j-1)\mathbf{e}_2} = J^\ell_{u+j\mathbf{e}_2} \quad \text{for } j \in \mathbb{Z}. \tag{A.15}$$

Note that this means $L^\ell_{u+j\mathbf{e}_2}$ is negative for $j < 0$. Now, we define the LPP values for $x \in u + \mathbb{Z}_{>0} \times \mathbb{Z}$:

$$L^\ell_x = \sup_{j : j \leqslant x_2 - u_2} \big\{L^\ell_{u+j\mathbf{e}_2} + G_{u+\mathbf{e}_1+j\mathbf{e}_2, x}\big\}, \quad I^\ell_x = L^\ell_x - L^\ell_{x-\mathbf{e}_1}, \quad \text{and} \quad J^\ell_x = L^\ell_x - L^\ell_{x-\mathbf{e}_2}. \tag{A.16}$$

The supremum is achieved at a finite j because the boundary variables J^ℓ stochastically dominate the bulk weights ω, as we show next. Note that one has

$$I^\ell_x \wedge J^\ell_x = \omega_x \quad \text{for all } x \in u + \mathbb{Z}_{>0} \times \mathbb{Z} \text{ and } \ell \in \{1,2\}. \tag{A.17}$$

For $k \geqslant 0$ and $\ell \in \{1,2\}$ let $\mathbf{J}^{\ell,k} = \{J^{\ell,k}_j\}_{j\in\mathbb{Z}} = \{J^\ell_{u+k\mathbf{e}_1+j\mathbf{e}_2}\}_{j\in\mathbb{Z}}$ and $\mathbf{s}^k = \{s^k_j\}_{j\in\mathbb{Z}} = \{\omega_{u+k\mathbf{e}_1+j\mathbf{e}_2}\}_{j\in\mathbb{Z}}$. Then $\mathbf{J}^{\ell,0}$ is the original boundary sequence on the vertical axis. In the notation of Lemma A.7, with $\sigma = r$, $\alpha_1 = q_1$, and $\alpha_2 = q_2$, setting $\mathbf{b}^\ell = \mathbf{Y}^\ell$ gives $(\mathbf{a}^1, \mathbf{a}^2) = (\mathbf{J}^{q_1}, \mathbf{J}^{q_2})$. Then, for any $\ell \in \{1,2\}$, (A.8) is satisfied, $G^\ell_j = L^\ell_{u+j\mathbf{e}_2}$, $j \in \mathbb{Z}$, is a sequence of arrival times, and $\widetilde{G}^\ell_j = L^\ell_{u+\mathbf{e}_1+j\mathbf{e}_2}$, $j \in \mathbb{Z}$, is the corresponding sequence of departure times. Consequently, $\mathbf{J}^{\ell,1} = D(\mathbf{J}^{\ell,0}, \mathbf{s}^1)$. Lemma A.7(d) then implies $(\mathbf{J}^{1,1}, \mathbf{J}^{2,1}) \overset{\mathrm{d}}{=} (\mathbf{J}^1, \mathbf{J}^2)$. Repeating this inductively gives that $\mathbf{J}^{\ell,k+1} = D(\mathbf{J}^{\ell,k}, \mathbf{s}^{k+1})$ and $(\mathbf{J}^{1,k}, \mathbf{J}^{2,k}) \overset{d}{=} (\mathbf{J}^1, \mathbf{J}^2)$ for all $k \geqslant 0$. This and the first inequality in (A.14) imply

$$J^1_x \leqslant J^2_x \quad \text{for all } x \in u + \mathbb{Z}_+ \times \mathbb{Z}. \tag{A.18}$$

Furthermore, Lemma A.7(e) implies that for any $x \in u + \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}$,

$$\big\{J^2_{x+j\mathbf{e}_2} : j \leqslant 0\big\} \quad \text{and} \quad \big\{J^1_{x+j\mathbf{e}_2} : j \geqslant 1\big\} \quad \text{are mutually independent.} \tag{A.19}$$

The definition (A.16) satisfies a semi-group property: For each $k \geqslant 0$, the values L^ℓ_x for x such that $x_1 > u_1 + k + 1$ satisfy

$$L^\ell_x = \sup_{j:j\leqslant x_2-u_2} \left\{ L^\ell_{u+k\mathbf{e}_1+j\mathbf{e}_2} + G_{u+(k+1)\mathbf{e}_1+j\mathbf{e}_2,x} \right\}. \tag{A.20}$$

This and the distributional equality $(\mathbf{J}^{1,k}, \mathbf{J}^{2,k}) \stackrel{d}{=} (\mathbf{J}^1, \mathbf{J}^2)$ imply that for any $z \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}$,

$$\left\{ I^1_{z+x+\mathbf{e}_1}, I^2_{z+x+\mathbf{e}_1}, J^1_{z+x}, J^2_{z+x} : x \in u + \mathbb{Z}_{\geqslant 0} \times \mathbb{Z} \right\} \stackrel{d}{=} \left\{ I^1_{x+\mathbf{e}_1}, I^2_{x+\mathbf{e}_1}, J^1_x, J^2_x : x \in u + \mathbb{Z}_{\geqslant 0} \times \mathbb{Z} \right\}. \tag{A.21}$$

The index on the I increments requires $x + \mathbf{e}_1$ because the increments are not defined on the boundary, where $x_1 = u_1$.

Next, we claim that for $\ell \in \{1, 2\}$, and for any $u \in \mathbb{Z}^2$,

$$\begin{aligned} &\left\{ I^\ell_{u+i\mathbf{e}_1}, J^\ell_{u+j\mathbf{e}_2} : i, j \in \mathbb{Z}_{>0} \right\} \text{ are mutually independent with marginal distributions} \\ &I^\ell_{u+i\mathbf{e}_1} \sim \mathrm{Geom}(q_\ell) \text{ and } J^\ell_{u+j\mathbf{e}_2} \sim \mathrm{Geom}(r/q_\ell). \end{aligned} \tag{A.22}$$

We have already shown that $\mathbf{J}^\ell$ are i.i.d. $\mathrm{Geom}(r/q_\ell)$ random variables. Also notice that $\{I^\ell_{u+i\mathbf{e}_1} : i \geqslant 1\}$ are a function of only $\{J^\ell_{u+j\mathbf{e}_2}, \omega_{u+i\mathbf{e}_1+j\mathbf{e}_2} : i \geqslant 1, j \leqslant 0\}$ which are independent of $\{J^\ell_{u+j\mathbf{e}_2} : j \geqslant 1\}$. What remains to prove is that the horizontal increments are i.i.d. and to determine their maginal distribution. For this, we prove the following claim inductively in $n \geqslant 1$:

$$\begin{aligned} &\left\{ I^\ell_{u+i\mathbf{e}_1}, J^\ell_{u+n\mathbf{e}_1+j\mathbf{e}_2} : 1 \leqslant i \leqslant n, j \leqslant 0 \right\} \text{ are mutually independent with} \\ &\text{marginal distributions } I^\ell_{u+i\mathbf{e}_1} \sim \mathrm{Geom}(q_\ell) \quad \text{and} \quad J^\ell_{u+n\mathbf{e}_1+j\mathbf{e}_2} \sim \mathrm{Geom}(r/q_\ell). \end{aligned} \tag{A.23}$$

This and the fact that $I^\ell_{u+(n+1)\mathbf{e}_1}$ is a function of $\{J^\ell_{u+n\mathbf{e}_1+j\mathbf{e}_2}, \omega_{u+(n+1,j)} : j \leqslant 0\}$ imply the mutual independence of the horizontal increments.

We now prove (A.22). For the base case $n = 1$, consider inter-arrival times $\{a_j = J^\ell_{u+j\mathbf{e}_2} : j \leqslant 0\}$ and service times $\{s_j = \omega_{u+\mathbf{e}_1+j\mathbf{e}_2} : j \leqslant 0\}$. The inter-departure times are $\{d_j = J^\ell_{u+\mathbf{e}_1+j\mathbf{e}_2} : j \leqslant 0\}$. The sojourn time is $t_0 = I^\ell_{u+\mathbf{e}_1}$. Lemma A.7(b) then gives the above claim for $n = 1$.

For the inductive step, assume the claim holds for a fixed $n \geqslant 1$. Then use inter-arrival times $\{a_j = J^\ell_{u+n\mathbf{e}_1+j\mathbf{e}_2} : j \leqslant 0\}$ and service times $\{s_j = \omega_{u+(n+1)\mathbf{e}_1+j\mathbf{e}_2} : j \leqslant 0\}$ which are independent of $\{I^\ell_{u+i\mathbf{e}_1} : 1 \leqslant i \leqslant n\}$ by the inductive hypothesis. Then compute the corresponding inter-departure times $\{d_j = J_{u+(n+1)\mathbf{e}_1+j\mathbf{e}_2} : j \leqslant 0\}$ and the sojourn time $t_n = I^\ell_{u+(n+1)\mathbf{e}_1}$. Lemma A.7(b) again gives the validity of the claim for $n + 1$, completing the proof of the claim (A.23).

Combining (A.11) with observation that I^ℓ_x are sojourn times gives

$$I^\ell_{x+\mathbf{e}_1} + J^\ell_{x+\mathbf{e}_1+\mathbf{e}_2} = J^\ell_{x+\mathbf{e}_2} + I^\ell_{x+\mathbf{e}_1+\mathbf{e}_2} \qquad \text{for all } x \in u + \mathbb{Z}_{\geqslant 0} \times \mathbb{Z} \text{ and } \ell \in \{1, 2\}. \tag{A.24}$$

And with the second inequality in (A.14) we get

$$I^1_x \geqslant I^2_x \quad \text{for all } x \in \mathbb{Z}_{>0} \times \mathbb{Z}. \tag{A.25}$$

Lastly, observe that $\{L^\ell_x : x \in u + \mathbb{Z}^2_{\geqslant 0}\}$ are last passage times with boundary weights $\{I^\ell_{u+i\mathbf{e}_1}, J^\ell_{u+j\mathbf{e}_2} : i, j \in \mathbb{Z}_{>0}\}$ and bulk weights ω_x, $x \in u + \mathbb{N}^2$. Indeed, if we denote by $G^\ell_{u,x}$ the passage time from u to $x \in u + \mathbb{N}^2$ with these boundary and bulk weights, then as in (3.2)

$$\begin{aligned} G^\ell_{u,x} &= \max_{1\leqslant k\leqslant x_1-u_1} \Big\{ \sum_{i=1}^k I^\ell_{u+i\mathbf{e}_1} + G_{u+k\mathbf{e}_1+\mathbf{e}_2,x} \Big\} \bigvee \max_{1\leqslant m\leqslant x_2-u_2} \Big\{ \sum_{j=1}^\ell J^\ell_{u+j\mathbf{e}_2} + G_{u+\mathbf{e}_1+m\mathbf{e}_2,x} \Big\} \\ &= \max_{1\leqslant k\leqslant x_1-u_1} \left\{ L^\ell_{u+k\mathbf{e}_1} + G_{u+k\mathbf{e}_1+\mathbf{e}_2,x} \right\} \bigvee \max_{1\leqslant m\leqslant x_2-u_2} \left\{ L^\ell_{u+m\mathbf{e}_2} + G_{u+\mathbf{e}_1+m\mathbf{e}_2,x} \right\} \\ &= \sup_{j\leqslant 0} \Big\{ L^\ell_{u+j\mathbf{e}_2} + \max_{1\leqslant k\leqslant x_1-u_1} \left[G_{u+\mathbf{e}_1+j\mathbf{e}_2,u+k\mathbf{e}_1} + G_{u+k\mathbf{e}_1+\mathbf{e}_2,x} \right] \Big\} \bigvee \max_{1\leqslant m\leqslant x_2-u_2} \left\{ L^\ell_{u+m\mathbf{e}_2} + G_{u+\mathbf{e}_1+\ell\mathbf{e}_2,x} \right\} \\ &= \sup_{j:j\leqslant x_2-u_2} \left\{ L^\ell_{u+j\mathbf{e}_2} + G_{u+\mathbf{e}_1+j\mathbf{e}_2,x} \right\} = L^\ell_x. \end{aligned}$$

By (A.17), the weights ω_x can be recovered from the edge weights I^ℓ_x and J^ℓ_x. Then, due to (A.21) we can extend the process $\left\{ \omega_x, I^1_{x+\mathbf{e}_1}, I^2_{x+\mathbf{e}_1}, J^1_x, J^2_x : x \in u + \mathbb{Z}_{\geqslant 0} \times \mathbb{Z} \right\}$ to a stationary process on the whole lattice.

This produces a $\overline{T}$-invariant probability measure $\overline{\mathbb{P}}_{q_1,q_2}$ on $(\overline{\Omega}, \overline{\mathcal{G}})$ whose marginal on Ω is exactly $\mathbb{P}$. We now verify that all the claims in the theorem hold for this choice of measure.

Property (a.i) holds by construction and the independence property (a.ii) follows from the definition (A.16). Recall that an $\overline{\omega} \in \overline{\Omega}$ has coordinate projections $\{\omega_x, I^1_x, I^2_x, J^1_x, J^2_x : x \in \mathbb{Z}^2\}$. Thus, the shift-covariance in (a.iv.1) holds trivially. The recovery and monotonicity properties in (a.iv.2) follow from (A.17), (A.18), and (A.25). The additivity property (A.7) is given in (A.24) and (A.6) follows from that. Then, as it was the case for Theorem A.1(e), property (a.v) follows from (A.6) and the shift-invariance of $\overline{\mathbb{P}}_{q_1,q_2}$.

Observe that (I^ℓ_x, J^ℓ_x) has mean $\big(q_\ell/(1-q_\ell), r/(q_\ell - r)\big)$. A direct computation using the explicit formulas (3.11) and (3.12) shows that this is equal to $\nabla\gamma(\overline{\xi}(q_\ell))$. This completes the proof of part (a) of the theorem. Part (b) follows from (A.19) and parts (c) and (d) from (A.22) and (A.23). □

Appendix B. Verifying Assumption 4.1 for the geometric LPP

This appendix is dedicated to the proof of an exponential tail bound for the location of exit points. It can be read independently of the rest of the paper. We assume throughout the section that $\omega_0 \sim \mathrm{Geom}(r)$ for a given $r \in (0,1)$.

For $\delta \in (0,1)$ recall the definition of the cone

$$S_\delta = \{x \in \mathbb{R}^2_{>0} : x\cdot \mathbf{e}_1 \geqslant \delta x \cdot \mathbf{e}_2 \text{ and } x\cdot\mathbf{e}_2 \geqslant \delta x\cdot \mathbf{e}_1\}.$$

Theorem B.1. — *Assume $\omega_0 \sim \mathrm{Geom}(r)$ for some $r \in (0,1)$. For any $\delta \in (0,r)$ and $\kappa \geqslant 0$ there exist positive finite constants $c_0 = c_0(\delta, r)$, $N_0 = N_0(\delta, r, \kappa)$, and $s_0 = s_0(\delta, r, \kappa)$ such that*

$$\mathbb{P}\big\{|Z^{\xi,\mathbf{e}_1}(m,n)| \vee |Z^{\xi,\mathbf{e}_2}(m,n)| \geqslant s(m+n)^{2/3}\big\} \leqslant \exp\{-c_0 s^3\}$$

for all $(m,n) \in S_\delta \cap \mathbb{Z}^2_{\geqslant N_0}$, $s \geqslant s_0$, and $\xi \in \mathrm{ri}\,\mathcal{U}$ such that $\xi_1 \in (\delta, 1-\delta)$ and $|\xi_1 - \frac{m}{m+n}| \leqslant \kappa(m+n)^{-1/3}$.

For $p, q \in (r,1)$ consider random variables $\{I^p_{i\mathbf{e}_1}, I^q_{i\mathbf{e}_1}, J^p_{i\mathbf{e}_2}, J^q_{i\mathbf{e}_2} : i, i \in \mathbb{N}\}$ that are mutually independent and independent of the weights ω and such that the I^p variables are $\mathrm{Geom}(p)$, the I^q variables are $\mathrm{Geom}(q)$, the J^p variables are $\mathrm{Geom}(r/p)$, and the J^q variables are $\mathrm{Geom}(r/q)$. Note that this parametrization in terms of p and q does not agree with the parametrization of the I and J random variables elsewhere in the paper. This abuse of notation is to simplify the formulas in this section; we will also abuse notation and continue to write $\mathbb{P}$ and $\mathbb{E}$ for the probability and expectation on the larger probability space on which this collection of random variables is defined.

We will write

$$G^{p,q}_x = G^{\mathrm{SW}}_x(\omega, I^p, J^q), \quad G^p_x = G^{\mathrm{SW}}_x(\omega, I^p, J^p), \quad \text{and} \quad G^q_x = G^{\mathrm{SW}}_x(\omega, I^q, J^q). \tag{B.1}$$

The quantities $\mathrm{Exit}^{p,q}_x$, $Z^{p,q,\mathbf{e}_k}_x$, Exit^p_x, $Z^{p,\mathbf{e}_k}_x$, Exit^q_x, and $Z^{q,\mathbf{e}_k}_x$ are defined similarly.

From (3.9) (which follows from Theorem A.2(c)) we see that $\{G^p_x : x \in \mathbb{Z}^2_{\geqslant 0}\}$ has the same distribution as $\{G^{\overline{\xi}(p)}_x : x \in \mathbb{Z}^2_{\geqslant 0}\}$. The same, of course, holds when p is replaced by q.

One of our major uses of the exact solvability of the model comes through an exact formula for a particular log moment generating function of the increment stationary passage time.

Proposition B.2. — *Let $m, n \in \mathbb{Z}^2_{\geqslant 0}$, and $p, q \in (r,1)$. Then*

$$\log \mathbb{E}\Big[\exp\Big\{\log\big(\frac{q}{p}\big) G^{p,q}(m,n)\Big\}\Big] = m\log\big(\frac{1-p}{1-q}\big) + n\log\Big(\frac{1-\frac{r}{q}}{1-\frac{r}{p}}\Big).$$

Proof. — Start by writing

$$\begin{aligned}\log \mathbb{E}\Big[\exp\Big\{\log\big(\frac{q}{p}\big) G^{p,q}(m,n)\Big\}\Big] &= \log \mathbb{E}\Big[\prod_{i=1}^m (q/p)^{I^p_{(i,0)}}\, e^{\log(q/p)(G^{p,q}(m,n) - G^{p,q}(m,0))}\Big] \\ &= m\log\big(\frac{1-p}{1-q}\big) + \log\mathbb{E}\Big[\prod_{i=1}^m \big(\frac{1-q}{1-p}\cdot (q/p)^{I^p_{(i,0)}}\big) e^{\log(q/p)(G^{p,q}(m,n)-G^{p,q}(m,0))}\Big].\end{aligned} \tag{B.2}$$

Next, note that $\mathbb{E}[\frac{1-q}{1-p}(q/p)^{I^p_{i\mathbf{e}_1}}] = 1$ and for any $n \in \mathbb{Z}_{\geqslant 0}$

$$\mathbb{E}\Big[\frac{1-q}{1-p}(q/p)^{I^p_{i\mathbf{e}_1}}\mathbb{1}\{I^p_{i\mathbf{e}_1} = n\}\Big] = q^n(1-q).$$

This means that the product inside the expectation on the right-hand side in (B.2) is a Radon-Nikodym derivative and using it to change the measure $\mathbb{P}$ switches the distribution of the boundary I^p weights to have the same distribution as the I^q weights. Consequently, (B.2) is equal to

$$\begin{aligned} &m\log\big(\frac{1-p}{1-q}\big) + \log\mathbb{E}\Big[e^{\log(q/p)(G^q(m,n)-G^q(m,0))}\Big] \\ &= m\log\big(\frac{1-p}{1-q}\big) + \log\mathbb{E}\Big[e^{\log(q/p)(G^{\overline{\xi}(q)}(m,n)-G^{\overline{\xi}(q)}(m,0))}\Big] \\ &= m\log\big(\frac{1-p}{1-q}\big) + \log\mathbb{E}\Big[e^{\log(q/p)G^{\overline{\xi}(q)}(0,n)}\Big]. \end{aligned}$$

For the last equality we used the additivity (3.5) and the shift-invariance (3.8). Now, simply compute

$$\begin{aligned} m\log\big(\frac{1-p}{1-q}\big) + \log\mathbb{E}\Big[e^{\log(q/p)G^{\overline{\xi}(q)}(0,n)}\Big] &= m\log\big(\frac{1-p}{1-q}\big) + \log\mathbb{E}\Big[e^{\log(q/p)J^q_{(0,1)}}\Big]^n \\ &= m\log\big(\frac{1-p}{1-q}\big) + n\log\Big(\frac{1-\frac{r}{q}}{1-\frac{r}{p}}\Big). \end{aligned}$$

□

We prove Theorem B.1 after a series of calculus lemmas. The following lemma is immediate from the definitions. Recall (3.10-3.13).

Lemma B.3. — *Fix $a, b > 0$. The function $p \mapsto M^p(a,b)$ is continuous and strictly convex on $(r,1)$, decreasing on $(r,\overline{p}(a,b)]$ with range $[\gamma(a,b),\infty)$, and increasing on $[\overline{p}(a,b),1)$ with range $[\gamma(a,b),\infty)$.*

Consequently, for each $\lambda \in (1,1/r)$, there exists a unique pair $\overline{p}^\lambda_-(a,b) \in (r,\overline{p}(a,b))$ and $\overline{p}^\lambda_+(a,b) \in (\overline{p}(a,b),1)$ such that $\overline{p}^\lambda_+(a,b) = \lambda\overline{p}^\lambda_-(a,b)$ and $M^{\overline{p}^\lambda_-}(a,b) = M^{\overline{p}^\lambda_+}(a,b)$. Precisely, using a little bit of calculus, we get that if $a \neq rb$, then

$$\overline{p}^\lambda_-(a,b) = \frac{r(\lambda+1)(a-b) + \sqrt{r^2(\lambda+1)^2(a-b)^2 - 4r\lambda(ra-b)(a-rb)}}{2\lambda(a-rb)} \tag{B.3}$$

and if $a = rb$, then

$$\overline{p}^\lambda_-(a,b) = \frac{r+1}{\lambda+1}.$$

This extends continuously to $\lambda = 1$ and $\lambda = 1/r$ with $\overline{p}^1_\pm(a,b) = \overline{p}(a,b)$, $\overline{p}^{1/r}_-(a,b) = r$, and $\overline{p}^{1/r}_+(a,b) = 1$.

For $\xi \in \mathbb{R}^2_{>0}$ and $p, q \in (r,1)$, define

$$L^{p,q}(\xi) = L^{p,q}(\xi_1,\xi_2) = \xi_1\log\big(\frac{1-p}{1-q}\big) + \xi_2\log\Big(\frac{1-\frac{r}{q}}{1-\frac{r}{p}}\Big).$$

Then for $\xi \in \mathbb{R}_{>0}$ and $q \in [r,1)$ set

$$\mathcal{L}^{\lambda,q}(\xi) = \inf_{q<s<1/\lambda} L^{s,\lambda s}(\xi) \tag{B.4}$$

when $\lambda \in [1,1/q)$ and $\mathcal{L}^{\lambda,q}(\xi) = \infty$ when $\lambda \geqslant 1/q$. In the special case where $q = r$ we abbreviate $\mathcal{L}^\lambda(\xi) = \mathcal{L}^{\lambda,r}(\xi)$.

Lemma B.4. — *Let $\xi \in \mathbb{R}_{>0}$, $q \in [r,1)$, and $\lambda \in (1,1/q)$. Then the infimum in* (B.4) *is uniquely achieved at $s = \max\{q,\overline{p}^\lambda_-(\xi)\}$.*

Proof. — A direct computation gives

$$\frac{\partial}{\partial s}L^{s,\lambda s}(\xi) = \frac{1}{s}\big(M^{\lambda s}(\xi) - M^s(\xi)\big) \quad \text{for } s \in (r,1/\lambda).$$

By Lemma B.3, this is a continuous strictly increasing function of s with range $\mathbb{R}$. It is equal to zero at $s = \overline{p}^{\lambda}_{-}(\xi)$. If $\overline{p}^{\lambda}_{-}(\xi) \in (q, 1/\lambda)$, then the unique infinimum in (B.4) is attained at $\overline{p}^{\lambda}_{-}(\xi)$. If $\overline{p}^{\lambda}_{-}(\xi) \in (r, q]$, then $L^{s,\lambda s}(\xi)$ strictly increases on $(q, 1/\lambda)$ and is thus minimized at $s = q$. □

For $k, \ell, m, n \in \mathbb{Z}$ with $m > k$ and $n \geqslant \ell$ let $G^{q}_{(k,\ell)}(m, n|1, 0)$ be the last-passage time for paths which start at (k, ℓ), immediately take an $\mathbf{e}_1$ step, and then go to (m, n), while collecting the weights $\{I^{q}_{k+j\mathbf{e}_1,\ell} : j \in \mathbb{N}\}$ on the south boundary. Precisely,

$$G^{q}_{(k,\ell)}(m, n|1, 0) = \max_{1 \leqslant j \leqslant m-k} \Big\{ \sum_{i=1}^{j} I^{q}_{k+i\mathbf{e}_1,\ell} + G_{(k+j)\mathbf{e}_1+(\ell+1)\mathbf{e}_2, m\mathbf{e}_1+n\mathbf{e}_2} \Big\}.$$

When $(k, \ell) = \mathbf{0}$ we omit it from the index.

Lemma B.5. — *Let $m, n \in \mathbb{N}$, $q \in [r, 1)$, and $\lambda \geqslant 1$. Then*

$$\log \mathbb{E}\big[e^{\log(\lambda) G^{q}(m,n|1,0)}\big] \leqslant \mathcal{L}^{\lambda,q}(m, n).$$

Proof. — The case $\lambda \geqslant 1/q$ is trivial because the right-hand side is infinite. When $\lambda = 1$ we have $L^{s,s}(m, n) = 0$ for all $s \in (r, 1)$ and the claim is again trivial. Therefore, assume $\lambda \in (1, 1/q)$.

Using

$$G^{q,\lambda q}(m, n) \geqslant I^{q}_{\mathbf{e}_1} + G^{q}(m, n|1, 0) \geqslant G^{q}(m, n|1, 0)$$

and Proposition B.2 we see that

$$\log \mathbb{E}\big[e^{\log(\lambda) G^{q}(m,n|1,0)}\big] \leqslant \log \mathbb{E}\big[e^{\log(\lambda) G^{q,\lambda q}(m,n)}\big] = L^{q,\lambda q}(m, n). \tag{B.5}$$

Geometric random variables are stochastically increasing in the parameter. Therefore, if $q \leqslant \overline{p}^{\lambda}_{-}(m, n)$,

$$\log \mathbb{E}\big[e^{\log(\lambda) G^{q}(m,n|1,0)}\big] \leqslant \log \mathbb{E}\big[e^{\log(\lambda) G^{\overline{p}^{\lambda}_{-}(m,n)}(m,n|1,0)}\big] \leqslant L^{\overline{p}^{\lambda}_{-},\lambda \overline{p}^{\lambda}_{-}}(m, n)$$

where the last inequality follows from applying (B.5) with $\overline{p}^{\lambda}_{-}(m, n)$ in place of q.

We have thus shown that

$$\log \mathbb{E}\big[e^{\log(\lambda) G^{q}(m,n|1,0)}\big] \leqslant L^{\max\{q,\overline{p}^{\lambda}_{-}\},\lambda \max\{q,\overline{p}^{\lambda}_{-}\}}(m, n) = \mathcal{L}^{\lambda,q}(m, n),$$

where the equality holds by Lemma B.4. □

Lemma B.6. — *For all $a, b > 0$, $\varepsilon \in (0, \min(r, 1-s, (1-r)/2)$, $s \in (r, 1)$, and $\lambda \in \big[\max((r+\varepsilon)/s, 1), (1-\varepsilon)/s\big]$*

$$\Big| L^{s,\lambda s}(a, b) - (\lambda-1)\Big(\frac{as}{1-s} + \frac{br}{s-r}\Big) - \frac{1}{2}(\lambda-1)^2\Big(\frac{as^2}{(1-s)^2} - \frac{br(2s-r)}{(s-r)^2}\Big)\Big| \leqslant 2\varepsilon^{-3}(a+b)(\lambda-1)^3.$$

Proof. — Fix a, b, ε, and s as in the claim and perform a Taylor expansion of $\lambda \mapsto L^{s,\lambda s}(a, b)$, defined on $(r/s, 1/s)$, at $\lambda = 1$. For the error term write

$$\Big| \frac{\partial^3}{\partial \lambda^3} L^{s,\lambda s}(a, b)\Big| = \Big| \frac{2br(3\lambda^2 s^2 - 3\lambda r s + r^2)}{\lambda^3(\lambda s - r)^3} + \frac{2as^3}{(1-\lambda s)^3}\Big|$$

and use $\lambda \geqslant 1$, $\lambda s - r \geqslant \varepsilon$, $\lambda s \leqslant 1$, $r \leqslant 1$, $s \leqslant 1$, and $1 - \lambda s \geqslant \varepsilon$ to bound the above by $8\varepsilon^{-3}(a+b)$. □

Lemma B.7. — *Fix $\delta \in (0, r)$. Let $C_0 = C_0(\delta, r)$ be given by*

$$C_0 = \max\Big\{r+1, \frac{(\delta^{-1}+1)[(1+r)^2\delta + 2r^2 + 2]}{8(1-r)^2\delta} + \frac{r(r+1)(\delta^{-1}+1)}{4(1-r)\sqrt{r\delta}} + \frac{r\delta^{-1}+1}{2(1-r)\sqrt{r\delta}}\Big\}.$$

Then for any $\lambda \in (1, 1/r)$ and $a, b > 0$ such that $(a, b) \in S_\delta$

$$\overline{p}^{\lambda}_{-}(a, b) - \overline{p}(a, b) \geqslant -C_0(\lambda - 1).$$

Proof. — Fix positive a and b with $(a,b) \in S_\delta$. Let

$$f(\lambda) = r(a-b)\frac{1}{\lambda} + \frac{1}{\lambda}\sqrt{r^2(\lambda+1)^2(a-b)^2 - 4r\lambda(ra-b)(a-rb)}\,.$$

Then

$$\begin{aligned} f'(\lambda) &= \frac{-r(a-b)}{\lambda^2} + \frac{-2r^2(\lambda+1)(a-b)^2 + 4r\lambda(ra-b)(a-rb)}{2\lambda^2\sqrt{r^2(\lambda+1)^2(a-b)^2 - 4r\lambda(ra-b)(a-rb)}} \\ &= \frac{r(1-t)\left[2\sqrt{r^2(\lambda+1)^2(t-1)^2 - 4r\lambda(rt-1)(t-r)} - 2r(\lambda+1)(1-t)\right] + 4r\lambda(rt-1)(t-r)}{2\lambda^2\sqrt{r^2(\lambda+1)^2(t-1)^2 - 4r\lambda(rt-1)(t-r)}} \cdot b \end{aligned}$$

where $t = a/b$. Let

$$g_\lambda(t) = 2\sqrt{r^2(\lambda+1)^2(t-1)^2 - 4r\lambda(rt-1)(t-r)} - 2r(\lambda+1)(1-t).$$

Then

$$g_\lambda'(t) = 2r\Big(\frac{r(\lambda+1)^2(t-1) - 2r\lambda(t-r) - 2\lambda(rt-1)}{\sqrt{r^2(\lambda+1)^2(t-1)^2 - 4r\lambda(rt-1)(t-r)}} + \lambda + 1\Big).$$

The quadratic equation in t inside the radical is minimized at

$$t = \frac{(\lambda+1)^2 r - 2\lambda(r^2+1)}{(\lambda-1)^2 r} = 1 - \frac{2\lambda(1-r)^2}{r(\lambda-1)^2} \leqslant 1 - \frac{2r^{-1}(1-r)^2}{r(r^{-1}-1)^2} = -1$$

Since this value for t is negative, the quadratic is smallest at $t = \delta$. With $t = \delta$, the quadratic as a function of λ is minimized at

$$\lambda = \frac{2(r\delta-1)(\delta-r)}{r(\delta-1)^2} - 1 = 1 - \frac{2(1-r)^2\delta}{r(1-\delta)^2}\,.$$

Since this is strictly below 1, the minimum over the interval $[1, 1/r)$ is achieved at $\lambda = 1$. The resulting minimum is thus

$$4r^2(1-\delta)^2 - 4r(r\delta-1)(\delta-r) = 4\delta r(1-r)^2 > 0. \tag{B.6}$$

This yields

$$\begin{aligned} |g_\lambda'(t)| &\leqslant 2r\Big(\frac{r(r^{-1}+1)^2\delta^{-1} + 2r + 2r^{-1}}{\sqrt{4\delta r(1-r)^2}} + r^{-1} + 1\Big) \\ &= \frac{(1+r)^2\delta^{-1} + 2r^2 + 2}{(1-r)\sqrt{\delta r}} + 2(1+r) = C(\delta, r) = C \end{aligned}$$

for all $t \in [\delta, 1/\delta]$ and $\lambda \in (1, 1/r)$.

Since $g_\lambda(r) = 0$ the Mean Value Theorem implies that $g_\lambda(t) = g_\lambda'(s)(t-r)$ for some s between t and r. In particular, since $\delta < r$, $s \in [\delta, 1/\delta]$, and $|g_\lambda(r)| \leqslant C|t-r|$. Returning to $f'(\lambda)$ we get

$$\begin{aligned} |f'(\lambda)| &= \Big|\frac{r(1-t)g_\lambda(t) + 4r\lambda(rt-1)(t-r)}{2\lambda^2\sqrt{r^2(\lambda+1)^2(t-1)^2 - 4r\lambda(rt-1)(t-r)}} \cdot b\Big| \\ &\leqslant \frac{Cr(\delta^{-1}+1) + 4(r\delta^{-1}+1)}{2\lambda^2\sqrt{r^2(\lambda+1)^2(t-1)^2 - 4r\lambda(rt-1)(t-r)}} \cdot |t-r|b \\ &\leqslant \frac{Cr(\delta^{-1}+1) + 4(r\delta^{-1}+1)}{4(1-r)\sqrt{\delta r}} \cdot |a-rb| \leqslant 2C_0|a-rb|, \end{aligned}$$

where in the second-to-last inequality we used $\lambda \geqslant 1$ and the lower bound (B.6) on the expression under the radical.

Now, if $a \neq rb$, then

$$\begin{aligned}\overline{p}^{\lambda}_{-}(a,b) - \overline{p}(a,b) &= \frac{(a + rb + 2\sqrt{rab})(r(a-b) + f(\lambda)) - 2(a-rb)(r(a+b) + (r+1)\sqrt{rab})}{2(a-rb)(a+rb+2\sqrt{rab})} \\ &= \frac{f(\lambda) - f(1)}{2(a-rb)}\,.\end{aligned}$$

By the Mean Value Theorem, $f(\lambda) = f(1) + f'(c)(\lambda - 1)$ for some $c \in (1, \lambda)$. In particular, $c \in (1, 1/r)$. Therefore, $|f(\lambda) - f(1)| \leqslant 2C_0|a - rb|(\lambda - 1)$ and

$$\overline{p}^{\lambda}_{-}(a,b) - \overline{p}(a,b) \geqslant -C_0(\lambda - 1).$$

If, on the other hand, $a = rb$, then

$$\begin{aligned}\overline{p}^{\lambda}_{-}(a,b) - \overline{p}(a,b) &= \frac{r+1}{\lambda+1} - \frac{r(a+b) + (r+1)\sqrt{rab}}{a + rb + 2\sqrt{rab}} \\ &= -\frac{(r+1)(\lambda-1)}{2(\lambda+1)} \geqslant -(r+1)(\lambda-1) \geqslant -C_0(\lambda-1)\end{aligned}$$

and the claim holds again. □

Lemma B.8. — *Let $0 < \delta < 1$. Let*

$$C_1 = C_1(\delta, r) = \frac{(1+\delta)^2 r(1-r)}{2\delta^2\sqrt{r}(1+\sqrt{r})^2} \quad \text{and} \quad C_2 = C_2(r) = \frac{2(r+1)^2}{r(1-r)}\,.$$

Then for all $\xi, \zeta \in S_\delta$ we have

$$|\overline{p}(\xi) - \overline{p}(\zeta)| \leqslant C_1\Big|\frac{\xi_1}{|\xi|_1} - \frac{\zeta_1}{|\zeta|_1}\Big|. \tag{B.7}$$

And for all $\xi \in \mathbb{R}^2_{>0}$ and $q \in (r, 1)$ we have

$$\Big|\overline{\xi}(q)\cdot \mathbf{e}_1 - \frac{\xi_1}{|\xi|_1}\Big| \leqslant C_2|q - \overline{p}(\xi)|. \tag{B.8}$$

Proof. — Note that $(t, 1-t) \in S_\delta$ if and only if $t \in (\frac{\delta}{1+\delta}, \frac{1}{1+\delta})$. For such t,

$$\frac{d}{dt}\overline{p}(t, 1-t) = -\frac{r(1-r)}{2\sqrt{rt(1-t)}\left(\sqrt{t} + \sqrt{r(1-t)}\right)^2} \geqslant -\frac{(1+\delta)^2 r(1-r)}{2\delta^2\sqrt{r}(1+\sqrt{r})^2} = -C_1(\delta, r).$$

(B.7) follows from this bound and the fact that $\overline{p}(\xi) = \overline{p}(c\xi)$ for any $\xi \in \mathbb{R}^2_{>0}$ and $c > 0$.

For the second claim, differentiate $\overline{\xi}(q)\cdot\mathbf{e}_1$ to get

$$-\frac{2r(1-r)(1-q)(q-r)}{\big((q^2(r+1) - 4qr + r(r+1)\big)^2} \geqslant -\frac{2r(1-r)^3}{\big((q^2(r+1) - 4qr + r(r+1)\big)^2} \geqslant -\frac{2(r+1)^2}{r(1-r)} = -C_2(r).$$

(B.8) follows from this bound and the fact that $\overline{\xi}(\overline{p}(\xi_1/|\xi|_1)) = \xi_1/|\xi|_1$ for all $\xi \in \mathbb{R}^2_{>0}$. □

The following estimates are immediate from (3.10) and (3.12).

Lemma B.9. — *For $x \in \mathbb{R}^2_{\geqslant 0}$,*

$$\frac{r}{1-r}|x|_1 \leqslant \gamma(x) \leqslant \frac{r+\sqrt{r}}{1-r}|x|_1.$$

Lemma B.10. — *For any $\delta \in (0,1)$ and $x \in S_\delta$ we have*

$$r + \frac{(1-r)\delta\sqrt{r\delta}}{(1+\sqrt{r})^2} \leqslant \overline{p}(x) \leqslant 1 - \frac{(1-\sqrt{r})\delta}{1+\sqrt{r}}\,.$$

Proof. — Let $t = x_2/x_1$. Then

$$\overline{p}(x) = \frac{r(1+t)+(r+1)\sqrt{rt}}{1+rt+2\sqrt{rt}} = r + \frac{(1-r)(rt+\sqrt{rt})}{1+rt+2\sqrt{rt}} \geqslant r + \frac{(1-r)(r\delta+\sqrt{r\delta})}{(1+\sqrt{r/\delta})^2} \geqslant r + \frac{(1-r)\delta\sqrt{r\delta}}{(1+\sqrt{r})^2}\,.$$

Similarly,

$$\overline{p}(x) = 1 - \frac{(1-r)(1+\sqrt{rt})}{(1+\sqrt{rt})^2} \leqslant 1 - \frac{(1-r)(1+\sqrt{r\delta})\delta}{(\sqrt{\delta}+\sqrt{r})^2} \leqslant 1 - \frac{(1-r)\delta}{(1+\sqrt{r})^2}\,. \qquad \square$$

We are now ready to prove Theorem B.1.

Proof of Theorem B.1. — Fix $\delta \in (0,r)$, $\kappa > 0$, and $\varepsilon \in (0, \frac{\delta}{2(1+\delta^{-1})})$. Take integers $(m,n) \in S_\delta$ and $k \leqslant \varepsilon(m+n)$ and write

$$\begin{aligned}\overline{p}(m-k,n) - \overline{p}(m,n) &= \frac{r(m+n-k)+(r+1)\sqrt{rn(m-k)}}{m-k+rn+2\sqrt{rn(m-k)}} - \frac{r(m+n)+(r+1)\sqrt{rmn}}{m+rn+2\sqrt{rmn}} \\ &= \frac{(1-r)\left(rnk+\sqrt{rn}(rn-m)(\sqrt{m}-\sqrt{m-k})+k\sqrt{rmn}\right)}{(m-k+rn+2\sqrt{rn(m-k)})(m+rn+2\sqrt{rmn})} \\ &= \frac{rnk+\sqrt{rn}(rn-m)(\sqrt{m}-\sqrt{m-k})+k\sqrt{rmn}}{(\gamma(m-k,n)+m-k)(m+rn+2\sqrt{rmn})}\,. \qquad \text{(B.9)}\end{aligned}$$

If $m - rn > 0$, dropping the $\sqrt{m-k}$ from the denominator and using $rn > 0$, we have

$$(m-rn)(\sqrt{m}-\sqrt{m-k}) = \frac{k(m-rn)}{\sqrt{m}+\sqrt{m-k}} \leqslant k\sqrt{m}\,.$$

The same inequality holds trivially if $m-rn \leqslant 0$. In the next computation use the above inequality to bound the numerator of (B.9), then bound the denominator using the upper bound from Lemma B.9 and the facts that $2\sqrt{mn} \leqslant m+n$ and $(m,n) \in S_\delta$:

$$\overline{p}(m-k,n) - \overline{p}(m,n) \geqslant \frac{(1-\sqrt{r})rnk}{2(1+\sqrt{r})(m+n)^2} \geqslant \frac{(1-\sqrt{r})rk}{2(1+\sqrt{r})(\delta^{-1}+1)(m+n)} = \frac{a_0(\delta,r)k}{m+n}\,. \qquad \text{(B.10)}$$

Next, take $\xi \in \mathrm{ri}\,\mathcal{U}$ with $\xi_1 \in (\delta, 1-\delta)$ and such that $|\xi_1 - \frac{m}{m+n}| \leqslant \kappa(m+n)^{-1/3}$. Abbreviate $q = \overline{p}(\xi)$. Note that $\xi \in S_\delta$ and therefore Lemma B.8 implies that

$$|q - \overline{p}(m,n)| \leqslant C_1\kappa(m+n)^{-1/3}. \qquad \text{(B.11)}$$

Let $s_0 = s_0(\delta,r,\kappa) = \max(1, 16C_1\kappa/a_0)$. Let

$$\epsilon = \epsilon(\delta,r) = \min\Big(\frac{r}{2}, \frac{(1-r)\delta\sqrt{r\delta}}{(1+\sqrt{r})^2}\,,\, \frac{(1-\sqrt{r})^2\delta^2}{2(1+\sqrt{r})^2}\Big).$$

Take η so that

$$\begin{aligned}0 < \eta < \min\Big\{&-\frac{\log r}{\varepsilon}\,, \frac{1}{2\varepsilon}\log\Big(1+\frac{(1-\sqrt{r})\delta}{1+\sqrt{r}}\Big)\,,\, 1\,,\, \frac{a_0}{32(1+C_0(\delta/2,r))}\,, \\ &\varepsilon^{-1}\log\Big(1+\frac{\epsilon}{\sqrt{2}}\Big)\,,\, \frac{\varepsilon\delta^2a_0^2}{100\big(4\epsilon^2+4a_0/5+16\epsilon\big)}\Big\}\,. \qquad \text{(B.12)}\end{aligned}$$

Let $N_0 = (s_0/\varepsilon)^3$ and take $(m,n) \in S_\delta \cap \mathbb{Z}^2_{\geqslant N_0}$. Take $s \in [s_0, \varepsilon(m+n)^{1/3}]$ and set

$$\lambda = \exp\{\eta s(m+n)^{-1/3}\} \leqslant e^{\eta\varepsilon} \in (1, 1/r). \qquad \text{(B.13)}$$

Then by Lemma B.10 and the choice of η

$$r+\epsilon \leqslant r + \frac{(1-r)\delta\sqrt{r\delta}}{(1+\sqrt{r})^2} \leqslant q \leqslant \lambda q \leqslant \lambda^2 q \leqslant e^{2\eta\varepsilon}\Big(1-\frac{(1-\sqrt{r})\delta}{1+\sqrt{r}}\Big) \leqslant 1 - \frac{(1-\sqrt{r})^2\delta^2}{4(1+\sqrt{r})^2} \leqslant 1-\epsilon. \qquad \text{(B.14)}$$

Since $\eta s(m+n)^{-1/3} \leqslant \eta\varepsilon \leqslant 1$ and $e^x - 1 \leqslant 2x$ for $x \in [0,1]$,

$$\lambda q - q \leqslant \lambda - 1 \leqslant \frac{2\eta s}{(m+n)^{1/3}}\,. \qquad \text{(B.15)}$$

The choices of ε and k and that $(m,n)\in S_\delta$ imply that $(m-k,n)\in S_{\delta/2}$. Then Lemma B.7 implies

$$\overline{p}^{\lambda}_{-}(m-k,n)-\overline{p}(m-k,n)\geqslant -C_0(\delta/2,r)(\lambda-1)\geqslant -\frac{2C_0\eta s}{(m+n)^{1/3}}\,. \tag{B.16}$$

Take $k=\lceil s(m+n)^{2/3}\rceil-1$ and abbreviate $\overline{p}_{\pm}=\overline{p}^{\lambda}_{\pm}(m-k,n)$ and $\overline{p}=\overline{p}(m-k,n)$. Note that

$$s_0/2\leqslant 2^{2/3}s_0-1\leqslant s(m+n)^{2/3}-1\leqslant k\leqslant s(m+n)^{2/3}\leqslant\varepsilon(m+n). \tag{B.17}$$

Putting this, (B.10-B.12), and (B.15-B.17) together we get

$$\begin{aligned}\overline{p}-q\geqslant\overline{p}^{\lambda}_{-}-q\geqslant\overline{p}^{\lambda}_{-}-\lambda q&\geqslant\frac{(a_0-2\eta-2C_0\eta)s-C_1\kappa}{(m+n)^{1/3}}-\frac{a_0}{m+n}\\&\geqslant\frac{7a_0 s}{8(m+n)^{1/3}}-\frac{a_0 s}{2^{2/3}(m+n)^{1/3}}\\&\geqslant\frac{a_0 s}{5(m+n)^{1/3}}>0.\end{aligned} \tag{B.18}$$

Thus, by Lemma B.4, $\mathcal{L}^{\lambda,\lambda q}(m-k,n)=L^{\overline{p}^{\lambda}_{-},\overline{p}^{\lambda}_{+}}(m-k,n)$. Also, Lemma B.10 and the choice of η in (B.12) imply

$$r+\epsilon\leqslant r+\frac{(1-r)\delta\sqrt{r\delta}}{(1+\sqrt{r})^2}\leqslant q\leqslant\overline{p}^{\lambda}_{-}\leqslant\lambda\overline{p}^{\lambda}_{-}\leqslant\lambda\overline{p}\leqslant e^{\eta\varepsilon}\Big(1-\frac{(1-\sqrt{r})\delta}{2(1+\sqrt{r})}\Big)\leqslant 1-\frac{(1-\sqrt{r})^2\delta^2}{4(1+\sqrt{r})^2}\leqslant 1-\epsilon.$$

In particular, $\overline{p}-r\geqslant q-r\geqslant\varepsilon$ and $1-\overline{p}\geqslant 1-\lambda\overline{p}\geqslant\varepsilon$. Using this, $(m-k,n)\in S_{\delta/2}$, $(m,n)\in S_\delta$, and the identity $a(\overline{p}(a,b)-r)^2-rb(1-\overline{p}(a,b))^2=0$, we get

$$\begin{aligned}&(m-k)(\overline{p}^{\lambda}_{-}-r)(q-r)-nr(1-\overline{p}^{\lambda}_{-})(1-q)\\&\quad\leqslant(m-k)(\overline{p}-r)(q-r)-nr(1-\overline{p})(1-q)\\&\quad=(m-k)(\overline{p}-r)^2+(m-k)(\overline{p}-r)(q-\overline{p})-nr(1-\overline{p})^2-nr(1-\overline{p})(\overline{p}-q)\\&\quad=-(\overline{p}-q)((m-k)(\overline{p}-r)+nr(1-\overline{p}))\\&\quad\leqslant-\varepsilon\delta^2(\overline{p}-q)(m+n)/2.\end{aligned} \tag{B.19}$$

We have now collected all the necessary pieces to be able to bound the probability of interest. The first line below uses the stochastic monotonicty of geometric random variables in their inverse mean parameter and the monotonicity of the exit points in the boundary weights. The second line uses that, on the event in the indicator function the value inside the exponent is 0. The third line drops the indicator function and uses the Cauchy-Schwartz inequality. The fourth line uses independence and shift-invariance. Write

$$\begin{aligned}&\mathbb{P}\{Z^{q,\mathbf{e}_1}(m,n)>k\}^2\leqslant\mathbb{P}\{Z^{\lambda q,q,\mathbf{e}_1}(m,n)>k\}^2\\&=\mathbb{E}\Big[\mathbb{1}\{Z^{\lambda q,q,\mathbf{e}_1}(m,n)>k\}\exp\Big\{\frac{\log(\lambda)}{2}\big(G^{\lambda q}(k,0)+G^{\lambda q}_{(k,0)}(m,n|1,0)-G^{\lambda q,q}(m,n)\big)\Big\}\Big]^2\\&\leqslant\mathbb{E}\Big[\exp\Big\{\log(\lambda)\big(G^{\lambda q}(k,0)+G^{\lambda q}_{(k,0)}(m,n|1,0)\big)\Big\}\Big]\,\mathbb{E}\Big[\exp\Big\{-\log(\lambda)G^{\lambda q,q}(m,n)\Big\}\Big]\\&=\mathbb{E}\Big[\exp\Big\{\log(\lambda)G^{\lambda q}(k,0)\Big\}\Big]\,\mathbb{E}\Big[\exp\Big\{\log(\lambda)G^{\lambda q}(m-k,n|1,0)\Big\}\Big]\,\mathbb{E}\Big[\exp\Big\{-\log(\lambda)G^{\lambda q,q}(m,n)\Big\}\Big].\end{aligned}$$

Bound the third expectation on the last line using Proposition B.2, the second expectation using Lemma B.5, and compute the first expectation explicitly using the moment generating function of the Geometric distribution, to get:

$$\begin{aligned}2\log\mathbb{P}\{Z^{q,\mathbf{e}_1}(m,n)>k\} &\leqslant k\log\Big(\frac{1-\lambda q}{1-\lambda^2 q}\Big)+\mathcal{L}^{\lambda,\lambda q}(m-k,n)+L^{\lambda q,q}(m,n)\\ &= k\log\Big(\frac{1-\lambda q}{1-\lambda^2 q}\Big)+L^{\overline{p}^\lambda_-,\overline{p}^\lambda_+}(m-k,n)-L^{q,\lambda q}(m,n)\\ &= k\log\Big(\frac{1-\lambda q}{1-\lambda^2 q}\Big)+L^{\overline{p}^\lambda_-,\overline{p}^\lambda_+}(m-k,n)-k\log\Big(\frac{1-q}{1-\lambda q}\Big)\\ &= -k\log\Big(1-\frac{(\lambda-1)^2 q}{(1-\lambda q)^2}\Big)+L^{\overline{p}^\lambda_-,\overline{p}^\lambda_+}(m-k,n)-L^{q,\lambda q}(m-k,n).\end{aligned}$$

Next, use (B.13), (B.14), and (B.12) to deduce

$$\frac{(\lambda-1)^2 q}{(1-\lambda q)^2}\leqslant \epsilon^{-2}(e^{\eta\varepsilon}-1)^2\leqslant 1/2.$$

Use this, the fact that $-\log(1-t)\leqslant 2t$ for $t\in\big[0,\frac12\big]$, (B.14), (B.17), and (B.15) to continue with the bound

$$\begin{aligned}2\log\mathbb{P}(Z^{q,\mathbf{e}_1}(m,n)>k) &\leqslant 2k\frac{(\lambda-1)^2 q}{(1-\lambda q)^2}+L^{\overline{p}^\lambda_-,\overline{p}^\lambda_+}(m-k,n)-L^{q,\lambda q}(m-k,n)\\ &\leqslant 2\epsilon^{-2}k(\lambda-1)^2+L^{\overline{p}^\lambda_-,\overline{p}^\lambda_+}(m-k,n)-L^{q,\lambda q}(m-k,n)\\ &\leqslant 4\epsilon^{-2}\eta^2 s^3+L^{\overline{p}^\lambda_-,\overline{p}^+_\lambda}(m-k,n)-L^{q,\lambda q}(m-k,n).\end{aligned} \tag{B.20}$$

Using Lemma B.6, (B.19), (B.18), and (B.15) we get

$$\begin{aligned}&L^{\overline{p}_-,\overline{p}_+}(m-k,n)-L^{q,\lambda q}(m-k,n)\\ &\quad\leqslant (\lambda-1)\Big(\frac{(m-k)\overline{p}_-}{1-\overline{p}_-}-\frac{(m-k)q}{1-q}+\frac{nr}{\overline{p}_--r}-\frac{nr}{q-r}\Big)\\ &\qquad+\frac12(\lambda-1)^2\Big(\frac{(m-k)\overline{p}_-^2}{(1-\overline{p}_-)^2}-\frac{(m-k)q^2}{(1-q)^2}-\frac{nr(2\overline{p}_--r)}{(\overline{p}_--r)^2}+\frac{nr(2q-r)}{(q-r)^2}\Big)\\ &\qquad+2\epsilon^{-3}(m+n)(\lambda-1)^3\\ &\quad=(\lambda-1)(\overline{p}_--q)\frac{(m-k)(\overline{p}_--r)(q-r)-nr(1-\overline{p}_-)(1-q)}{(1-\overline{p}_-)(1-q)(\overline{p}_--r)(q-r)}\\ &\qquad+\frac12(\lambda-1)^2(\overline{p}_--q)\Big(\frac{(m-k)(\overline{p}_-+q-2\overline{p}_-q)}{(1-\overline{p}_-)^2(1-q)^2}+\frac{nr(-\overline{p}_-r+2\overline{p}_-q-rq)}{(\overline{p}_--r)^2(q-r)^2}\Big)\\ &\qquad+2\epsilon^{-3}(\lambda-1)^3(m+n)\\ &\quad\leqslant -\frac{\varepsilon\delta^2}{2\epsilon^4}(\lambda-1)(\overline{p}_--q)(\overline{p}-q)(m+n)+\epsilon^{-4}(\lambda-1)^2(\overline{p}_--q)(m+n)+2\epsilon^{-3}(\lambda-1)^3(m+n)\\ &\quad\leqslant -\Big(\frac{\varepsilon\delta^2a_0^2\eta}{50\epsilon^4}-\frac{4\eta^2a_0}{5\epsilon^4}-\frac{16\eta^3}{\epsilon^3}\Big)s^3\leqslant -\Big(\frac{\varepsilon\delta^2a_0^2\eta}{50\epsilon^4}-\frac{4\eta^2a_0}{5\epsilon^4}-\frac{16\eta^2}{\epsilon^3}\Big)s^3.\end{aligned}$$

Setting $c_1=c_1(\delta,r)=\epsilon^{-4}\varepsilon\delta^2a_0^2\eta/200$ and using (B.20) and the choice of η in (B.12) we get

$$\mathbb{P}\{Z^{q,\mathbf{e}_1}(m,n)\geqslant s(m+n)^{2/3}\}\leqslant e^{-c_1s^3}$$

for $s\in[s_0,\varepsilon(m+n)^{1/3}]$. When $s\geqslant(m+n)^{1/3}$, the above probability is 0 and the bound holds trivially. When $s\in[\varepsilon(m+n)^{1/3},(m+n)^{1/3}]$ we have

$$\mathbb{P}\{Z^{q,\mathbf{e}_1}(m,n)\geqslant s(m+n)^{2/3}\}\leqslant \mathbb{P}\{Z^{q,\mathbf{e}_1}(m,n)\geqslant \varepsilon(m+n)\}\leqslant e^{-c_1\varepsilon(m+n)^{1/3}}\leqslant e^{-c_1\varepsilon s^3}.$$

The claim of the theorem is thus proved for the case of $Z^{q,\mathbf{e}_1}$. The equivalent bound for vertical exit points follows by symmetry. □

Appendix C. Verifying Assumption 5.2 for the geometric LPP

We first prove a bound on the probability that an i.i.d. random walk with non-positive drift remains non-positive for its first n steps, given some control over the step's higher moments.

Lemma C.1. — *Let $\{X_i, i \in \mathbb{N}\}$ be i.i.d. random variables. Suppose $\mu = E[X_1] \leqslant 0$ and $\sqrt{Var(X_1)} \geqslant \varepsilon$ and $E[|X_1-\mu|^p] \leqslant D$ for some $p \geqslant 3$ and $\varepsilon, D \in (0,\infty)$. Call $S_k = \sum_{i=1}^k X_i$. There exists a finite $C = C(p,D,\varepsilon) > 0$ such that for all $n \in \mathbb{N}$,*

$$P(S_1 \leqslant 0, S_1 \leqslant 0, \ldots, S_n \leqslant 0) \leqslant C\big(n^{-\frac{p-2}{2(p+1)}} \vee |\mu|\big) \quad \text{and} \tag{C.1}$$

$$P(S_1 \geqslant 0, S_2 \geqslant 0, \ldots, S_n \geqslant 0) \leqslant \frac{C}{\sqrt{n}}. \tag{C.2}$$

Proof. — Since $\mu \leqslant 0$ the probability in (C.2) is bounded above by the probability of $\{S_1 \geqslant \mu, S_2 \geqslant 2\mu, \ldots, S_n \geqslant n\mu\}$. The bound (C.2) then follows from Theorem 5.1.7 in [**?**].

Let $n \in \mathbb{N}$ be sufficiently large that $t_n = n^{-\frac{p-2}{2(p+1)}} < \epsilon$ and let $\nu_n = (\mu + t_n)^+$. Note that $\nu_n - \mu = t_n \vee |\mu| > 0$. Let $\overline{S}_{k,n} = S_k - k\nu_n$. Then

$$P(S_1 \leqslant 0, S_2 \leqslant 0, \ldots, S_n \leqslant 0) \leqslant P(\overline{S}_{1,n} \leqslant 0, \overline{S}_{2,n} \leqslant 0, \ldots, \overline{S}_{n,n} \leqslant 0).$$

For $k \in \mathbb{N}$ define $p_{k,n} = P(\overline{S}_{1,n} \leqslant 0, \overline{S}_{2,n} \leqslant 0, \ldots, \overline{S}_{k-1,n} \leqslant 0, \overline{S}_{k,n} > 0)$ and $\tau_n = \inf\{k : \overline{S}_{k,n} > 0\}$. For $s \in [0,1]$, set

$$p_n(s) = \sum_{k=1}^{\infty} s^k p_{k,n}$$

and observe that $P(\tau_n = \infty) = 1 - p_n(1)$. By the Sparre-Andersen Theorem, Theorem XII.7.1 in [**?**], for $s \in [0,1)$,

$$\log \frac{1}{1-p_n(s)} = \sum_{k=1}^{\infty} \frac{s^k}{k} P(\overline{S}_{k,n} > 0). \tag{C.3}$$

Denote by $\Phi(x)$ and $\phi(x)$ the cumulative distribution function and probability density function of a standard Normal random variable. Recall that $p \geqslant 3$. By the Berry-Esseen theorem [20, Theorem 3.4.9], for all $x \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$\Big|P\Big\{\frac{\overline{S}_{k,n} - \mu k + \nu_n k}{\varepsilon\sqrt{k}} \leqslant x\Big\} - \Phi(x)\Big| \leqslant \frac{3D^{3/p}}{\varepsilon^3\sqrt{k}}$$

Set $m_n = \frac{\varepsilon\sqrt{2\pi}}{2(\nu_n-\mu)} = \frac{\varepsilon\sqrt{2\pi}}{2(t_n \vee |\mu|)} > 0$. Since ϕ is decreasing on $[0,\infty)$ with $\phi(0) = 1/\sqrt{2\pi}$,

$$1 - \Phi\Big((\nu_n-\mu)\frac{\sqrt{k}}{\varepsilon}\Big) = \frac{1}{2} - \int_0^{(\nu_n-\mu)\frac{\sqrt{k}}{\varepsilon}} \phi(x)dx \geqslant \frac{1}{2} - \frac{(\nu_n-\mu)\sqrt{k}}{\varepsilon\sqrt{2\pi}} = \frac{1}{2} - \frac{\sqrt{k}}{2m_n}\,.$$

Then we have

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k} P(\overline{S}_{k,n} > 0) &\geqslant \sum_{1 \leqslant k < m_n^2} \frac{1}{k} P(\overline{S}_{k,n} > 0) \geqslant \sum_{1 \leqslant k < m_n^2} \frac{1}{k}\Big(1 - \Phi\Big(-(\mu-\nu_n)\frac{\sqrt{k}}{\varepsilon}\Big) - \frac{3D^{3/p}}{\varepsilon^3\sqrt{k}}\Big)\\ &\geqslant \sum_{1 \leqslant k < m_n^2} \frac{1}{2k} - \frac{1}{2m_n}\sum_{1 \leqslant k < m_n^2} \frac{1}{\sqrt{k}} - \frac{3D^{3/p}}{\varepsilon^3}\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}\\ &\geqslant \log m_n - 1 - \frac{9D^{3/p}}{\varepsilon^3},\end{aligned}$$

where the empty sum is, as usual, equal to 0. When $m_n > 1$, each of the three bounds in the last line comes from integral comparison. The bounds are trivial when $0 < m_n < 1$. It then follows from (C.3) that

$$P(\tau_n = \infty) \leqslant \frac{2e^{\frac{9D^{3/p}}{\varepsilon^3}+1}}{\varepsilon\sqrt{2\pi}}(t_n \vee |\mu|).$$

On the other hand, by the Markov and Burkholder-Davis-Gundy [**?** , Theorem 4.2.12] inequalities followed by integral comparison,

$$P(\exists k \geqslant n : \overline{S}_{k,n} > 0) \leqslant \sum_{k=n}^{\infty} P(\overline{S}_{k,n} > 0) \leqslant \sum_{k=n}^{\infty} P(|S_k - k\mu|^p > |k(\mu - \nu_n)|^p)$$
$$\leqslant (2/e)^{p/2} p^p D \sum_{k=n}^{\infty} \frac{1}{k^{p/2}|\mu - \nu_n|^p} \leqslant \frac{2(2/e)^{p/2}p^p D}{p-2} n^{1-p/2}(t_n \vee |\mu|)^{-p}.$$

Combining these results, we have

$$P(S_1 \leqslant 0, \ldots, S_n \leqslant 0) \leqslant P(\tau_n = \infty) + P(\exists k \geqslant n \text{ such that } \overline{S}_{k,n} > 0)$$
$$\leqslant \frac{2e^{\frac{9D^{3/p}}{\varepsilon^3}+1}}{\varepsilon\sqrt{2\pi}}(t_n \vee |\mu|) + \frac{2(2/e)^{p/2}p^p D}{p-2} n^{1-p/2}(t_n \vee |\mu|)^{-p}.$$

Note that $n^{1-p/2}t_n^{-p} = t_n$ and, if $t_n \leqslant |\mu|$, then we have $n^{1-p/2}|\mu|^{-p} = t_n^{1+p}|\mu|^{-p} \leqslant |\mu|$; in this case, we have $n^{1-p/2}(t_n \vee |\mu|)^{-p} = |\mu| = t_n \vee |\mu|$. On the other hand, if $t_n \geqslant |\mu|$, then $t_n = n^{1-p/2}t_n^{-p} \leqslant n^{1-p/2}|\mu|^{-p}$ and, consequently, $n^{1-p/2}(t_n \vee |\mu|)^{-p} = (n^{1-p/2}t_n^{-p}) \wedge (n^{1-p/2}|\mu|^{-p}) = t_n = t_n \vee |\mu|$. Bound (C.1) follows. ☐

Lemma C.2. — *If* $\omega_0 \sim \mathrm{Geom}(r)$*, then Assumption 5.2 holds for any* $a_0 \in (1/3, 2/3)$*.*

Proof. — Fix $a_0 \in (1/3, 2/3)$. The steps of the random walk $S_k^{\xi_\star,\eta^\star}$ for $k \in [\![0, N^{2/3}]\!]$ are i.i.d. differences of independent geometric random variables with parameters $r/\overline{p}(\xi_\star)$ and $r/\overline{p}(\eta^\star)$. Under the conditions of Assumption 5.2 on $\xi_\star$ and $\eta^\star$, Lemma B.8 implies that

$$-C_1 r^{-1} N^{-a_0/2} \leqslant \mu = \mathbb{E}[S_1^{\xi_\star,\eta^\star}] = \frac{\overline{p}(\xi_\star)}{r} - \frac{\overline{p}(\eta^\star)}{r} \leqslant 0.$$

Since $\overline{p}(\xi_\star)$ and $\overline{p}(\eta^\star)$ are both above r, the variance of $S_1^{\xi_\star,\eta^\star}$ is bounded below by $\varepsilon = 2r/(1-r)^2$. Since $\xi_\star \cdot \mathbf{e}_1$ and $\eta^\star \cdot \mathbf{e}_1$ are assumed to be in $(\delta, 1-\delta)$, $\overline{p}(\xi_\star)$ and $\overline{p}(\eta^\star)$ are bounded away from r, uniformly in N, and thus for any $p \geqslant 1$ there exists a finite constant $D = D(\delta, p)$ such that $\mathbb{E}[|S_1^{\xi_\star,\eta^\star} - \mu|^p] \leqslant D(\delta, p)$ for all $N \in \mathbb{N}$. Take $p \geqslant 3$ large enough so that $\frac{p-2}{3(p+1)} > a_0/2$. The conditions of Lemma C.1 are satisfied. If we take $n = \lfloor N^{2/3} \rfloor$, then (C.1) gives, for N large enough,

$$\mathbb{P}\left\{S_1^{\xi_\star,\eta^\star} \leqslant 0, S_2^{\xi_\star,\eta^\star} \leqslant 0, \ldots, S_{\lfloor N^{2/3} \rfloor}^{\xi_\star,\eta^\star} \leqslant 0\right\} \leqslant C\big((C_1 r^{-1} N^{-a_0/2}) \vee (\lfloor N^{2/3} \rfloor)^{-\frac{p-2}{2(p+1)}}\big) \leqslant CC_1 r^{-1} N^{-a_0/2}.$$

Repeating this same argument for $S_k^{\xi^\star,\eta_\star}$, $k \in [\![-N^{2/3}, 0]\!]$, yields

$$\mathbb{P}(S_{-1}^{\xi^\star,\eta_\star} \leqslant 0, S_{-2}^{\xi^\star,\eta_\star} \leqslant 0, \ldots, S_{-N^{2/3}}^{\xi^\star,\eta_\star} \leqslant 0) \leqslant CC_1 r^{-1} N^{-a_0/2}.$$

Bound (5.5) follows from the independence proved in Corollary A.4. The lemma is proved. ☐

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