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Variational formulas and disorder regimes of random walks in random potentials

FIRAS RASSOUL-AGHA[1], TIMO SEPPÄLÄINEN[2] and ATILLA YILMAZ[3]

[1] *Department of Mathematics, University of Utah, 155S 1400E, Salt Lake City, UT 84112, USA. E-mail: firas@math.utah.edu; url: http://www.math.utah.edu/~firas*
[2] *Department of Mathematics, University of Wisconsin–Madison, Van Vleck Hall, 480 Lincoln Dr., Madison WI 53706-1388, USA. E-mail: seppalai@math.wisc.edu; url: http://www.math.wisc.edu/~seppalai*
[3] *Department of Mathematics, Koç University, 34450 Sarıyer, Istanbul, Turkey. E-mail: atillayilmaz@ku.edu.tr; url: http://home.ku.edu.tr/~atillayilmaz/*

We give two variational formulas (qVar1) and (qVar2) for the quenched free energy of a random walk in random potential (RWRP) when (i) the underlying walk is directed or undirected, (ii) the environment is stationary and ergodic, and (iii) the potential is allowed to depend on the next step of the walk which covers random walk in random environment (RWRE). In the directed i.i.d. case, we also give two variational formulas (aVar1) and (aVar2) for the annealed free energy of RWRP. These four formulas are the same except that they involve infima over different sets, and the first two are modified versions of a previously known variational formula (qVar0) for which we provide a short alternative proof. Then, we show that (qVar0) always has a minimizer, (aVar2) never has any minimizers unless the RWRP is an RWRE, and (aVar1) has a minimizer if and only if the RWRP is in the weak disorder regime. In the latter case, the minimizer of (aVar1) is unique and it is also the unique minimizer of (qVar1), but (qVar2) has no minimizers except for RWRE. In the case of strong disorder, we give a sufficient condition for the nonexistence of minimizers of (qVar1) and (qVar2) which is satisfied for the log-gamma directed polymer with a sufficiently small parameter. We end with a conjecture which implies that (qVar1) and (qVar2) have no minimizers under very strong disorder.

Keywords: directed polymer; KPZ universality; large deviation; quenched free energy; random environment; random potential; random walk; strong disorder; variational formula; very strong disorder; weak disorder

1. Introduction

1.1. The model

Random walk in random potential (RWRP) on $\mathbb{Z}^d$, with $d \geq 1$, has three ingredients.

(i) *The underlying walk*: Fix a finite set $\mathcal{R} \subset \mathbb{Z}^d$ with $|\mathcal{R}| \geq 2$. Define $p : \mathbb{Z}^d \to [0,1]$ by $p(z) = 1/|\mathcal{R}|$ if $z \in \mathcal{R}$ and $p(z) = 0$ otherwise. Consider random walk on $\mathbb{Z}^d$ with i.i.d. steps that have p as their common distribution. This walk induces a probability measure P_x on paths starting at $x \in \mathbb{Z}^d$. Expectations under P_x are denoted by E_x.

(ii) *The environment*: Let $\mathcal{G}$ be the additive subgroup of $\mathbb{Z}^d$ generated by $\mathcal{R}$. Take a probability space $(\Omega, \mathfrak{S}, \mathbb{P})$ equipped with an Abelian group $\{T_x : x \in \mathcal{G}\}$ of measurable transformations such that (i) $T_{x+y} = T_x \circ T_y$ and (ii) T_0 is the identity. Assume that $\mathbb{P}$ is invariant and ergodic w.r.t. this group. Expectations under $\mathbb{P}$ are denoted by $\mathbb{E}$, and sample points from $(\Omega, \mathfrak{S}, \mathbb{P})$ are referred to as environments.

(iii) *The potential*: Take a measurable function $V : \Omega \times \mathcal{R} \to \mathbb{R}$. For every $\omega \in \Omega$, $x \in \mathbb{Z}^d$ and $z \in \mathcal{R}$, the quantity $V(T_x\omega, z)$ is referred to as the potential at the ordered pair $(x, x+z)$ in the environment ω.

Given $n \geq 1$ and $\omega \in \Omega$, we define the quenched RWRP probability measure

$$Q^\omega_{n,x}\big((X_i)_{i\geq 0} \in \cdot\big) = \frac{1}{\mathcal{Z}^\omega_{n,x}} E_x\big[e^{\sum_{i=0}^{n-1} V(T_{X_i}\omega, Z_{i+1})} \mathbb{1}_{\{(X_i)_{i\geq 0}\in\cdot\}}\big]$$

on paths starting at any $x \in \mathbb{Z}^d$. Here, $(X_i)_{i\geq 0}$ denotes the random path with increments $Z_{i+1} = X_{i+1} - X_i$, and

$$\mathcal{Z}^\omega_{n,x} = E_x\big[e^{\sum_{i=0}^{n-1} V(T_{X_i}\omega, Z_{i+1})}\big]$$

is the normalizing factor, called the quenched partition function.

Remark 1.1. We have fixed p to be the uniform distribution on $\mathcal{R}$, but we can easily incorporate more general cases. Indeed, consider a measurable $\hat{p} : \Omega \times \mathbb{Z}^d \to [0, 1]$ such that, for $\mathbb{P}$-a.e. ω: (i) $\hat{p}(\omega, z) > 0$ if and only if $z \in \mathcal{R}$; and (ii) $\sum_{z\in\mathcal{R}} \hat{p}(\omega, z) = 1$. Then, the discrete-time Markov chain on $\mathbb{Z}^d$, with transition probabilities $\pi^\omega_{x,y} := \hat{p}(T_x\omega, y - x)$ for $x, y \in \mathbb{Z}^d$, is a quenched random walk in random environment (RWRE). Taking the underlying walk to be this RWRE is equivalent to adding $-\log \hat{p}(\omega, z) - \log |\mathcal{R}|$ to the potential $V(\omega, z)$.

Remark 1.2. We have given a rather abstract definition of the environment space. The canonical setting is as follows: there is a Borel set $\Gamma \subset \mathbb{R}$, and $(\Omega, \mathfrak{S})$ is $\Gamma^{\mathbb{Z}^d}$ equipped with the product Borel σ-algebra. In this case, environments are represented as $\omega = (\omega_x)_{x\in\mathbb{Z}^d}$, and elements of the group $\{T_x : x \in \mathcal{G}\}$ are translations defined by $(T_x\omega)_y = \omega_{x+y}$.

In the initial parts of this paper, we will consider RWRP with the abstract environment formulation. However, in later parts, we will adopt the canonical model and make the following extra assumptions.

(Dir) *Directed nearest-neighbor walk*: $\mathcal{R} = \{e_1, \dots, e_d\}$, the standard basis for $\mathbb{R}^d$, with $d \geq 2$.

(Ind) *Independent environment*: The components of $\omega = (\omega_x)_{x\in\mathbb{Z}^d}$ are i.i.d. under $\mathbb{P}$.

(Loc) *Local potential*: There exists a $V_o : \Gamma \times \mathcal{R} \to \mathbb{R}$ such that $V(\omega, z) = V_o(\omega_0, z)$ for every $\omega = (\omega_x)_{x\in\mathbb{Z}^d} \in \Omega = \Gamma^{\mathbb{Z}^d}$ and $z \in \mathcal{R}$.

These assumptions enable us to use martingale techniques in the analysis of the asymptotic behaviour of RWRP, see Section 1.3. If V_o does not depend on z, then RWRP is also referred to as a directed polymer. However, we prefer to keep the z dependence because, this way, the results on the quenched free energy of RWRP have implications regarding large deviations, see Remark 1.7.

There is a vast literature on RWRP, RWRE and directed polymers: see the lectures/surveys [3,9,15,19,33,41] and the references therein. In what follows, we will focus only on the parts of the literature that are directly relevant to our results.

1.2. Quenched free energy and large deviations

In a recent paper [29], we prove the $\mathbb{P}$-a.s. existence of the quenched free energy

$$\Lambda_q(V) := \lim_{n\to\infty} \frac{1}{n} \log \mathcal{Z}^{\omega}_{n,0}. \tag{1.1}$$

In order to give the precise statement of this result, we need two definitions.

Definition 1.3. *A measurable function $F : \Omega \times \mathcal{R} \to \mathbb{R}$ is said to be a centered cocycle if it satisfies the following conditions.*

(i) *Centered*: $\mathbb{E}[|F(\cdot,z)|] < \infty$ *and* $\mathbb{E}[F(\cdot,z)] = 0$ *for every* $z \in \mathcal{R}$.
(ii) *Cocycle*:

$$\sum_{i=0}^{m-1} F(T_{x_i}\omega, z_{i+1}) = \sum_{j=0}^{n-1} F\big(T_{x'_j}\omega, z'_{j+1}\big)$$

for $\mathbb{P}$-a.e. ω, every $m,n \geq 1$, $(x_i)_{i=0}^m$ and $(x'_j)_{j=0}^n$ such that $z_{i+1} := x_{i+1} - x_i \in \mathcal{R}$, $z'_{j+1} := x'_{j+1} - x'_j \in \mathcal{R}$, $x_0 = x'_0$ and $x_m = x'_n$.

The class of centered cocycles is denoted by $\mathcal{K}_0$.

Definition 1.4. *A measurable function $V : \Omega \times \mathcal{R} \to \mathbb{R}$ is said to be in class $\mathcal{L}$ if $\mathbb{E}[|V(\cdot,z)|] < \infty$ and*

$$\limsup_{\delta\to 0} \limsup_{n\to\infty} \max_{x \in \bigcup_{j=1}^n D_j} \frac{1}{n} \sum_{0\leq i\leq \delta n} \big|V(T_{x+iz'}\omega, z)\big| = 0$$

for $\mathbb{P}$-a.e. ω and every $z, z' \in \mathcal{R}$ such that $z' \neq 0$, where

$$D_j = \big\{z_1 + \cdots + z_j \in \mathbb{Z}^d : z_i \in \mathcal{R} \text{ for every } i = 1, \ldots, j\big\} \tag{1.2}$$

denotes the set of points accessible from the origin in exactly j steps chosen from $\mathcal{R}$.

Theorem 1.5. *Assume that $\mathfrak{S}$ is countably generated and $V \in \mathcal{L}$. Then, the limit in (1.1) exists $\mathbb{P}$-a.s., is deterministic, and satisfies*

$$\Lambda_q(V) = \inf_{F\in\mathcal{K}_0} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \Big\{\log\Big(\sum_{z\in\mathcal{R}} p(z) e^{V(\omega,z)+F(\omega,z)}\Big)\Big\} \in (-\infty, \infty]. \tag{1.3}$$

This result was initially obtained in [39] for bounded potentials under the assumption that $\{\pm e_1, \ldots, \pm e_d\} \subset \mathcal{R}$. The version in Theorem 1.5 is part of [29], Theorem 2.3, which is valid for potentials of the form $V : \Omega \times \mathcal{R}^\ell \to \mathbb{R}$ with arbitrary $\ell \geq 1$. Actually, the latter result contains two variational formulas for $\Lambda_q(V)$, but the second one is not directly relevant for our purposes in this paper, so we omit it for the sake of brevity.

The proof of Theorem 1.5 is based on a rather technical approach involving careful applications of ergodic and minimax theorems, which was developed in [23,24] in the context of stochastic homogenization of viscous Hamilton–Jacobi equations and was first adapted in [30] to large deviations for RWRE. However, the existence of the a.s. limit in (1.1) can be shown more easily (without giving any formulas for $\Lambda_q(V)$) by subadditivity arguments and additional estimates such as concentration inequalities or lattice animal bounds. This has been done in [5,8,36] for directed polymers under various moment assumptions on the potential, and more recently in [27], Theorem 2.2(b), in the setting of Theorem 1.5. In fact, the latter result drops the assumption that $\mathfrak{S}$ is countably generated and only requires $V \in \mathcal{L}$. We record it below for future reference.

Theorem 1.6. *Assume that $V \in \mathcal{L}$. Then, the limit in (1.1) exists $\mathbb{P}$-a.s., is deterministic, and satisfies $\Lambda_q(V) \in (-\infty, \infty]$.*

The hypotheses of Theorem 1.5 are satisfied in the commonly studied examples. First of all, in the canonical setting, the product Borel σ-algebra is countably generated. Second, bounded potentials are in class $\mathcal{L}$ under arbitrary stationary and ergodic $\mathbb{P}$, and so is any V with $\mathbb{E}[|V(\cdot, z)|] < \infty$ when $d = 1$. In the multidimensional case under (Ind) and (Loc), it suffices to have $\mathbb{E}[|V(\cdot, z)|^p] < \infty$ for some $p > d$. In general, there is a tradeoff between the degree of mixing in $\mathbb{P}$ and the moment of $V(\cdot, z)$ required. See [29], Lemma A.4, for further details and proofs. Note that these assumptions do not rule out $\Lambda_q(V) = \infty$. Indeed, it is easy to see that the latter holds under (Ind) and (Loc) when $\mathcal{R}$ allows multiple visits to points and V is unbounded.

Assume additionally that Ω is a compact metric space and $\mathfrak{S}$ is its Borel σ-algebra. (These assumptions are valid in the canonical setting if Γ is compact.) Let $\mathcal{M}_s(\Omega \times \mathcal{R})$ be the space of Borel probability measures μ on $\Omega \times \mathcal{R}$ such that

$$\sum_{z\in\mathcal{R}} \int_\Omega \varphi(\omega)\mu(d\omega, z) = \sum_{z\in\mathcal{R}} \int_\Omega \varphi(T_z\omega)\mu(d\omega, z)$$

for every $\varphi \in C_b(\Omega)$, where $C_b(\cdot)$ denotes the space of bounded continuous functions. It is shown in [29], Theorem 3.1, that, when $\Lambda_q(V) < \infty$, Theorem 1.5 implies a large deviation principle (LDP) for the quenched distributions $Q^\omega_{n,0}(R_n \in \cdot)$ on $\mathcal{M}_s(\Omega \times \mathcal{R})$ of the empirical measure

$$R_n = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{T_{X_i}\omega, Z_{i+1}}.$$

The rate function of this LDP has the following formula:

$$\mathfrak{I}_q(\mu) = \sup_{f\in C_b(\Omega\times\mathcal{R})} \left\{ \sum_{z\in\mathcal{R}} \int_\Omega f(\omega, z)\mu(d\omega, z) - \Lambda_q(f + V) \right\} + \Lambda_q(V). \tag{1.4}$$

As a corollary, we get an LDP for $Q^\omega_{n,0}(X_n/n \in \cdot)$ with the rate function

$$I_q(v) = \sup_{\lambda\in\mathbb{R}^d} \{\lambda \cdot v - \Lambda_q(f_\lambda + V)\} + \Lambda_q(V). \tag{1.5}$$

Here, $f_\lambda : \Omega \times \mathcal{R} \to \mathbb{R}$ is defined by $f_\lambda(\omega, z) = \lambda \cdot z$.

Remark 1.7. Observe that $f_\lambda + V$ depends on z even if V does not. This is why it is important to allow potentials that depend on z in Theorem 1.5.

Remark 1.8. Theorem 1.6 also implies the aforementioned LDPs, but does not provide formulas for $\Lambda_q(V)$, $\Lambda_q(f + V)$ and $\Lambda_q(f_\lambda + V)$ appearing in the rate functions (1.4) and (1.5).

In the theory of large deviations, the former LDP is referred to as level-2, and gives the latter one (known as level-1) via the so-called contraction principle. See [13,14,16,28,34] for the definitions of these concepts as well as general background on large deviations. The highest level is level-3 (also known as process level) and is established for RWRP in [29], Theorem 3.2. This last LDP covers and strengthens various previous results on the quenched large deviations for RWRP and RWRE such as [1,6,7,20,26,30,35,38,39,42,43]. See [29], Section 1.3, for a detailed account.

1.3. Directed i.i.d. case: Disorder regimes

Assume that the conditions (Dir), (Ind) and (Loc) from Section 1.1 are satisfied. In this case, the σ-algebras

$$\mathfrak{S}_0^n = \sigma\big(\omega_x : x \in \mathbb{Z}_+^d, |x|_1 \le n-1\big) \quad \text{and}$$
$$\mathfrak{S}_0^\infty = \sigma\big(\omega_x : x \in \mathbb{Z}_+^d\big)$$

are relevant. Here and throughout, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ is the set of nonnegative integers and $|x|_1 := |x_1| + \cdots + |x_d|$ denotes the ℓ_1-norm. Functions that are measurable w.r.t. $\mathfrak{S}_0^\infty$ are sometimes referred to as future measurable.

Define the annealed free energy

$$\Lambda_a(V) := \log\Bigg(\sum_{z \in \mathcal{R}} p(z)\mathbb{E}\big[e^{V(\cdot,z)}\big]\Bigg) \in (-\infty, \infty]. \tag{1.6}$$

It is straightforward to check that

$$W_n(\omega) := \frac{\mathcal{Z}_{n,0}^\omega}{\mathbb{E}[\mathcal{Z}_{n,0}^\omega]} = E_0\big[e^{\sum_{i=0}^{n-1} V(T_{X_i}\omega, Z_{i+1}) - n\Lambda_a(V)}\big]$$

holds and $(W_n)_{n\ge1}$ is a nonnegative martingale w.r.t. the filtration $(\mathfrak{S}_0^n)_{n\ge1}$. Therefore,

$$W_\infty := \lim_{n\to\infty} W_n \qquad \mathbb{P}\text{-a.s.} \tag{1.7}$$

exists. Moreover, the event $\{W_\infty = 0\}$ is measurable w.r.t. the tail σ-algebra

$$\bigcap_{n\ge1} \sigma\big(\omega_x : x \in \mathbb{Z}_+^d, |x|_1 \ge n\big)$$

and the Kolmogorov zero-one law implies the following dichotomy:

$$\text{either} \quad \mathbb{P}(W_\infty = 0) = 0 \quad \text{(the \textit{weak disorder} regime);}$$
$$\text{or} \quad \mathbb{P}(W_\infty = 0) = 1 \quad \text{(the \textit{strong disorder} regime).}$$

This analysis is due to Bolthausen [2] in the case of directed polymers (i.e., for potentials that do not depend on z) and is easily adapted to our setting, which we leave to the reader. The terms weak disorder and strong disorder were coined in [8].

It follows from Jensen's inequality that $\Lambda_q(V) \le \Lambda_a(V)$ always holds. This is known as the annealing bound. Observe that, in the case of weak disorder, we have $\Lambda_a(V) < \infty$ (since otherwise $W_n = 0$) and

$$0 = \lim_{n\to\infty} \frac{1}{n} \log W_n(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{Z}^{\omega}_{n,0} - \Lambda_a(V) = \Lambda_q(V) - \Lambda_a(V) \tag{1.8}$$

for $\mathbb{P}$-a.e. ω. Therefore,

$$\Lambda_q(V) < \Lambda_a(V) \qquad \text{(the \textit{very strong disorder} regime)}$$

is a sufficient condition for strong disorder. However, it is not known whether it is necessary for strong disorder. We will say more about this and related open problems in Section 2.4.

The following theorem collects the results regarding the dependence of the disorder regimes of directed polymers on (i) the dimension d and (ii) an inverse temperature parameter β which is introduced to modify the strength of the potential.

Theorem 1.9. *Assume that* (Dir), (Ind) *and* (Loc) *are satisfied*, V *does not depend on* z, *and* $\Lambda_a(\beta V) < \infty$ *for every* $\beta \ge 0$. *Then, we have the following results.*

(a) *There exist* $0 \le \beta_c = \beta_c(V,d) \le \beta_c' = \beta_c'(V,d) \le \infty$ *such that the RWRP* (*or directed polymer*) *with potential* βV *is in*:

  (i) *the weak disorder regime if* $\beta \in \{0\} \cup (0, \beta_c)$,
  (ii) *the strong disorder regime if* $\beta \in (\beta_c, \infty)$, *and*
  (iii) *the very strong disorder regime if* $\beta \in (\beta_c', \infty)$.

(b) *The critical inverse temperatures* $\beta_c = \beta_c(V,d)$ *and* $\beta_c' = \beta_c'(V,d)$ *satisfy*

  (i) $\beta_c > 0$ *if* $d \ge 4$, *and*
  (ii) $\beta_c' = 0$ *if* $d = 2, 3$.

Part (a) of this theorem is proved in [11], Theorem 3.2; item (i) of part (b) is established in a series of papers [2,21,32]; and item (ii) of part (b) is shown in [10] for $d = 2$ and [25] for $d = 3$. In fact, [25] covers $d = 2, 3$ and is valid under the weaker assumption of $\Lambda_a(\beta_o V) < \infty$ for some $\beta_o > 0$. As far as we know, these results have not been adapted to the RWRP model with potentials that depend on z. However, the analogs of items (i) and (ii) of part (b) have been established in [38,40] in the context of large deviations for directed RWRE.

1.4. Organization of the article

In Section 2, we present our results along with remarks and open problems. The subsequent sections contain the proofs of our results.

2. Results

2.1. Quenched free energy in the general case

In order to abbreviate the variational formula (1.3) given in Theorem 1.5 for the quenched free energy $\Lambda_q(V)$, we define

$$K(V,F) := \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega}\left\{\log\left(\sum_{z\in\mathcal{R}} p(z)e^{V(\omega,z)+F(\omega,z)}\right)\right\}$$

for every measurable function $F:\Omega\times\mathcal{R}\to\mathbb{R}$. Observe that $K(V,F)$ is equal to

$$K'(V,g) := \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega}\left\{\log\left(\sum_{z\in\mathcal{R}} \frac{p(z)e^{V(\omega,z)}g(T_z\omega)}{g(\omega)}\right)\right\}$$

when F is of the form

$$F(\omega,z) = \big(\nabla^* g\big)(\omega,z) := \log\left(\frac{g(T_z\omega)}{g(\omega)}\right) \tag{2.1}$$

for some $g\in L^+(\Omega,\mathfrak{S},\mathbb{P})$. Here and throughout,

$$L^+\big(\Omega,\mathfrak{S}',\mathbb{P}\big) := \big\{g:\Omega\to\mathbb{R} : g \text{ is } \mathfrak{S}'\text{-measurable and } 0<g(\omega)<\infty \text{ for } \mathbb{P}\text{-a.e. } \omega\big\} \quad \text{and}$$
$$L^{++}\big(\Omega,\mathfrak{S}',\mathbb{P}\big) := \big\{g:\Omega\to\mathbb{R} : g \text{ is } \mathfrak{S}'\text{-measurable and } \exists c>0 \text{ s.t. } c<g(\omega)<\infty \text{ for } \mathbb{P}\text{-a.e. } \omega\big\}$$

for every σ-algebra $\mathfrak{S}'\subset\mathfrak{S}$ on Ω. We start our analysis by showing that the logarithmic gradient (as in (2.1)) of any $g\in L^+(\Omega,\mathfrak{S},\mathbb{P})$ is in $\mathcal{K}_0$ whenever $K'(V,g)<\infty$, see Lemma 3.1. Then, we give a short alternative proof of (1.3) and provide two modified versions of it.

Theorem 2.1. *Assume that $V\in\mathcal{L}$. Then, we have the following variational formulas.*

$$\Lambda_q(V) = \inf_{F\in\mathcal{K}_0} K(V,F), \tag{qVar0}$$

$$\Lambda_q(V) = \inf_{g\in L^+} K'(V,g), \tag{qVar1}$$

$$\Lambda_q(V) = \inf_{g\in L^{++}} K'(V,g). \tag{qVar2}$$

Here, the spaces L^+ and L^{++} stand for (i) $L^+(\Omega,\mathfrak{S},\mathbb{P})$ *and* $L^{++}(\Omega,\mathfrak{S},\mathbb{P})$ *in the general case and* (ii) $L^+(\Omega,\mathfrak{S}_0^\infty,\mathbb{P})$ *and* $L^{++}(\Omega,\mathfrak{S}_0^\infty,\mathbb{P})$ *under* (Dir) *and* (Loc).

Remark 2.2. It is shown in [29], Lemma C.3, that $\mathcal{K}_0$ is the $L^1(\Omega, \mathfrak{S}, \mathbb{P})$-closure of

$$\left\{\nabla^* g : \exists C > 0 \text{ s.t. } C^{-1} < g(\omega) < C \text{ for } \mathbb{P}\text{-a.e. } \omega\right\}.$$

Unfortunately, our understanding of $\mathcal{K}_0$ does not go much beyond this characterization. Thus, for applications, (qVar0) is perhaps not very useful. (qVar1) and (qVar2) replace $\mathcal{K}_0$ by the much more concrete class of logarithmic gradients. This way, they simplify (qVar0) and thereby improve our understanding of the large deviation rate functions $\mathfrak{I}_q$ and I_q via (1.4) and (1.5), respectively.

The proof of Theorem 2.1 does not rely on the rather technical minimax approach taken in [29], Theorem 2.3. The lower bounds in (qVar0), (qVar1) and (qVar2) follow from a standard spectral argument, whereas the upper bounds hinge on a certain control on the minima of path integrals of centered cocycles on large sets which is implied by an ergodic theorem and is trivial in the case of (qVar2). Moreover, as we have recorded in Theorem 1.6, the existence of the a.s. limit in (1.1) is shown in [27], Theorem 2.2(b), for $V \in \mathcal{L}$ (without assuming that $\mathfrak{S}$ is countably generated) by subadditivity and elementary estimates. In short, the proof of Theorem 2.1 is completely independent of Theorem 1.5.

Now that we have three closely related variational formulas for $\Lambda_q(V)$, it is natural to ask whether they possess minimizers, that is, the infima in their definitions are attained. We provide a positive answer to this question for (qVar0). As its proof in Section 3.2 attests, the technical significance of this result is due to the lack of weak compactness of the unit ball in $L^1(\Omega, \mathfrak{S}, \mathbb{P})$.

Theorem 2.3. *Assume that $V \in \mathcal{L}$. Then*, (qVar0) *always has a minimizer.*

It turns out that, unlike (qVar0), the variational formulas (qVar1) and (qVar2) do not always have minimizers. In fact, this is one of the main results in this paper, see Section 2.3. The possible lack of minimizers might be seen as a shortcoming of our formulas. However, we will argue that it is actually an advantage since it carries valuable information about the disorder regime of the model, at least in the directed i.i.d. case.

2.2. Annealed free energy in the directed i.i.d. case

Assume that (Dir), (Ind) and (Loc) hold. Our analysis of the variational formulas (qVar1) and (qVar2) for $\Lambda_q(V)$ builds on its analog for the annealed free energy $\Lambda_a(V)$ defined in (1.6).

Theorem 2.4. *Assume* (Dir), (Ind), *and* (Loc). *Then*, *we have the following variational formulas*.

$$\Lambda_a(V) = \inf_{g \in L^+ \cap L^1} K'(V, g), \tag{aVar1}$$

$$\Lambda_a(V) = \inf_{g \in L^{++} \cap L^1} K'(V, g). \tag{aVar2}$$

Here, L^+, L^{++} *and* L^1 *stand for* $L^+(\Omega, \mathfrak{S}_0^\infty, \mathbb{P})$, $L^{++}(\Omega, \mathfrak{S}_0^\infty, \mathbb{P})$ *and* $L^1(\Omega, \mathfrak{S}_0^\infty, \mathbb{P})$, *respectively.*

Remark 2.5. The variational formulas (qVar1) and (aVar1) for $\Lambda_q(V)$ and $\Lambda_a(V)$ can be equivalently written as the infima of $K(V, F)$ over

$$\{F \in \mathcal{K}_0 : F = \nabla^* g \text{ for some } g \in L^+\} \quad \text{and} \quad \{F \in \mathcal{K}_0 : F = \nabla^* g \text{ for some } g \in L^+ \cap L^1\},$$

respectively. The presence of these different sets is not merely a technical artifact of our proofs, as we know that $\Lambda_q(V) < \Lambda_a(V)$ in the case of very strong disorder, cf. Theorem 1.9. We find this strict inequality to be particularly interesting because both of these sets are dense in $\mathcal{K}_0$ by [29], Lemma C.3, cf. Remark 2.2. The same comment applies to (qVar2) and (aVar2).

In the light of Theorem 2.3 and the paragraph below it, we ask if/when (aVar1) and (aVar2) have any minimizers. The answer to this question constitutes our first variational result on the disorder regimes of RWRP.

Theorem 2.6. *Assume* (Dir), (Ind), (Loc), *and* $\Lambda_a(V) < \infty$.

(a) (aVar1) *has a minimizer if and only if there is weak disorder. In this case, the minimizer is unique* (*up to a multiplicative constant*), *equal to* W_∞ *defined in* (1.7), *and there is no need for taking essential supremum in* $K'(V, W_\infty)$, *that is*,

$$\Lambda_a(V) = K'(V, W_\infty) = \log\biggl(\sum_{z\in\mathcal{R}} \frac{p(z)e^{V(\omega,z)} W_\infty(T_z\omega)}{W_\infty(\omega)}\biggr) \qquad \textit{for } \mathbb{P}\textit{-a.e. } \omega.$$

(b) (aVar2) *has no minimizers unless* $\mathcal{Z}^\omega_{1,0}$ *is* $\mathbb{P}$*-essentially constant*, *cf. Remark* 2.7.

Remark 2.7. Theorem 2.6(a) implies that the only minimizer candidate of (aVar2) is W_∞. However, we will show in Proposition 4.3 that $W_\infty \notin L^{++}$ unless $\mathcal{Z}^\omega_{1,0} = \sum_{z\in\mathcal{R}} p(z)e^{V(\omega,z)}$ is $\mathbb{P}$-essentially constant. In the latter case, $\mathcal{Z}^\omega_{n,0}$ is $\mathbb{P}$-essentially constant for every $n \geq 1$, and $\mathbb{P}(W_n = 1) = \mathbb{P}(W_\infty = 1) = 1$. By Theorems 2.6(a) and 2.8(a), W_∞ is the unique minimizer of (aVar1), (aVar2), (qVar1) and (qVar2). Observe that, in this case, the RWRP is nothing but an RWRE with transition kernel $\hat{p}(\omega, z) = p(z)e^{V(\omega,z)-\Lambda_a(V)}$, cf. Remark 1.1.

Other characterizations of weak disorder have been previously given in the literature on directed polymers. First of all, it is shown in [8], Theorem 2.1, that weak disorder is equivalent to the delocalization of the polymer in an appropriate sense. Precisely, when $\Lambda_a(V) < \infty$ and V is not $\mathbb{P}$-essentially constant, there is weak disorder if and only if

$$\sum_{n=1}^{\infty} \bigl(Q^\omega_{n,0}\bigr)^{\otimes 2}(X_n = \tilde{X}_n) < \infty.$$

Here, $\tilde{X}_n$ is an independent copy of X_n under $Q^\omega_{n,0}$. Second, [11], Proposition 3.1, collects some useful characterizations of weak disorder, e.g., the $L^1(\mathbb{P})$-convergence or uniform integrability of the martingale $(W_n)_{n\geq 1}$. As far as we know, part (a) of Theorem 2.6 is the first variational characterization of weak disorder for RWRP. Its proof builds on an earlier characterization given as part of [11], Proposition 3.1, for directed polymers, see Section 4.2 for details.

2.3. Analysis of (qVar1) and (qVar2) in the directed i.i.d. case

We continue working under (Dir), (Ind) and (Loc). In the case of weak disorder, $\Lambda_q(V) = \Lambda_a(V) < \infty$ by (1.8). Therefore, the unique minimizer W_∞ of (aVar1) in $L^+ \cap L^1$ is also a minimizer of (qVar1) in the larger space L^+. However, it is not a-priori clear whether W_∞ is the unique minimizer of (qVar1). The following theorem settles this issue.

Theorem 2.8. *Assume* (Dir), (Ind), (Loc), *and weak disorder.*

(a) *Up to a multiplicative constant, the unique minimizer W_∞ of* (aVar1) *is also the unique minimizer of* (qVar1).

(b) (qVar2) *has no minimizers unless $\mathcal{Z}^\omega_{1,0}$ is $\mathbb{P}$-essentially constant, cf. Remark* 2.7.

Note that Theorem 2.8 does not say anything about whether (qVar1) and (qVar2) have any minimizers in the case of strong disorder. This turns out to be a more difficult question. In order to address it, we introduce

$$h_n^\lambda(\omega) := E_0\big[e^{\sum_{i=0}^{n-1} V(T_{X_i}\omega, Z_{i+1}) - n\lambda}\big] \tag{2.2}$$

for every $n \geq 1$, $\lambda \in \mathbb{R}$ and $\omega \in \Omega$, and consider the future measurable functions

$$\underline{h}_\infty^\lambda(\omega) := \liminf_{n\to\infty} h_n^\lambda(\omega) \quad \text{and} \quad \bar{h}_\infty^\lambda(\omega) := \limsup_{n\to\infty} h_n^\lambda(\omega).$$

With this notation, $W_n = h_n^\lambda$ and $W_\infty = \underline{h}_\infty^\lambda = \bar{h}_\infty^\lambda$ when $\lambda = \Lambda_a(V) < \infty$. For general $\lambda \in \mathbb{R}$, we know that

$$\lim_{n\to\infty} \frac{1}{n} \log h_n^\lambda(\omega) = \Lambda_q(V) - \lambda$$

holds for $\mathbb{P}$-a.e. ω. Therefore,

$$\mathbb{P}\big(\underline{h}_\infty^\lambda = \bar{h}_\infty^\lambda = 0\big) = 1 \qquad \text{if } \lambda > \Lambda_q(V) \quad \text{and}$$
$$\mathbb{P}\big(\underline{h}_\infty^\lambda = \bar{h}_\infty^\lambda = \infty\big) = 1 \qquad \text{if } \lambda < \Lambda_q(V).$$

Hence, the only nontrivial choice of parameter is $\lambda = \Lambda_q(V)$. In the latter case, each of the events

$$\begin{aligned} &\{\underline{h}_\infty^\lambda = 0\}, \qquad \{0 < \underline{h}_\infty^\lambda < \infty\}, \qquad \{\underline{h}_\infty^\lambda = \infty\}, \\ &\{\bar{h}_\infty^\lambda = 0\}, \qquad \{0 < \bar{h}_\infty^\lambda < \infty\}, \qquad \{\bar{h}_\infty^\lambda = \infty\} \end{aligned} \tag{2.3}$$

has $\mathbb{P}$-probability zero or one, see Lemmas 5.1 and 5.3. To provide some insight, we make a slight digression from the variational analysis and use one of these events to give a quenched characterization of weak disorder.

Theorem 2.9. *Assume* (Dir), (Ind), (Loc), $V \in \mathcal{L}$, *and* $\Lambda_q(V) < \infty$. *Then, there is weak disorder if and only if* $\mathbb{P}(0 < \underline{h}_\infty^\lambda < \infty) = 1$, *i.e.,* $\underline{h}_\infty^\lambda \in L^+$, *for* $\lambda = \Lambda_q(V)$.

Next, we use another event in (2.3) to conditionally prove that (qVar1) and (qVar2) do not always have any minimizers under strong disorder.

Theorem 2.10. *Assume* (Dir), (Ind), (Loc), $V \in \mathcal{L}$, *and* $\Lambda_q(V) < \infty$. *If there is strong disorder and* $\mathbb{P}(\bar{h}^{\lambda}_{\infty} = 0) = 0$ *for* $\lambda = \Lambda_q(V)$, *then* (qVar1) *and* (qVar2) *have no minimizers.*

Finally, we provide a sufficient condition for the key hypothesis of Theorem 2.10. To this end, we fix $\lambda = \Lambda_q(V) < \infty$ and let

$$H_n(\omega) := E_0\big[e^{\sum_{i=0}^{n-1} V(T_{X_i}\omega, Z_{i+1}) - n\Lambda_q(V)} \mathbb{1}_{\{X_n = (n/d,\dots,n/d)\}}\big] \tag{2.4}$$

be the "bridge" analog of h^{λ}_n. (For convenience, we assume that n is divisible by d.) With this notation, we clearly have $h^{\lambda}_n \geq H_n$.

Proposition 2.11. *Assume* (Dir), (Ind), (Loc), $V \in \mathcal{L}$, *and* $\Lambda_q(V) < \infty$. *If there exists an increasing sequence* $(a(n))_{n\geq 1}$ *such that*

$$\lim_{n\to\infty} a(n) = \infty, \qquad \lim_{n\to\infty} \frac{a(n-1)}{a(n)} = 1 \quad \textit{and} \quad \limsup_{n\to\infty} \mathbb{P}\big(\log H_n \geq a(n)\big) > 0, \tag{2.5}$$

then

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{\log H_n}{a(n)} \geq 1\right) = 1. \tag{2.6}$$

In particular, $\mathbb{P}(\bar{h}^{\lambda}_{\infty} = \infty) = 1$ *for* $\lambda = \Lambda_q(V)$.

It has been recently shown in [4] that $n^{-1/3} \log H_n$ has an F_{GUE} distributional limit for the log-gamma directed polymer model on $\mathbb{Z}^2$ with parameter $\gamma \in (0, \gamma^*)$ for some $\gamma^* > 0$. In particular, the conditions in Proposition 2.11 are satisfied with $a(n) = n^{1/3}$. On the other hand, since $d = 2$ in this example, it is in the very strong disorder regime by [10,25]. (Technically, [10] assumes that $\Lambda_a(\beta V) < \infty$ for every $\beta > 0$, and [25] weakens this assumption to $\Lambda_a(\beta_o V) < \infty$ for some $\beta_o > 0$. The log-gamma model satisfies only this weaker condition.) We thereby conclude that (qVar1) and (qVar2) do not always have any minimizers in the case of very strong disorder. We record this as a remark for future reference.

Remark 2.12. Assume (Dir), (Ind), (Loc), $V \in \mathcal{L}$, and $\Lambda_q(V) < \infty$. Then, as explained in the paragraph above, (qVar1) and (qVar2) do not always have any minimizers in the case of very strong disorder.

2.4. Additional remarks and open problems

We know from Theorem 1.9 that the critical inverse temperatures $\beta_c = \beta_c(V,d)$ and $\beta_c' = \beta_c'(V,d)$ satisfy $\beta_c = \beta_c' = 0$ for $d = 2, 3$, and it is natural to expect that $\beta_c = \beta_c'$ for every

$d \geq 2$. However, this is an open problem, see [11], Remark 3.2. Furthermore, it is generally believed that there is strong disorder at β_c for $d \geq 4$. The latter claim is supported by the analogous result in the context of directed polymers on trees which follows from [22].

With this background, here is our conjecture regarding the very strong disorder regime and the events in (2.3).

Conjecture 2.13. *Assume* (Dir), (Ind), (Loc), $V \in \mathcal{L}$, *and* $\Lambda_q(V) < \infty$. *Then,*

$$\mathbb{P}\big(0 = \underline{h}^\lambda_\infty < \bar{h}^\lambda_\infty = \infty\big) = 1$$

for $\lambda = \Lambda_q(V)$ *whenever there is very strong disorder.*

If this conjecture is indeed true, it would readily give the following quenched characterization of the disorder regimes.

(a) If there is weak disorder, then

$$\mathbb{P}\big(0 < \underline{h}^\lambda_\infty = \bar{h}^\lambda_\infty < \infty\big) = 1 \qquad \text{for } \lambda = \Lambda_q(V) = \Lambda_a(V) < \infty.$$

(b) If there is critically strong disorder, then

$$\mathbb{P}\big(\underline{h}^\lambda_\infty = \bar{h}^\lambda_\infty = 0\big) = 1 \qquad \text{for } \lambda = \Lambda_q(V) = \Lambda_a(V) < \infty.$$

(c) If there is very strong disorder, then

$$\mathbb{P}\big(0 = \underline{h}^\lambda_\infty < \bar{h}^\lambda_\infty = \infty\big) = 1 \qquad \text{for } \lambda = \Lambda_q(V) < \Lambda_a(V) \leq \infty.$$

This result would constitute a stronger version of Theorem 2.9. Note that parts (a) and (b) are trivial since $\underline{h}^\lambda_\infty = \bar{h}^\lambda_\infty = W_\infty$ for $\lambda = \Lambda_a(V)$.

As a second application, if Conjecture 2.13 is true, then very strong disorder would imply the hypotheses of Theorem 2.10, and (qVar1) and (qVar2) would never have any minimizers in that case. In other words, we could establish a stronger version of Remark 2.12.

The result of Borodin *et al.* [4] that we have used to satisfy the conditions of Proposition 2.11 is a form of Kardar–Parisi–Zhang (KPZ) universality and is expected to hold for a large class of models, see [12] for a survey. However, Proposition 2.11 is much more modest since it does not require any sharp estimates such as the $n^{1/3}$ scaling in KPZ universality. Indeed, slowly growing sequences, for example, $a(n) = \log\log\log n$, satisfy the first two conditions in (2.5).

Finally, observe that Theorem 2.10 is not applicable in the (hypothetical) case of critically strong disorder since, then, $\mathbb{P}(\bar{h}^\lambda_\infty = 0) = 1$ for $\lambda = \Lambda_q(V) = \Lambda_a(V)$. Therefore, we refrain from making any claims regarding the existence of any minimizers of (qVar1) and (qVar2) in that case.

3. Quenched free energy in the general case

3.1. Variational formulas (qVar0), (qVar1) and (qVar2) for $\Lambda_q(V)$

Lemma 3.1. *Assume that* $V(\cdot, z) \in L^1(\Omega, \mathfrak{S}, \mathbb{P})$ *for every* $z \in \mathcal{R}$. *If* $K(V, F) < \infty$ *with* $F = \nabla^* g$ *as defined in* (2.1) *for some* $g \in L^+(\Omega, \mathfrak{S}, \mathbb{P})$, *then* $F \in \mathcal{K}_0$.

Proof. It follows from the definition of $K(V,F)$ that

$$F(\cdot,z) \le \big|V(\cdot,z)\big| + \log|\mathcal{R}| + K(V,F) \tag{3.1}$$

$\mathbb{P}$-a.s. for every $z \in \mathcal{R}$. Therefore, $F^+(\cdot,z)$ is integrable and $\mathbb{E}[F(\cdot,z)]$ is well defined, even though it might a-priori be $-\infty$. Note that $\mathbb{E}[F(\cdot,z)] = -\infty$ is equivalent to $\mathbb{E}[|F(\cdot,z)|] = \infty$.

As a consequence of telescoping, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} F(T_{iz}\omega, z) = \frac{1}{n}\log\frac{g(T_{nz}\omega)}{g(\omega)} = \frac{1}{n}\log g(T_{nz}\omega) - \frac{1}{n}\log g(\omega). \tag{3.2}$$

By Birkhoff's ergodic theorem, the LHS of (3.2) converges $\mathbb{P}$-a.s. (and hence also in $\mathbb{P}$-probability) to $\mathbb{E}[F(\cdot,z)] \in [-\infty,\infty)$. However, the RHS of (3.2) converges to 0 in $\mathbb{P}$-probability. Indeed, for every $\varepsilon > 0$,

$$\mathbb{P}\bigg(\bigg|\frac{1}{n}\log g \circ T_{nz}\bigg| > \varepsilon\bigg) = \mathbb{P}\bigg(\bigg|\frac{1}{n}\log g\bigg| > \varepsilon\bigg) = \mathbb{P}\big(|\log g| > n\varepsilon\big) \to 0 \qquad \text{as } n \to \infty.$$

We conclude that $\mathbb{E}[F(\cdot,z)] = 0$ and $F(\cdot,z) \in L^1(\mathbb{P})$. Finally, the cocycle property is obvious from the definition of F. This finishes the proof. □

Proof of Theorem 2.1 (The upper bounds). We start by considering (qVar0). Take any $F \in \mathcal{K}_0$ and assume WLOG that $K(V,F) < \infty$ since the desired upper bound is otherwise trivial. Observe that

$$\begin{aligned}
&E_0\big[e^{\sum_{i=0}^{n-1} V(T_{X_i}\omega, Z_{i+1}) + F(T_{X_i}\omega, Z_{i+1})}\big] \\
&\quad = \sum_{x \in D_{n-1}} E_0\big[e^{\sum_{i=0}^{n-2} V(T_{X_i}\omega, Z_{i+1}) + F(T_{X_i}\omega, Z_{i+1})} \mathbb{1}_{\{X_{n-1}=x\}}\big] \sum_{z\in\mathcal{R}} p(z) e^{V(T_x\omega,z) + F(T_x\omega,z)} \\
&\quad \le \sum_{x \in D_{n-1}} E_0\big[e^{\sum_{i=0}^{n-2} V(T_{X_i}\omega, Z_{i+1}) + F(T_{X_i}\omega, Z_{i+1})} \mathbb{1}_{\{X_{n-1}=x\}}\big] e^{K(V,F)} \\
&\quad = E_0\big[e^{\sum_{i=0}^{n-2} V(T_{X_i}\omega, Z_{i+1}) + F(T_{X_i}\omega, Z_{i+1})}\big] e^{K(V,F)} \le \cdots \le e^{nK(V,F)},
\end{aligned}$$

where D_{n-1} is defined in (1.2). Therefore,

$$\begin{aligned}
\Lambda_q(V) &= \lim_{n\to\infty} \frac{1}{n}\log \mathcal{Z}^{\omega}_{n,0} \\
&\le \limsup_{n\to\infty} \frac{1}{n}\log E_0\big[e^{\sum_{i=0}^{n-1} V(T_{X_i}\omega, Z_{i+1}) + F(T_{X_i}\omega, Z_{i+1})}\big] \le K(V,F)
\end{aligned} \tag{3.3}$$

if

$$\liminf_{n\to\infty} \min_{x\in D_n} \frac{1}{n}\sum_{i=0}^{n-1} F(T_{x_i}\omega, z_{i+1}) \ge 0. \tag{3.4}$$

Here, by the cocycle property, $(x_i)_{i=0}^n$ is allowed to be any path such that $z_{i+1} = x_{i+1} - x_i \in \mathcal{R}$, $x_0 = 0$ and $x_n = x$.

We see from (3.1) that F is $\mathbb{P}$-a.s. bounded from above by a function in $\mathcal{L}$. Under this assumption, it has been recently shown in [18], Theorem 9.3, that

$$\lim_{n\to\infty} \max_{x\in D_n} \frac{1}{n} \left| \sum_{i=0}^{n-1} F(T_{x_i}\omega, z_{i+1}) \right| = 0.$$

This is an ergodic theorem for cocycles. In particular, we have (3.4), and therefore, (3.3). Taking infimum over all $F \in \mathcal{K}_0$ gives the upper bound in (qVar0).

The upper bounds in (qVar1) and (qVar2) are now easy. Indeed, take any $g \in L^+(\Omega, \mathfrak{S}, \mathbb{P})$ and assume WLOG that $K'(V, g) < \infty$ since the desired upper bounds are otherwise trivial. Then, $F := \nabla^* g \in \mathcal{K}_0$ by Lemma 3.1, and $\Lambda_q(V) \leq K(V, F) = K'(V, g)$ by the upper bound in (qVar0). Taking infimum over all $g \in L^+(\Omega, \mathfrak{S}, \mathbb{P})$ gives the upper bound in (qVar1), from which the upper bound in (qVar2) follows. □

Remark 3.2. Note that, for the logarithmic gradient $F = \nabla^* g$ of any $g \in L^+(\Omega, \mathfrak{S}, \mathbb{P})$, the condition in (3.4) can be written as

$$\liminf_{n\to\infty} \min_{x\in D_n} \frac{1}{n} \log \frac{g(T_x\omega)}{g(\omega)} \geq 0. \tag{3.5}$$

Consequently, the upper bound in (qVar2) does not rely on the aforementioned ergodic theorem for cocycles because (3.5) is obvious for $g \in L^{++}(\Omega, \mathfrak{S}, \mathbb{P})$.

Proof of Theorem 2.1 (The lower bounds). Assume WLOG that $\Lambda_q(V) < \infty$ since the desired lower bounds are otherwise trivial. For any $\lambda > \Lambda_q(V)$ and $n \geq 1$, recall the function h_n^λ which was introduced in (2.2). Set $h_0^\lambda = 1$ as a convention and define

$$g_\lambda := \sum_{n=0}^{\infty} h_n^\lambda \geq 1. \tag{3.6}$$

Since

$$\lim_{n\to\infty} \frac{1}{n} \log h_n^\lambda(\omega) = \Lambda_q(V) - \lambda < 0$$

for $\mathbb{P}$-a.e. ω, we have $g_\lambda \in L^{++}(\Omega, \mathfrak{S}, \mathbb{P})$. Moreover, under (Dir) and (Loc), g_λ is future measurable.

Decompose g_λ in the following way: for $\mathbb{P}$-a.e. ω,

$$\begin{aligned} g_\lambda(\omega) &= 1 + \sum_{n=1}^{\infty} h_n^\lambda(\omega) = 1 + \sum_{n=1}^{\infty} \sum_{z\in\mathcal{R}} p(z) e^{V(\omega,z)-\lambda} h_{n-1}^\lambda(T_z\omega) \\ &= 1 + \sum_{z\in\mathcal{R}} p(z) e^{V(\omega,z)-\lambda} \sum_{n=1}^{\infty} h_{n-1}^\lambda(T_z\omega) = 1 + \sum_{z\in\mathcal{R}} p(z) e^{V(\omega,z)-\lambda} g_\lambda(T_z\omega). \end{aligned}$$

Rearranging this, we see that

$$\lambda = \log\left(\frac{e^{\lambda}}{g_\lambda(\omega)} + \sum_{z\in\mathcal{R}} \frac{p(z)e^{V(\omega,z)}g_\lambda(T_z\omega)}{g_\lambda(\omega)}\right) > \log\left(\sum_{z\in\mathcal{R}} \frac{p(z)e^{V(\omega,z)}g_\lambda(T_z\omega)}{g_\lambda(\omega)}\right). \tag{3.7}$$

Therefore, $\lambda \geq K'(V, g_\lambda)$. First, taking infimum over all $g \in L^{++}$ and then taking infimum over $\lambda > \Lambda_q(V)$ gives the lower bound in (qVar2), from which the lower bounds in (qVar1) and (qVar0) follow since $\nabla^* g_\lambda \in \mathcal{K}_0$ by Lemma 3.1. □

3.2. Minimizing the variational formula (qVar0) for $\Lambda_q(V)$

Proof of Theorem 2.3. If $\Lambda_q(V) = \infty$, then every $F \in \mathcal{K}_0$ is trivially a minimizer of (qVar0). Therefore, in the rest of the proof, we will assume that $\Lambda_q(V) < \infty$.

Since (qVar0) involves an infimum, for every $i \geq 1$, there exists an $F_i \in \mathcal{K}_0$ such that

$$\sum_{z\in\mathcal{R}} p(z)e^{V(\cdot,z)+F_i(\cdot,z)} \leq e^{\Lambda_q(V)+1/i}$$

holds $\mathbb{P}$-a.s. Note that

$$F_i(\cdot, z) \leq \left|V(\cdot, z)\right| + \log|\mathcal{R}| + \Lambda_q(V) + 1/i$$

for every $z \in \mathcal{R}$. Since $V(\cdot, z)$ is in $L^1(\mathbb{P})$, we see that $F_i^+(\cdot, z)$ is uniformly integrable. The F_i are centered by definition, so we have $\mathbb{E}[F_i^-(\cdot, z)] = \mathbb{E}[F_i^+(\cdot, z)]$. Therefore, $\mathbb{E}[F_i^-(\cdot, z)]$ is uniformly bounded. By [24], Lemma 4.3, we can write

$$F_i^-(\cdot, z) = \hat{F}_i^-(\cdot, z) + R_i(\cdot, z),$$

where, up to a common subsequence, $\hat{F}_i^-(\cdot, z)$ is uniformly integrable and $R_i(\cdot, z) \geq 0$ converges to 0 in $\mathbb{P}$-probability. Extracting a further subsequence, $\tilde{F}_i(\cdot, z) = F_i^+(\cdot, z) - \hat{F}_i^-(\cdot, z)$ is weakly convergent in $L^1(\mathbb{P})$ to some $\tilde{F}(\cdot, z)$, and $R_i(\cdot, z)$ converges $\mathbb{P}$-a.s. to 0. By [31], Theorem 3.12, $\tilde{F}(\cdot, z)$ is in the strong $L^1(\mathbb{P})$-closure of the convex hull of $\{\tilde{F}_i(\cdot, z) : i \geq 1\}$, that is, there exists a finite convex combination $\tilde{G}_i(\cdot, z) := \sum_{j=i}^{\infty} \alpha_{i,j}\tilde{F}_j(\cdot, z)$ that converges to $\tilde{F}(\cdot, z)$ strongly in $L^1(\mathbb{P})$. Up to a further subsequence, $\tilde{G}_i(\cdot, z)$ converges $\mathbb{P}$-a.s. to $\tilde{F}(\cdot, z)$. This ensures that $\tilde{F}(\cdot, z)$ satisfies the cocycle property. Moreover, since $R_i(\cdot, z) \geq 0$, we have $c(z) := \mathbb{E}[\tilde{F}(\cdot, z)] \geq 0$. Let $F(\cdot, z) = \tilde{F}(\cdot, z) - c(z)$ for every $z \in \mathcal{R}$. Then, $F \in \mathcal{K}_0$. By Jensen's inequality,

$$\sum_{z\in\mathcal{R}} p(z)e^{V(\cdot,z)+\tilde{G}_i(\cdot,z)-\sum_{j=i}^{\infty}\alpha_{i,j}R_j(\cdot,z)} \leq e^{\Lambda_q(V)+1/i}.$$

Sending $i \to \infty$, we get

$$\sum_{z\in\mathcal{R}} p(z)e^{V(\omega,z)+F(\omega,z)+c(z)} \leq e^{\Lambda_q(V)}$$

for $\mathbb{P}$-a.e. ω and conclude that F is a minimizer of (qVar0). Plus, we deduce that $c(z) = 0$ for every $z \in \mathcal{R}$ since, otherwise, the RHS of (qVar0) would be strictly less than $\Lambda_q(V)$. $\square$

4. Annealed free energy in the directed i.i.d. case

In the rest of the paper, L^+, L^{++} and L^1 stand for $L^+(\Omega, \mathfrak{S}_0^\infty, \mathbb{P})$, $L^{++}(\Omega, \mathfrak{S}_0^\infty, \mathbb{P})$ and $L^1(\Omega, \mathfrak{S}_0^\infty, \mathbb{P})$, respectively.

4.1. Variational formulas (aVar1) and (aVar2) for $\Lambda_a(V)$

Proof of Theorem 2.4 (The upper bounds). Take any $g \in L^+ \cap L^1$ and assume WLOG that $K'(V,g) < \infty$ since the desired upper bounds are otherwise trivial. Then, for $\mathbb{P}$-a.e. ω,

$$K'(V,g) \geq \log\left(\sum_{z\in\mathcal{R}} \frac{p(z)e^{V(\omega,z)}g(T_z\omega)}{g(\omega)}\right).$$

Rearranging this, we get

$$g(\omega) \geq \sum_{z\in\mathcal{R}} p(z)e^{V(\omega,z)-K'(V,g)}g(T_z\omega). \tag{4.1}$$

For every $z \in \mathcal{R}$, the random variables $V(\cdot, z)$ and $g \circ T_z$ are independent by (Dir), (Ind), (Loc) and the future measurability of g. Taking the expectation of both sides of (4.1), we see that

$$\begin{aligned}\mathbb{E}[g] &\geq \sum_{z\in\mathcal{R}} p(z)\mathbb{E}\big[e^{V(\cdot,z)-K'(V,g)}g \circ T_z\big] \\ &= \sum_{z\in\mathcal{R}} p(z)\mathbb{E}\big[e^{V(\cdot,z)-K'(V,g)}\big]\mathbb{E}[g \circ T_z] = e^{\Lambda_a(V)-K'(V,g)}\mathbb{E}[g]\end{aligned} \tag{4.2}$$

by stationarity, which implies $\Lambda_a(V) \leq K'(V,g)$. The infimum over all $g \in L^+ \cap L^1$ gives the upper bound in (aVar1), from which the upper bound in (aVar2) follows. $\square$

Proof of Theorem 2.4 (The lower bounds). Assume WLOG that $\Lambda_a(V) < \infty$ since the desired lower bounds are otherwise trivial. Take any $\lambda > \Lambda_a(V)$ and recall the function $g_\lambda \in L^{++}$ which is defined in (3.6). Its expected value is easy to compute:

$$\mathbb{E}[g_\lambda] = \sum_{n=0}^\infty \mathbb{E}\big[h_n^\lambda\big] = \sum_{n=0}^\infty e^{n(\Lambda_a(V)-\lambda)} = \frac{1}{1-e^{\Lambda_a(V)-\lambda}} < \infty.$$

Therefore, $g_\lambda \in L^{++} \cap L^1$. We have seen in (3.7) that $\lambda \geq K'(V, g_\lambda)$. Taking first infimum over all $g \in L^{++} \cap L^1$ and then infimum over $\lambda > \Lambda_a(V)$ gives the lower bound in (aVar2), from which the lower bound in (aVar1) follows. $\square$

4.2. An annealed variational characterization of weak disorder

Lemma 4.1. *Assume* (Dir), (Ind), *and* (Loc). *Then, weak disorder is equivalent to the existence of a function* $g \in L^+ \cap L^1$ *such that*

$$g = \sum_{z \in \mathcal{R}} p(z) e^{V(\cdot,z)-\lambda} g \circ T_z \tag{4.3}$$

$\mathbb{P}$-*a.s. for some* $\lambda \in \mathbb{R}$. *In that case*, $\lambda = \Lambda_a(V)$, *and* g *is equal* (*up to a multiplicative constant*) *to* W_∞ *which is defined in* (1.7).

Remark 4.2. This result has been previously obtained as part of [11], Proposition 3.1, in the case of directed polymers, that is, for potentials that do not depend on z. Our proof below is a straightforward adaptation, which we include for the sake of completeness as well as for demonstrating a technique that we will use in the rest of the paper.

Proof of Lemma 4.1. If there is weak disorder, then $\Lambda_a(V) < \infty$ and $W_\infty \in L^+$ by definition. Observe that

$$\mathbb{E}[W_\infty] \le \liminf_{n\to\infty} \mathbb{E}[W_n] = 1$$

by Fatou's lemma, so in fact $W_\infty \in L^+ \cap L^1$. Decompose W_n with respect to the first step of the underlying random walk and see that

$$W_n = \sum_{z \in \mathcal{R}} p(z) e^{V(\cdot,z)-\Lambda_a(V)} W_{n-1} \circ T_z. \tag{4.4}$$

Taking $n \to \infty$ gives (4.3) with $g = W_\infty$ and $\lambda = \Lambda_a(V)$.

Conversely, if there exists some $g \in L^+ \cap L^1$ and $\lambda \in \mathbb{R}$ such that (4.3) is satisfied, then we take the expectation of both sides of (4.3) and get

$$\mathbb{E}[g] = \sum_{z \in \mathcal{R}} p(z) \mathbb{E}\big[e^{V(\cdot,z)-\lambda}\big] \mathbb{E}[g \circ T_z] = e^{\Lambda_a(V)-\lambda} \mathbb{E}[g]$$

which implies that $\Lambda_a(V) = \lambda < \infty$. Here, as in (4.2), we used (Dir), (Ind), (Loc) and the future measurability of g. Iterating (4.3) for $n \ge 1$ times, we get

$$\begin{aligned} g &= E_0\left[\exp\left(\sum_{i=0}^{n-1} V(T_{X_i}\cdot, Z_{i+1}) - n\Lambda_a(V)\right) g \circ T_{X_n}\right] \\ &= \sum_x h_n^\lambda(\cdot,x) g \circ T_x \end{aligned}$$

with $\lambda = \Lambda_a(V)$ and

$$h_n^\lambda(\cdot,x) = E_0\big[e^{\sum_{i=0}^{n-1} V(T_{X_i}\cdot, Z_{i+1}) - n\lambda} \mathbb{1}_{\{X_n = x\}}\big].$$

Observe that $h_n^\lambda(\cdot, x)$ is $\mathfrak{S}_0^n$-measurable since X_n only takes values $x \in \mathbb{Z}_+^d$ such that $|x|_1 = n$. On the other hand, $g \circ T_x$ is independent of $\mathfrak{S}_0^n$ since g is future measurable. Therefore,

$$\mathbb{E}\big[g|\mathfrak{S}_0^n\big] = \sum_x h_n^\lambda(\cdot, x)\mathbb{E}[g \circ T_x] = \sum_x h_n^\lambda(\cdot, x)\mathbb{E}[g] = W_n \mathbb{E}[g]. \tag{4.5}$$

Finally,

$$W_\infty = \lim_{n\to\infty} W_n = \lim_{n\to\infty} \frac{\mathbb{E}[g|\mathfrak{S}_0^n]}{\mathbb{E}[g]} = \frac{g}{\mathbb{E}[g]} > 0$$

holds $\mathbb{P}$-a.s. and we conclude that there is weak disorder. □

Proof of Theorem 2.6. If there is weak disorder, then by Lemma 4.1, there exists a $g \in L^+ \cap L^1$ that satisfies (4.3) with $\lambda = \Lambda_a(V) < \infty$. Rearranging this equality, we immediately see that g is a minimizer of (aVar1) and there is no need for taking essential supremum in $K'(V, g)$.

Conversely, if $\Lambda_a(V) < \infty$ and (aVar1) has a minimizer $g \in L^+ \cap L^1$, then we have

$$g(\omega) \geq \sum_{z\in\mathcal{R}} p(z)e^{V(\omega,z)-\Lambda_a(V)} g(T_z\omega) \tag{4.6}$$

for $\mathbb{P}$-a.e. ω. If taking essential supremum in $K'(V, g)$ were indeed necessary, then the inequality in (4.6) would be strict on a set of positive $\mathbb{P}$-probability. In that case, we would have

$$\begin{aligned}\mathbb{E}[g] &> \sum_{z\in\mathcal{R}} p(z)\mathbb{E}\big[e^{V(\cdot,z)-\Lambda_a(V)} g \circ T_z\big] \\ &= \sum_{z\in\mathcal{R}} p(z)\mathbb{E}\big[e^{V(\cdot,z)-\Lambda_a(V)}\big]\mathbb{E}[g \circ T_z] = e^{\Lambda_a(V)-\Lambda_a(V)}\mathbb{E}[g] = \mathbb{E}[g]\end{aligned}$$

which is a contradiction. Hence, there is no need for taking essential supremum in $K'(V, g)$. Therefore, g satisfies (4.3) with $\lambda = \Lambda_a(V)$. By Lemma 4.1, we have weak disorder and g is equal (up to a multiplicative constant) to W_∞. This concludes the proof of part (a).

For part (b), note that any minimizer of (aVar2) would be a minimizer of (aVar1). Therefore, by part (a), (aVar2) has no minimizers under strong disorder, and has at most one minimizer under weak disorder, namely W_∞. However, the latter is ruled out by Proposition 4.3 below unless $\mathcal{Z}_{1,0}^\omega$ is $\mathbb{P}$-essentially constant. □

Proposition 4.3. *Assume* (Dir), (Ind), (Loc), *and weak disorder. Then,*

$$\text{(a)} \quad \mathbb{P}\text{-}\operatorname*{ess\,inf}_\omega W_\infty(\omega) = 0 \quad \textit{and} \quad \text{(b)} \quad \mathbb{P}\text{-}\operatorname*{ess\,sup}_\omega W_\infty(\omega) = \infty$$

unless

$$\mathcal{Z}_{1,0}^\omega = \sum_{z\in\mathcal{R}} p(z)e^{V(\omega,z)}$$

is $\mathbb{P}$-essentially constant, cf. Remark 2.7.

Proof. Let us prove part (a) by contradiction. Suppose $\exists c \in (0,1)$ such that $\mathbb{P}(W_\infty > c) = 1$. Then, we have $\mathbb{P}(W_n \le c) = 0$ for every $n \ge 1$ because, otherwise,

$$\begin{aligned} c\mathbb{P}(W_n \le c) &< \mathbb{E}[W_\infty \mathbb{1}_{\{W_n \le c\}}] = \mathbb{E}\big[\mathbb{E}\big[W_\infty \mathbb{1}_{\{W_n \le c\}} | \mathfrak{S}_0^n\big]\big] = \mathbb{E}\big[\mathbb{E}\big[W_\infty | \mathfrak{S}_0^n\big] \mathbb{1}_{\{W_n \le c\}}\big] \\ &= \mathbb{E}[W_n \mathbb{1}_{\{W_n \le c\}}] \le c\mathbb{P}(W_n \le c). \end{aligned}$$

On the other hand, if $\mathcal{Z}_{1,0}^\omega$ is not $\mathbb{P}$-essentially constant, then $\mathbb{P}(W_n \le c) > 0$ for large $n \ge 1$. Indeed, $\mathbb{E}[\mathcal{Z}_{1,0}^\omega] = e^{\Lambda_a(V)}$ and there exists a $\delta > 0$ such that $\mathbb{P}(\mathcal{Z}_{1,0}^\omega \le e^{\Lambda_a(V)-\delta}) > 0$. By the assumptions (Ind) and (Loc), the event

$$\bigcap_{|x|_1 \le n-1} \{\omega : \mathcal{Z}_{1,x}^\omega \le e^{\Lambda_a(V)-\delta}\}$$

has positive $\mathbb{P}$-probability. On this event,

$$\begin{aligned} W_n(\omega) &= \sum_x W_{n-1}(\omega, x) \sum_{z \in \mathcal{R}} p(z) e^{V(T_x\omega, z) - \Lambda_a(V)} = \sum_x W_{n-1}(\omega, x) \mathcal{Z}_{1,x}^\omega e^{-\Lambda_a(V)} \\ &\le W_{n-1}(\omega) e^{-\delta} \le \cdots \le e^{-n\delta} \le c \end{aligned}$$

for $n \ge |\log c|/\delta$. Here, $W_{n-1}(\omega, x) := h_{n-1}^\lambda(\omega, x)$ with $\lambda = \Lambda_a(V)$. The proof of part (b) is similar. □

5. Analysis of (qVar1) and (qVar2) in the directed i.i.d. case

5.1. Quenched variational analysis of weak disorder

Proof of Theorem 2.8. First, without assuming weak disorder, suppose $V \in \mathcal{L}$, $\Lambda_q(V) < \infty$, and $g \in L^+$ is a minimizer of (qVar1). Then, it satisfies

$$g(\omega) \ge \sum_{z \in \mathcal{R}} p(z) e^{V(\omega, z) - \Lambda_q(V)} g(T_z\omega) \tag{5.1}$$

for $\mathbb{P}$-a.e. ω. Iterating this inequality for $n \ge 1$ times, we see that

$$g(\omega) \ge \sum_x h_n^\lambda(\omega, x) g(T_x\omega)$$

holds with $\lambda = \Lambda_q(V)$. Dividing both sides by $h_n^\lambda(\omega)$, we get

$$\frac{g(\omega)}{h_n^\lambda(\omega)} \ge \sum_x \mu_n(\omega, x) g(T_x\omega),$$

where

$$\mu_n(\omega, x) := \frac{h_n^\lambda(\omega, x)}{h_n^\lambda(\omega)} = Q_{n,0}^\omega(X_n = x)$$

does not depend on λ. For any $0 < M < \infty$,

$$\frac{g(\omega)}{h_n^\lambda(\omega)} \wedge M \ge \sum_x \mu_n(\omega, x)\big(g(T_x\omega) \wedge M\big)$$

by Jensen's inequality since $u \mapsto u \wedge M$ is a concave function. Note that, as in the proof of Lemma 4.1, for every $x \in \mathbb{Z}_+^d$ with $|x|_1 = n$, the random variable $\mu_n(\cdot, x)$ (resp., $g \circ T_x$) is measurable w.r.t. (resp., independent of) the σ-algebra $\mathfrak{S}_0^n$. Therefore,

$$\begin{aligned}\mathbb{E}\left[\frac{g}{h_n^\lambda} \wedge M \middle| \mathfrak{S}_0^n\right] &\ge \sum_x \mu_n(\cdot, x)\mathbb{E}\big[(g \circ T_x) \wedge M\big] \\ &= \sum_x \mu_n(\cdot, x)\mathbb{E}[g \wedge M] = \mathbb{E}[g \wedge M].\end{aligned} \tag{5.2}$$

In the case of weak disorder, we know that $\lambda = \Lambda_q(V) = \Lambda_a(V)$ by (1.8), h_n^λ converges $\mathbb{P}$-a.s. to $W_\infty \in L^+ \cap L^1$ as $n \to \infty$, and W_∞ is a minimizer of (qVar1). By the dominated convergence theorem for conditional expectations (see [17], Theorem 5.5.9), the LHS of (5.2) converges $\mathbb{P}$-a.s. to $(g/W_\infty) \wedge M$ as $n \to \infty$. Therefore,

$$\mathbb{E}[g \wedge M] \le \frac{g}{W_\infty} \wedge M \le \frac{g}{W_\infty} < \infty$$

holds $\mathbb{P}$-a.s. Sending $M \to \infty$ and applying the monotone convergence theorem, we see that $\mathbb{E}[g] < \infty$. So, $g \in L^+ \cap L^1$ and it is a minimizer of (aVar1). By Theorem 2.6, g is equal (up to a multiplicative constant) to W_∞. This concludes the proof of part (a). Finally, part (b) follows from Proposition 4.3 since $W_\infty \notin L^{++}$ unless $\mathcal{Z}_{1,0}^\omega$ is $\mathbb{P}$-essentially constant. □

5.2. A quenched characterization of weak disorder

Lemma 5.1. *For $\lambda = \Lambda_q(V) < \infty$, each of the events $\{\underline{h}_\infty^\lambda = 0\}$, $\{0 < \underline{h}_\infty^\lambda < \infty\}$ and $\{\underline{h}_\infty^\lambda = \infty\}$ has $\mathbb{P}$-probability zero or one.*

Proof. For every $m, n \ge 1$ and $x \in \mathbb{Z}_+^d$ such that $|x|_1 = m$, we have

$$h_{m+n}^\lambda = \sum_y h_m^\lambda(\cdot, y) h_n^\lambda \circ T_y \ge h_m^\lambda(\cdot, x) h_n^\lambda \circ T_x, \tag{5.3}$$

where the equality follows from decomposing the LHS w.r.t. the possible values of X_m. Taking liminf of both sides as $n \to \infty$, we get

$$\underline{h}_\infty^\lambda \ge h_m^\lambda(\cdot, x)\underline{h}_\infty^\lambda \circ T_x.$$

Therefore,

$$\big\{\omega : \underline{h}_\infty^\lambda(\omega) = 0\big\} \subset \bigcap_{m=1}^\infty \bigcap_{|x|_1 = m} \big\{\omega : \underline{h}_\infty^\lambda(T_x\omega) = 0\big\} \tag{5.4}$$

and

$$\{\omega : \underline{h}^\lambda_\infty(\omega) = \infty\} \supset \bigcup_{m=1}^{\infty} \bigcup_{|x|_1=m} \{\omega : \underline{h}^\lambda_\infty(T_x\omega) = \infty\}. \tag{5.5}$$

If $\mathbb{P}(\underline{h}^\lambda_\infty = 0) < 1$, then by ergodicity the RHS of (5.4) is a $\mathbb{P}$-probability zero event and, therefore, we in fact have $\mathbb{P}(\underline{h}^\lambda_\infty = 0) = 0$. Similarly, if $\mathbb{P}(\underline{h}^\lambda_\infty = \infty) > 0$, then by ergodicity the RHS of (5.5) is a $\mathbb{P}$-probability one event and, therefore, we in fact have $\mathbb{P}(\underline{h}^\lambda_\infty = \infty) = 1$. □

Proof of Theorem 2.9. One direction is immediate. Indeed, if there is weak disorder, then

$$\underline{h}^\lambda_\infty = \liminf_{n\to\infty} h^\lambda_n = \lim_{n\to\infty} W_n = W_\infty \in L^+$$

for $\lambda = \Lambda_q(V) = \Lambda_a(V)$.

Conversely, assume that $\underline{h}^\lambda_\infty \in L^+$ for $\lambda = \Lambda_q(V) < \infty$. Then, for every $n \geq 1$, we have

$$h^\lambda_{n+1} = \sum_{z\in\mathcal{R}} p(z) e^{V(\cdot,z)-\Lambda_q(V)} h^\lambda_n \circ T_z.$$

Taking liminf of both sides as $n \to \infty$, we get

$$\underline{h}^\lambda_\infty \geq \sum_{z\in\mathcal{R}} p(z) e^{V(\cdot,z)-\Lambda_q(V)} \underline{h}^\lambda_\infty \circ T_z.$$

Multiplying both sides of this inequality by $e^{\Lambda_q(V)}/\underline{h}^\lambda_\infty$ and then taking logarithm, we see that $\Lambda_q(V) \geq K'(V, \underline{h}^\lambda_\infty)$, so $\underline{h}^\lambda_\infty$ is a minimizer of (qVar1). The proof of Theorem 2.8 carries over until (5.2) and we have

$$\mathbb{E}\left[\frac{\underline{h}^\lambda_\infty}{h^\lambda_n} \wedge M \middle| \mathfrak{S}^n_0\right] \geq \mathbb{E}\big[\underline{h}^\lambda_\infty \wedge M\big]$$

for every $0 < M < \infty$. Observe that

$$\limsup_{n\to\infty} \frac{\underline{h}^\lambda_\infty}{h^\lambda_n} = \left(\liminf_{n\to\infty} \frac{h^\lambda_n}{\underline{h}^\lambda_\infty}\right)^{-1} = 1.$$

By a simple modification of the dominated convergence theorem for conditional expectations (see Lemma 5.2 below), we have

$$\mathbb{E}\big[\underline{h}^\lambda_\infty \wedge M\big] \leq \limsup_{n\to\infty} \mathbb{E}\left[\frac{\underline{h}^\lambda_\infty}{h^\lambda_n} \wedge M \middle| \mathfrak{S}^n_0\right] \leq 1 \wedge M.$$

We send $M \to \infty$ and get $\mathbb{E}[\underline{h}^\lambda_\infty] \leq 1$ by the monotone convergence theorem. Therefore, $\underline{h}^\lambda_\infty \in L^+ \cap L^1$ and it is a minimizer of (aVar1) since $\Lambda_a(V) \leq K'(V, \underline{h}^\lambda_\infty) = \Lambda_q(V) \leq \Lambda_a(V)$. In particular, $\Lambda_a(V) < \infty$. Finally, we use Theorem 2.6 to conclude that there is weak disorder. □

Lemma 5.2. *Let Y_n, Y and Z be future measurable functions such that $Y = \limsup_{n\to\infty} Y_n$, $|Y_n| \le Z$ for all $n \ge 1$, and $\mathbb{E}[Z] < \infty$. Then, $\limsup_{n\to\infty} \mathbb{E}[Y_n|\mathfrak{S}_0^n] \le Y$ holds $\mathbb{P}$-a.s.*

Proof. Let $U_N = \sup\{Y_n - Y : n \ge N\}$ for every $N \ge 1$. Then, $|U_N| \le 2Z$, so $\mathbb{E}[|U_N|] < \infty$. Now,

$$\limsup_{n\to\infty} \mathbb{E}\big[Y_n - Y|\mathfrak{S}_0^n\big] \le \lim_{n\to\infty} \mathbb{E}\big[U_N|\mathfrak{S}_0^n\big] = U_N.$$

Sending $N \to \infty$, we see that

$$\limsup_{n\to\infty} \mathbb{E}\big[Y_n - Y|\mathfrak{S}_0^n\big] \le \lim_{N\to\infty} U_N = \limsup_{n\to\infty} Y_n - Y = 0.$$

We conclude that

$$\limsup_{n\to\infty} \mathbb{E}\big[Y_n|\mathfrak{S}_0^n\big] \le \lim_{n\to\infty} \mathbb{E}\big[Y|\mathfrak{S}_0^n\big] = Y. \qquad \square$$

5.3. Quenched variational analysis of strong disorder

Lemma 5.3. *For $\lambda = \Lambda_q(V) < \infty$, each of the events $\{\bar{h}_\infty^\lambda = 0\}$, $\{0 < \bar{h}_\infty^\lambda < \infty\}$ and $\{\bar{h}_\infty^\lambda = \infty\}$ has $\mathbb{P}$-probability zero or one.*

Proof. Taking limsup as $n \to \infty$ of both sides of the inequality in (5.3), we get

$$\bar{h}_\infty^\lambda \ge h_m^\lambda(\cdot, x)\bar{h}_\infty^\lambda \circ T_x$$

for every $m \ge 1$ and $x \in \mathbb{Z}_+^d$ such that $|x|_1 = m$. Therefore, the set relations (5.4) and (5.5) hold with $\underline{h}_\infty^\lambda$ replaced by $\bar{h}_\infty^\lambda$. The rest of the proof is identical to that of Lemma 5.1. $\square$

Proof of Theorem 2.10. Fix $\lambda = \Lambda_q(V)$ and assume that $\mathbb{P}(0 < \bar{h}_\infty^\lambda \le \infty) = 1$. Take any future measurable function g satisfying $\mathbb{P}(0 \le g < \infty) = 1$ and (5.1). Our strategy will be to show that $g \in L^1$. The proof of Theorem 2.8 carries over until (5.2) and we have

$$\mathbb{E}\left[\frac{g}{h_n^\lambda} \wedge M \middle| \mathfrak{S}_0^n\right] \ge \mathbb{E}[g \wedge M]$$

for every $0 < M < \infty$. Pick a sufficiently small $\delta > 0$ such that $\mathbb{P}(\bar{h}_\infty^\lambda > \delta) > 0$. Let

$$\mathbf{n}_1 = \mathbf{n}_1(\omega) = \inf\big\{n \ge 1 : h_n^\lambda(\omega) \ge \delta\big\}$$

be the first time that $h_n^\lambda(\omega) \ge \delta$ (if such a time exists, otherwise it is infinite). Similarly, for every $k \ge 2$, let

$$\mathbf{n}_k = \mathbf{n}_k(\omega) = \inf\big\{n > \mathbf{n}_{k-1} : h_n^\lambda(\omega) \ge \delta\big\}.$$

Each $\mathbf{n}_k$ is an $\mathbb{N} \cup \{\infty\}$-valued stopping time and we can consider the σ-algebras

$$\mathfrak{S}_0^{\mathbf{n}_k} := \big\{A \in \mathfrak{S}_0^\infty : A \cap \{\mathbf{n}_k \le n\} \in \mathfrak{S}_0^n \text{ for every } n \ge 1\big\}.$$

For every $k \geq 1$, we have

$$\begin{aligned}
\mathbb{E}\left[\frac{g}{h^\lambda_{\mathbf{n}_k}} \wedge M \middle| \mathfrak{S}_0^{\mathbf{n}_k}\right]\mathbb{1}_{\{\mathbf{n}_k<\infty\}} &= \sum_{n=1}^{\infty}\mathbb{E}\left[\frac{g}{h^\lambda_n} \wedge M \middle| \mathfrak{S}_0^{\mathbf{n}_k}\right]\mathbb{1}_{\{\mathbf{n}_k=n\}} \\
&= \sum_{n=1}^{\infty}\mathbb{E}\left[\frac{g}{h^\lambda_n} \wedge M \middle| \mathfrak{S}_0^{n}\right]\mathbb{1}_{\{\mathbf{n}_k=n\}} \qquad (5.6)\\
&\geq \sum_{n=1}^{\infty}\mathbb{E}[g \wedge M]\mathbb{1}_{\{\mathbf{n}_k=n\}} = \mathbb{E}[g \wedge M]\mathbb{1}_{\{\mathbf{n}_k<\infty\}}.
\end{aligned}$$

Here, (5.6) follows from Lemma 5.5 below. Now, on the set

$$\left\{\bar{h}^\lambda_\infty > \delta\right\} \subset \bigcap_{k\geq 1}\{\mathbf{n}_k < \infty\},$$

we have $h^\lambda_{\mathbf{n}_k} \geq \delta$ for every $k \geq 1$, and therefore

$$\mathbb{E}[g \wedge M]\mathbb{1}_{\{\bar{h}^\lambda_\infty>\delta\}} \leq \mathbb{E}\left[\frac{g}{\delta} \wedge M \middle| \mathfrak{S}_0^{\mathbf{n}_k}\right]\mathbb{1}_{\{\bar{h}^\lambda_\infty>\delta\}}.$$

Since $\mathfrak{S}_0^{\mathbf{n}_k} \uparrow \mathfrak{S}_0^{\infty}$ as $k \to \infty$, we deduce that

$$\mathbb{E}[g \wedge M] \leq (g/\delta) \wedge M \qquad (5.7)$$

on the set $\{\bar{h}^\lambda_\infty > \delta\}$. Next, send $M \to \infty$ and apply the monotone convergence theorem to get $\mathbb{E}[g] \leq g/\delta$ on the same set. This means that $g \in L^1$. On the other hand, if g were in $L^+ \cap L^1$, then it would be a minimizer of (aVar1) since it would satisfy $\Lambda_a(V) \leq K'(V,g) \leq \Lambda_q(V) \leq \Lambda_a(V)$ by rearranging (5.1), and there would be weak disorder by Theorem 2.6. Therefore, $g \notin L^+$. We conclude that (qVar1) has no minimizers since every minimizer of (qVar1) must satisfy (5.1) and be in L^+. In particular, (qVar2) also has no minimizers. □

Remark 5.4. We can strengthen the argument at the end of the proof of Theorem 2.10 in the following way. Since $g \notin L^+$, the set $\{g = 0\}$ has positive $\mathbb{P}$-probability. We can pick a sufficiently small $\delta > 0$ such that $\{g = 0\}$ and $\{\bar{h}^\lambda_\infty > \delta\}$ have a nontrivial intersection. Then, $\mathbb{E}[g \wedge M] = 0$ by (5.7) and, sending $M \to \infty$, we see that $\mathbb{E}[g] = 0$, that is, $\mathbb{P}(g = 0) = 1$.

Lemma 5.5. *Let τ be an $\mathbb{N} \cup \{\infty\}$-valued stopping time for the filtration $(\mathfrak{S}_0^n)_{n\geq 1}$. Consider the σ-algebra*

$$\mathfrak{S}_0^\tau := \left\{A \in \mathfrak{S}_0^\infty : A \cap \{\tau \leq n\} \in \mathfrak{S}_0^n \text{ for every } n \geq 1\right\}.$$

Then, for every $Y \in L^1(\Omega, \mathfrak{S}_0^\infty, \mathbb{P})$ and $n \geq 1$,

$$\mathbb{E}\big[Y|\mathfrak{S}_0^\tau\big]\mathbb{1}_{\{\tau=n\}} = \mathbb{E}\big[Y\mathbb{1}_{\{\tau=n\}}|\mathfrak{S}_0^\tau\big] = \mathbb{E}\big[Y|\mathfrak{S}_0^n\big]\mathbb{1}_{\{\tau=n\}}.$$

Proof. The first equality is automatic since $\{\tau = n\}$ is $\mathfrak{S}_0^\tau$-measurable. For the second equality, start by noting that $\mathbb{E}[Y|\mathfrak{S}_0^n]\mathbb{1}_{\{\tau=n\}}$ is $\mathfrak{S}_0^\tau$-measurable. Indeed, for $m \geq n$,

$$\mathbb{E}\big[Y|\mathfrak{S}_0^n\big]\mathbb{1}_{\{\tau=n\}}\mathbb{1}_{\{\tau\leq m\}} = \mathbb{E}\big[Y|\mathfrak{S}_0^n\big]\mathbb{1}_{\{\tau=n\}}$$

is $\mathfrak{S}_0^n$- and hence $\mathfrak{S}_0^m$-measurable. The case $m < n$ is trivial since then

$$\mathbb{E}\big[Y|\mathfrak{S}_0^n\big]\mathbb{1}_{\{\tau=n\}}\mathbb{1}_{\{\tau\leq m\}} = 0.$$

Now, for every bounded $\mathfrak{S}_0^\tau$-measurable test function φ,

$$\mathbb{E}[Y\mathbb{1}_{\{\tau=n\}}\varphi] = \mathbb{E}\big[\mathbb{E}\big[Y|\mathfrak{S}_0^n\big]\mathbb{1}_{\{\tau=n\}}\varphi\big]$$

since $\varphi\mathbb{1}_{\{\tau=n\}}$ is $\mathfrak{S}_0^n$-measurable. This concludes the proof. □

5.4. A sufficient condition for the nonexistence of minimizers of (qVar1) and (qVar2)

Proof of Proposition 2.11. It is clear from (2.4) that

$$\log H_{m+n} \geq \log H_m + \log H_n \circ T_{(m/d,\ldots,m/d)}$$

for every $m, n \geq 1$ (and divisible by d). By (Dir), (Ind) and (Loc), the summands on the RHS are independent. Let

$$\varepsilon = \limsup_{n\to\infty} \mathbb{P}\big(\log H_n \geq a(n)\big) > 0.$$

Fix an arbitrary $\delta \in (0, 1)$. There exists an $m_1 \geq 1$ such that $\mathbb{P}(\log H_{m_1} \geq a(m_1)) > \varepsilon/2$. Inductively pick $m_2, m_3, \ldots$ as follows. Given $m_1, m_2, \ldots, m_{k-1}$, let $n_{k-1} = m_1 + \cdots + m_{k-1}$. For sufficiently large $m_k \geq 1$, we have

$$\frac{\log H_{n_{k-1}}}{a(n_k)} \geq -\delta/2, \qquad \frac{a(m_k)}{a(n_k)} \geq 1 - \delta/2 \quad \text{and} \quad \mathbb{P}\big(\log H'_{m_k} \geq a(m_k)|\mathfrak{S}_0^{n_{k-1}}\big) > \varepsilon/2.$$

Here, $n_k = n_{k-1} + m_k$ and $H'_{m_k} = H_{m_k} \circ T_{(n_{k-1}/d,\ldots,n_{k-1}/d)}$. Note that such an m_k always exists, but depends on n_{k-1} and $\log H_{n_{k-1}}$, so it is a $\mathfrak{S}_0^{n_{k-1}}$-measurable random integer. Now, observe that

$$\frac{\log H_{n_k}}{a(n_k)} \geq \frac{\log H_{n_{k-1}}}{a(n_k)} + \frac{\log H'_{m_k}}{a(n_k)} \geq -\delta/2 + (1-\delta/2)\frac{\log H'_{m_k}}{a(m_k)} \geq 1-\delta$$

if $\log H'_{m_k} \geq a(m_k)$. Therefore,

$$\mathbb{P}\left(\frac{\log H_{n_k}}{a(n_k)} \geq 1-\delta\,\middle|\,\mathfrak{S}_0^{n_{k-1}}\right) \geq \mathbb{P}\big(\log H'_{m_k} \geq a(m_k)|\mathfrak{S}_0^{n_{k-1}}\big) > \varepsilon/2.$$

By Lévy's extension of the second Borel–Cantelli lemma (see [37], page 124),

$$\mathbb{P}\left(\frac{\log H_{n_k}}{a(n_k)} \geq 1 - \delta \text{ i.o.}\right) = 1.$$

Since $\delta > 0$ is arbitrary, this gives (2.6). In particular, for $\lambda = \Lambda_q(V)$,

$$\mathbb{P}\big(\bar{h}_\infty^\lambda = \infty\big) = \mathbb{P}\Big(\limsup_{n\to\infty} \log h_n^\lambda = \infty\Big) \geq \mathbb{P}\Big(\limsup_{n\to\infty} \log H_n = \infty\Big) = 1. \qquad \square$$

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