Random Growth Models

Michael Damron

Firas Rassoul-Agha

Timo Seppäläinen

School of Mathematics, Georgia Institute of Technology, 686 Cherry St., Atlanta, GA, USA
E-mail address: mdamron6@gatech.edu

University of Utah, Mathematics Department, 155 S 1400 E, Salt Lake City, UT 84109, USA
E-mail address: firas@math.utah.edu

Department of Mathematics, 480 Lincoln Drive, University of Wisconsin–Madison, Madison, WI 53711, USA
E-mail address: seppalai@math.wisc.edu

2010 *Mathematics Subject Classification.* 60K35, 60K37, 82B43

Contents

Preface

Irregular and stochastic growth is all around us: tumors, bacterial colonies, infections, fluid spreading in a porous medium, and propagating flame fronts. The study of simplified mathematical models of stochastic growth began in probability theory in the 1960s. Quite serendipitously, these models have turned out to be extremely hard to analyze. They have inspired innovative probability theory and have led to new connections between probability and other parts of mathematics.

Much progress has been made in a few exactly solvable special cases of random growth models. Out of this effort has arisen a new distinct subject, *integrable probability*, with a deep algebraic component. However, thus far the tools of integrable probability apply only in the exactly solvable setting where special structures are present. On the broader class of models, researchers have had success with approaches that combine probability with ideas of functional analytic and geometric nature. Examples include the use of concentration of measure and Busemann functions to study geodesics and prove fluctuation bounds, and curvature bounds in the derivation of the Kardar-Parisi-Zhang (KPZ) scaling relation.

This book consists of a collection of expanded notes from the AMS Short Course "Random Growth Models," which took place in Atlanta, GA, at the AMS Joint Mathematics Meetings in January 2017. The objective is to showcase recent breakthroughs in a class of stochastic growth models called first- and last-passage percolation and their relations to particle systems and the KPZ equation. The topics covered span aspects of both the general models and the exactly solvable ones.

This book is intended for an audience beyond specialists in the field. In particular, we hope to provide an opening into the area for graduate students and researchers who might consider studying these models. It is assumed that the reader is familiar with the basic notions of probability. A quick review of probability is given in [**2**].

Article [**2**] introduces various growth models and gives an overview of first- and last-passage percolation. After perusal of [**2**], the remaining chapters can be read independently of each other. Small changes in notation sometimes happen from chapter to chapter. In the remainder of this preface we highlight themes that run through more than one chapter and link together different parts of the book.

The first fundamental question of the subject is the limit shape of a growth model and the properties of the limit shape. Already here open questions are legion. Article [**2**] covers both known and conjectured general properties of the limit shape. These are then used and restated when needed in [**3**] and [**4**].

Rost's formula for the shape function of the corner growth model with exponential weights is stated in [**2**] and derived in both [**4**] and [**5**]. In [**4**], Rost's formula comes after a development of a theory for the corner growth model with general weights. Article [5] concentrates on the exactly solvable exponential case and derives the shape formula with a more direct argument. Continuity of the shape function up to the boundary is proved for directed last-passage percolation in [4]. Standard first-passage percolation is defined in the full space instead of a quadrant, so the boundary issue does not arise.

The second question is the question of fluctuations. There is a natural progression through the book. Article [6] treats the best known non-optimal bounds for general models with concentration inequalities. Article [5] proves sharp exponent bounds with coupling tools for the stationary version of an exactly solvable model. Finally [1] obtains the finest results with integrability techniques that go beyond probability and draw on ideas from quantum integrable systems. The fluctuation results proved in that chapter are widely believed to be universal and present in models for which the tools of exact solvability are no longer applicable. Though such a conjecture seems far from being proved, there is a weaker type of universality in which many systems under special scalings converge to the KPZ equation. Article [1] describes this weak KPZ universality conjecture and provides some of the stochastic analysis tools used in establishing it for certain models.

A third main theme is the structure of optimizing paths, or geodesics. These are studied in conjunction with Busemann functions. Article [3] takes up this topic in undirected first-passage percolation, and article [4] in directed last-passage percolation. The two chapters complement each other because most arguments can be transferred from one setting to the other. Both chapters show how Busemann functions are used to construct and study directional geodesics. The chapters take somewhat different approaches to the construction of Busemann functions. Article [3] uses subsequential weak limits in general dimensions while article [4] relies on a connection with queuing theory in two dimensions but shows how to go from the weak limit to an almost sure limit.

Both [3] and [4] utilize conjectured properties of the limit shape to derive results about Busemann functions. By contrast, in [5], the existence and precise distributional properties of Busemann functions can be proved without additional assumptions because the limit shape and invariant distributions of the exponential corner growth model are explicitly known.

We thank the American Mathematical Society for financial support of the 2017 AMS Short Course on Random Growth Models and for the opportunity to publish this proceedings volume.

Michael Damron
Firas Rassoul-Agha
Timo Seppäläinen

Bibliography

[1] Ivan Corwin, *Exactly solving the kpz equation*, Proc. Amer. Math. Soc. (2018). To appear.
[2] Michael Damron, *Random growth models: shape and convergence rate*, Proc. Amer. Math. Soc. (2018). To appear.
[3] Jack Hanson, *Infinite geodesics, asymptotic directions, and busemann functions in first-passage percolation*, Proc. Amer. Math. Soc. (2018). To appear.
[4] Firas Rassoul-Agha, *Busemann functions, geodesics, and the competition interface for directed last-passage percolation*, Proc. Amer. Math. Soc. (2018). To appear.
[5] Timo Seppäläinen, *The corner growth model with exponential weights*, Proc. Amer. Math. Soc. (2018). To appear.
[6] Philippe Sosoe, *Fluctuations in first-passage percolation*, Proc. Amer. Math. Soc. (2018). To appear.