JOINTLY INVARIANT MEASURES FOR THE KARDAR-PARISI-ZHANG EQUATION

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Abstract. We give an explicit description of the jointly invariant measures for the KPZ equation. These are couplings of Brownian motions with drift, and can be extended to a process defined for all drift parameters simultaneously. We term this process the KPZ horizon (KPZH). As a corollary of this description, we resolve a recent conjecture of Janjigian, and the second and third authors by showing the existence of a random, countably infinite dense set of directions at which the Busemann process of the KPZ equation is discontinuous. This signals instability and shows the failure of the one force–one solution principle and the existence of at least two extremal semi-infinite polymer measures in the exceptional directions. As the inverse temperature parameter β for the KPZ equation goes to ∞, the KPZH converges to the stationary horizon (SH) first introduced by Busani, and studied further by Busani and the third and fourth authors. As $\beta \searrow 0$, the KPZH converges to a coupling of Brownian motions that differ by linear shifts, which is a jointly invariant measure for the Edwards-Wilkinson fixed point.

Contents

1. Introduction

1.1. Invariant measures of the KPZ equation. For $t > s$, consider the KPZ equation

$$\partial_t h(t,x) = \frac{1}{2}\partial_{xx} h(t,x) + \frac{\beta}{2}(\partial_x h(t,x))^2 + W(t,x), \quad h(s,x) = h_s(x), \tag{1.1}$$

with inverse temperature $\beta > 0$, initial condition h_s at time s and space-time white noise W as driving force (see Section 3.1). Classically, the solution to this equation is ill-posed, but formally, one can solve the KPZ equation via the Cole-Hopf transformation $h(t,x) = \frac{1}{\beta}\log Z(t,x)$, where Z solves the stochastic heat equation (SHE) with multiplicative noise:

$$\partial_t Z(t,x) = \frac{1}{2}\partial_{xx} Z(t,x) + \beta Z(t,x) W(t,x), \quad Z(s,x) = e^{\beta h_s(x)}. \tag{1.2}$$

Rigorous solutions to this equation have been discussed in [BC95, BG97, CD14, CD15]. Recently, great progress has been made in understanding solutions of the KPZ equation in the work on Martin Hairer [Hai13, Hai14] on regularity structures. Another perspective through paracontrolled distributions has been studied in [GP17, PR19].

It is well-known that Brownian motion with diffusivity 1 and arbitrary drift is an invariant measure for (1.1). The notion of invariance requires the caveat that invariance only holds up to a global height shift. That is, we let $h(t,x|B)$ denote the solution to (1.1) a time $t > 0$ with $h(0,x) = B(x)$, where B is a Brownian motion. Then,

$$\{h(t,x|B) - h(t,0|B) : x \in \mathbb{R}\} \stackrel{d}{=} B.$$

See [JRAS23] and the references therein for a detailed discussion of this height shift. Using the work of [AKQ14a, AKQ14b, AJRS22], one can construct solutions to the KPZ equation with the same driving noise W but started from different initial conditions. The present paper is concerned with *jointly invariant*

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measures, namely couplings of Brownian motions $F^1,\dots,F^k$ with different drifts such that on $C(\mathbb{R})^k$ we have for all $t>0$ the distributional invariance

$$\big(h(t,\bullet\,|\,F^1)-h(t,0\,|\,F^1),\dots,h(t,\bullet\,|\,F^k)-h(t,0\,|\,F^k)\big)\overset{d}{=}\big(F^1(\bullet),\dots,F^k(\bullet)\big). \tag{1.3}$$

The existence, uniqueness, and ergodicity of such jointly invariant measures, up to an asymptotic slope condition, was established in [JRS22] (see Section 3.4 of that paper for a detailed discussion). We state this condition as follows:

$$\begin{aligned} -\infty \le \limsup_{x\to-\infty}\frac{F(x)}{|x|} < \lambda = \lim_{x\to\infty}\frac{F(x)}{x} \qquad && \text{if } \lambda>0\\ \lim_{x\to-\infty}\frac{F(x)}{|x|} = |\lambda| > \limsup_{x\to\infty}\frac{F(x)}{x}\ge -\infty \qquad && \text{if } \lambda<0\\ -\infty\le \limsup_{|x|\to\infty}\frac{F(x)}{|x|}\le 0 \qquad && \text{if } \lambda=0. \end{aligned} \tag{1.4}$$

Our first theorem gives an explicit description of these measures.

Theorem 1.1. *Let $\lambda_1<\cdots<\lambda_k$ be real. Let $Y^1,\dots,Y^k$ be independent two-sided Brownian motions with diffusivity 1 and drifts $\lambda_1,\dots,\lambda_k$, respectively. Set $F_\beta^1=Y^1$ and then for $j=2,\dots,k$, define*

$$\exp\big[\beta F_\beta^j(y)\big]=e^{\beta Y^1(y)}\cdot\frac{\displaystyle\int_{-\infty<x_{j-1}<\cdots<x_1<y}\prod_{i=1}^{j-1}e^{\beta(Y^{i+1}(x_i)-Y^i(x_i))}dx_i}{\displaystyle\int_{-\infty<x_{j-1}<\cdots<x_1<0}\prod_{i=1}^{j-1}e^{\beta(Y^{i+1}(x_i)-Y^i(x_i))}dx_i}.$$

Then, $(F_\beta^1,\dots,F_\beta^k)$ is distributed as the unique jointly stationary and ergodic measure for the KPZ equation (1.1) *such that, for $1\le j\le k$, each F_β^j satisfies almost surely the asymptotic slope condition* (1.4) *for $\lambda=\lambda_j$. In particular, F_β^j is a two-sided Brownian motion with diffusivity 1 and drift λ_j.*

In Section 2.3, we extend the measures of Theorem 1.1 to a process $\{F_\beta^\lambda\}_{\lambda\in\mathbb{R}}$, which we term the KPZ horizon with inverse temperature β (KPZH$_\beta$ for short, or sometimes simply KPZH). The path space of this process is the Skorokhod space $D(\mathbb{R},C(\mathbb{R}))$ of functions $\mathbb{R}\to C(\mathbb{R})$ that are right-continuous with left limits. $C(\mathbb{R})$ is endowed with its Polish topology of uniform convergence on compact sets. The term KPZ horizon is made in analogy to the stationary horizon (SH), which was first introduced by Busani in [Bus21] and studied by Busani and the third and fourth authors in [SS23b, BSS22b, BSS22a, BSS23]. The KPZ fixed point is the $1:2:3$ large-time scaling limit of the KPZ equation [QS23, Vir20, Wu23]. This is discussed more in Section 1.3. The SH gives the unique jointly invariant measure for the KPZ fixed point under the same asymptotic slope conditions. In fact, as $\beta\nearrow\infty$, the projections of KPZH$_\beta$ on $C(\mathbb{R},\mathbb{R}^k)$ converge to the SH.

The description in Theorem 1.1 gives rise to the following description of the difference function for the two jointly invariant measures.

Theorem 1.2. *Let $\beta>0$ and $\{F_\beta^\lambda\}_{\lambda\in\mathbb{R}}$ be the* KPZH$_\beta$. *For $\lambda_1<\lambda_2$ with $\lambda=\lambda_2-\lambda_1$,*

$$\{F_\beta^{\lambda_2}(y)-F_\beta^{\lambda_1}(y):y\ge0\}\overset{d}{=}\{\beta^{-1}\log\big(1+X_{\lambda,\beta}Y_{\lambda,\beta}(y)\big):y\ge0\}$$

where $X_{\lambda,\beta}\sim$ Gamma$(\lambda\beta^{-1},\beta^{-2})$, independent of the process $\{Y_{\lambda,\beta}(y):y\ge0\}$. The law of this latter process is given by

$$\{Y_{\lambda,\beta}(y):y\ge0\}\ \overset{d}{=}\ \Big\{\int_0^y\exp\big(\sqrt{2}\beta B(x)+\lambda\beta x\big)\,dx:y\ge0\Big\},$$

where B is a standard Brownian motion.

Qualitatively, a key feature is that the extension to the full KPZH process inherently produces discontinuities:

Theorem 1.3. *Let $F_\beta=\{F_\beta^\lambda\}_{\lambda\in\mathbb{R}}$ be the* KPZH$_\beta$ *and $\mathbb{P}_\beta$ its distribution on the space $D(\mathbb{R},C(\mathbb{R}))$. Then, $\mathbb{P}_\beta$-almost surely there exists a random countably infinite dense subset Λ_β of $\mathbb{R}$ such that whenever $x\ne y$, $\alpha\mapsto F_\beta^\alpha(y)-F_\beta^\alpha(x)$ is discontinuous at $\alpha=\lambda$ if and only if $\lambda\in\Lambda_\beta$.*

1.2. **Discontinuities of the Busemann process in the continuum directed random polymer.** The work of [AKQ14a, AKQ14b, AJRS22] constructs a four-parameter field $\{Z_\beta(t,y\,|\,s,x) : x,y \in \mathbb{R}, s<t\}$ on a single probability space so that, for each $s \in \mathbb{R}$ and any initial data h_s,

$$(t,y) \mapsto \frac{1}{\beta}\int_{\mathbb{R}} e^{\beta h_s(x)} Z_\beta(t,y\,|\,s,x)\,dx$$

solves the SHE (1.2) at times $t \in (s,\infty)$ and agrees with the notion of solution from [BC95, BG97, CD14, CD15]. This four-parameter family defines random probability measures Q_β^W on paths $g:[s,t]\to\mathbb{R}$ from (x,s) to (y,t) whose time-r distribution is given by

$$Q_\beta^{(s,x)\to(t,y)}(g(r)\in dz) = \frac{Z_\beta(t,y\,|\,r,z)Z_\beta(r,z\,|\,s,x)}{Z_\beta(t,y\,|\,s,x)}\,dz \qquad \text{for } s<r<t.$$

In this sense, we say that Z_β is the partition function for the *continuum directed random polymer* (CDRP) first introduced in [AKQ14a]. The measures $Q_\beta^{(s,x)\to(t,y)}$ extend in a Gibbsian sense to measures $Q_\beta^{(t,y)}$ on semi-infinite backward paths $g:(-\infty,t]\to\mathbb{R}$ rooted at (t,y). The Gibbs property is that, conditional on the path passing through (s,x) at time $s\in(-\infty,t)$, the portion of the path between (t,y) and (s,x) is distributed as $Q_\beta^{(s,x)\to(t,y)}$. See [JRS22, Section 9] for a more precise definition and detailed discussion. The infinite-path measure is said to be *strongly λ-directed* if

$$Q_\beta^{(t,y)}\left(\lim_{r\to-\infty}\frac{g(r)}{|r|}=\lambda\right)=1.$$

To study this collection of infinite-path measures, Janjigian and the second and third authors [JRS22] constructed *Busemann functions* for the SHE. For a fixed $\lambda\in\mathbb{R}$, these satisfy the almost sure locally uniform limits [JRS22, Theorem 3.16]

$$b_\beta^\lambda(s,x,t,y) = \lim_{r\to-\infty}\log\frac{Z_\beta(s,x\,|\,r,z_r)}{Z_\beta(t,y\,|\,r,z_r)}, \tag{1.5}$$

simultaneously for all paths $\{z_r : r<s\wedge t\}$ that satisfy $\lim_{r\to-\infty}\frac{z_r}{|r|}=\lambda$. Furthermore, the article [JRS22] constructs the Busemann process

$$\{b_\beta^{\lambda\Box}(s,x,t,y) : (s,x,t,y)\in\mathbb{R}^4,\ \lambda\in\mathbb{R},\ \Box\in\{-,+\}\}$$

on a single event of full probability. The sign parameter $\Box\in\{-,+\}$ is a necessary ingredient of the description. A fixed value $\lambda\in\mathbb{R}$ is almost surely not a discontinuity of this process, i.e. $b_\beta^{\lambda-}\not\equiv b_\beta^{\lambda+}$ (Theorem 3.2(iv) below). But the existence of random discontinuities across the uncountably many values λ was left open [JRS22, Open Problem 2]. In general, $\lambda\mapsto b_\beta^{\lambda-}(s,x,t,y)$ is left-continuous, while $\lambda\mapsto b_\beta^{\lambda+}(s,x,t,y)$ is right-continuous. The set of exceptional directions at which jumps occur is defined by

$$\Lambda_{b_\beta} := \{\lambda\in\mathbb{R} : b_\beta^{\lambda-}(s,x,t,y)\neq b_\beta^{\lambda+}(s,x,t,y) \text{ for some } (s,x,t,y)\in\mathbb{R}^4\}. \tag{1.6}$$

The set Λ_{b_β} is exactly the set of directions where the semi-infinite Gibbs measure supported on λ-directed paths is not unique [JRS22, Theorems 3.35, 3.38]. The Busemann process is a *eternal solution* to the KPZ equation, meaning that started from any initial time, the Busemann process evolves forward in time via the KPZ equation. When the Busemann process is discontinuous at λ, the *one force–one solution* principle fails because there are two eternal solutions to the equation satisfying the same asymptotic slope conditions. This is manifested in the dynamic programming principle proved in [JRS22] and recorded in the present paper as Theorem 3.2(x). Theorem 3.5 of [JRS22], recorded as Theorem 3.2(v) in the present paper, established the following dichotomy: either $\mathbb{P}(\Lambda_{b_\beta}=\varnothing)=1$ or $\mathbb{P}(\Lambda_{b_\beta}$ is countable and dense in $\mathbb{R})=1$. Our next theorem states that the latter is true.

Theorem 1.4. *Let $\beta>0$. Then, $\mathbb{P}(\Lambda_{b_\beta}=\varnothing)<1$.*

Theorem 1.4 is a direct consequence of Theorems 1.1 and 1.3. In particular, we deduce from Theorem 1.1 that the KPZH$_\beta$ is equal in law to the Busemann process for the SHE/KPZ equation (see Corollary 4.3).

The proof of the existence of discontinuities of the KPZH$_\beta$ comes in Corollary 2.12. Our proof exploits the explicit description of the distribution of $F_\beta^\lambda(y)-F_\beta^0(y)$ given in Theorem 1.2. It would be interesting to see if there is a proof of the condition above that uses softer properties of Busemann functions and can be generalized to other models. However, as a counterexample, consider the deterministic approximation of the Green's function for the KPZ equation with $\beta=1$ (see [JRS22, Section 1.5, Theorem 3.8])

$$\widetilde{\mathcal{H}}(t,y\,|\,s,x) = -\frac{t-s}{24}-\frac{(y-x)^2}{2(t-s)}.$$

In this setting, the Busemann function is equal to

$$\widetilde{b}^{\lambda}(s,x,t,y) = \lim_{r\to-\infty} \widetilde{\mathcal{H}}(t,y|r,-r\lambda) - \widetilde{\mathcal{H}}(s,x|r,-r\lambda) = \frac{(12\lambda^2-1)(t-s)}{24} + (y-x)\lambda,$$

which is continuous in the parameter λ. Thus, any more general condition to prove the existence of discontinuities would need deeper information about the noise present in the model and cannot rely only on curvature or strict convexity of the shape function.

1.3. **High and low temperature limits of the KPZ horizon.** The KPZ equation interpolates between two so-called *universality classes*. This phenomenon follows from the explicit formulas calculated in [ACQ11] and is explicitly noted in [Cor12, Theorem 1.1]. Setting $\beta = 1$ for simplicity (noting that the general equation can be obtained from this one by scaling, see [Cor12, Equation (3)]), we let $Z(T,X) = Z_1(T,X\,|\,0,0)$, and set

$$F_T(s) = P\Big(\log Z(T,X) + \frac{X^2}{2T} + \frac{T}{24} \le s\Big).$$

Theorems 1.1 and Corollary 1.2 of [Cor12] state that $F_T(s)$ does not depend on X and that

$$\lim_{T\to\infty} F_T(2^{-1/3}T^{1/3}s) = F_{\mathrm{GUE}}(s), \quad \text{and} \quad \lim_{T\searrow 0} F_T(2^{-1/2}\pi^{1/4}T^{1/4}(s-\log\sqrt{2\pi T})) = \Phi(s),$$

where F_{GUE} is the Tracy-Widom GUE distribution, and Φ is the standard Gaussian distribution. The Tracy-Widom distribution is central to the KPZ universality class, while the Gaussian distribution is central to the Edwards-Wilkinson class [EW82, Cor12]. On the KPZ side of things, much recent work has been devoted to stronger convergence on the level of the process [QS23, Vir20, Wu23, DZ22a, DZ22b]. See Section 1.4.2 for a more detailed discussion of the relevant literature.

The scaling relations for Z_β proved in [AKQ14b, AJRS22] (recorded in the present paper as Theorem 3.1) imply that $Z(T,0) = Z_1(T,0\,|\,0,0) \overset{d}{=} \frac{1}{\sqrt{T}} Z_{T^{1/4}}(1,0\,|\,0,0)$. Hence, large times T correspond to high inverse temperatures β, while short times T correspond to small values of β. In this same spirit, the results of this section show that the KPZH$_\beta$ interpolates between the jointly invariant measures in the KPZ and Edwards-Wilkinson universality classes, seen in the limits as $\beta \nearrow \infty$ and $\beta \searrow 0$, respectively.

We discuss here KPZH$_\beta$ in relation to its earlier-introduced zero temperature counterpart, the *stationary horizon*. The stationary horizon (SH) is a stochastic process $G = \{G^\lambda\}_{\lambda\in\mathbb{R}}$ with path space $D(\mathbb{R}, C(\mathbb{R}))$. Marginally, each $C(\mathbb{R})$-valued component G^λ is a Brownian motion with diffusivity $\sqrt{2}$ and drift 2λ. See Appendix B for a formal definition of the SH.

Figure 1 shows a simulation of the KPZH$_\beta$ for three different values of β, namely $0.1, 1$, and 20. In each case, we use the values $\lambda = -5, -2.5, 0, 2.5, 5$. For small β, we see the trajectories tend to look like affine shifts of one another. For large β, the trajectories appear to stick very closely together in a neighborhood of the origin. In fact, before the limit the paths do not actually touch outside the origin, but at $\beta = \infty$, each pair of paths coincide in a nondegenerate interval around the origin.

The next theorem states that SH is the diffusive scaling limit of KPZH$_\beta$, for a fixed temperature, or equivalently, the $\beta \nearrow \infty$ limit of KPZH$_\beta$. We also see a more trivial limit as $\beta \searrow 0$. The mode of convergence proved is on the level of finite-dimensional projections on the spaces $C(\mathbb{R}, \mathbb{R}^k)$ for $k \ge 1$. We conjecture that a convergence on the Skorokhod space $D(\mathbb{R}, C(\mathbb{R}))$ should also hold, as is proved for exponential LPP in [Bus21] and for the TASEP speed process in [BSS22a]. However, the topology of convergence on the space $D(\mathbb{R}, C(\mathbb{R}))$ needs to be adjusted because the set of discontinuities for the prelimiting object is not isolated in a compact window of space, as is the case in [Bus21, BSS22a]. We leave the investigation of tightness on $D(\mathbb{R}, C(\mathbb{R}))$ to future work. The convergence of parts (i) and (ii) below are equivalent by the scaling relations of 2.10(ii) followed by the change of variable $\gamma \mapsto \gamma\beta$.

Theorem 1.5. *Let $\{G^\lambda\}_{\lambda\in\mathbb{R}}$ be the* SH *and $\{F_\beta^\lambda\}_{\lambda\in\mathbb{R}}$ the* KPZH$_\beta$. *Fix two real parameters $\beta > 0$ and $\alpha \in \mathbb{R}$. For any finite increasing vector $\lambda_1 < \cdots < \lambda_k$, $\{G^{\lambda_i}\}_{1\le i\le k}$ is the limit in distribution on $C(\mathbb{R}, \mathbb{R}^k)$, as $\gamma \to \infty$, of the following two processes:*

(i) $\{F_\gamma^{\lambda_i}(2\,\bullet)\}_{1\le i\le k}$.

(ii) $\Big\{\gamma^{-1} F_\beta^{\alpha+\gamma^{-1}\lambda_i}(2\gamma^2\,\bullet) - 2\gamma\alpha\bullet\Big\}_{1\le i\le k}$.

Furthermore, let B be a standard two-sided Brownian motion (*diffusivity* 1 *and zero drift*)*. Then as $\gamma \searrow 0$, the processes in parts* (i) *and* (ii) *above converge in distribution, on $C(\mathbb{R}, \mathbb{R}^k)$, to $\{B(2\bullet) + 2\lambda_i\bullet\}_{1\le i\le k}$.*

For large $\gamma > 0$, the scaling in Item (ii) above fixes a temperature β and considers a direction perturbed from the drift α. This is the scaling of the initial data in the convergence of the KPZ equation to the KPZ fixed point, as in [QS23, Vir20, DZ22a, DZ22b, Wu23].

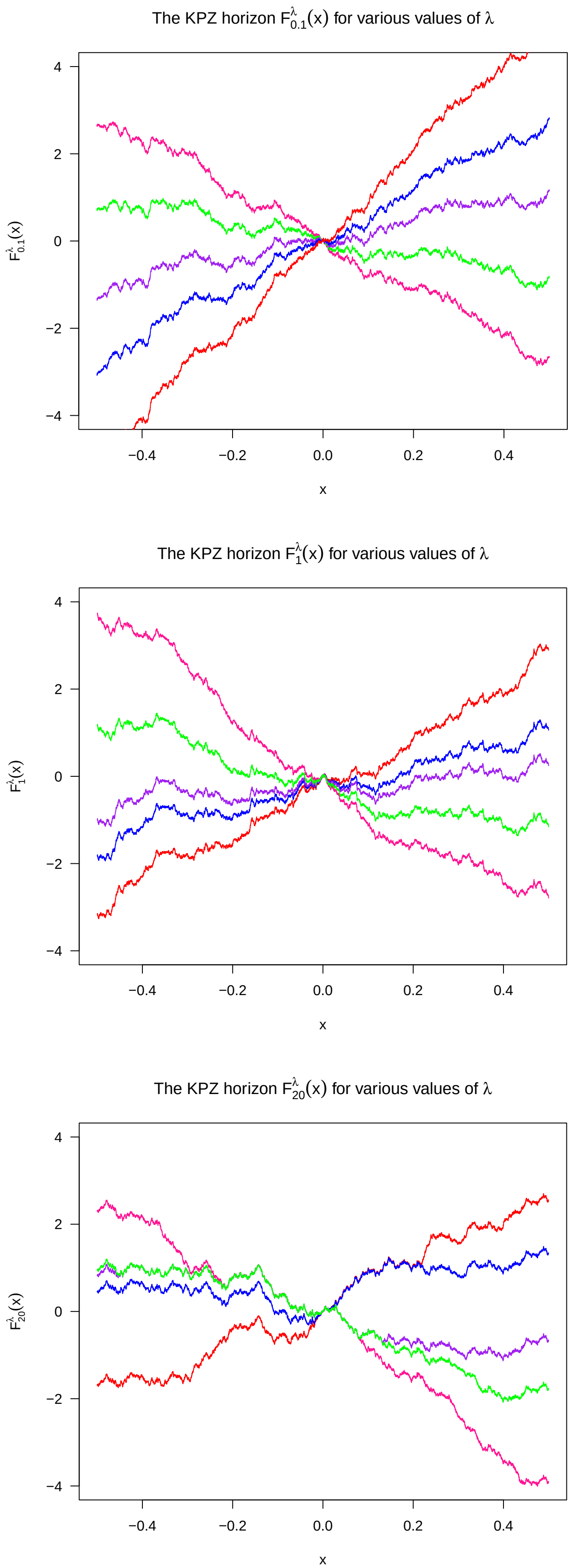

FIGURE 1. KPZH$_\beta$ for three inverse temperature values $\beta = 0.1$, 1, and 20 from top to bottom, and in each frame for the drift values $\lambda = -5$ (pink), $\lambda = -2.5$ (green), $\lambda = 0$ (purple), $\lambda = 2.5$ (blue), and $\lambda = 5$ (red).

Setting $\gamma = 2^{-1/3}T^{1/3}$, the sequence in part (ii) becomes

$$\Big\{2^{1/3}T^{-1/3}F_\beta^{\alpha+2^{1/3}T^{-1/3}\lambda_i}(2^{1/3}T^{2/3}\bullet) - 2^{2/3}T^{1/3}\alpha\bullet\Big\}_{1\le i\le k},$$

which, as $T\to\infty$, demonstrates the $1:2:3$ scaling in convergence to the KPZ fixed point. There are only two scaling parameters now because we are scaling initial data, so there is no time parameter. However, we can in fact strengthen our result to a process-level convergence using the recent results of Wu [Wu23]. Let $\mathcal{L} = \{\mathcal{L}(x,s;y,t) : x,y\in\mathbb{R}, s<t\}$ be the directed landscape (DL). For upper semicontinuous initial data $\mathfrak{h}:\mathbb{R}\to\mathbb{R}\cup\{-\infty\}$ satisfying, for some $a,b>0$, $\mathfrak{h}(x)\le a+b|x|$ for all $x\in\mathbb{R}$ and $\mathfrak{h}(x)>-\infty$ for some x, define

$$h_{\mathcal{L}}(t,y|s,\mathfrak{h}) = \sup_{x\in\mathbb{R}}\{\mathfrak{h}(x)+\mathcal{L}(x,s;y,t)\}. \tag{1.7}$$

Let $h_{Z_\beta}(t,y|s,f)$ be the solution of the KPZ equation (1.1) started at time s with initial data f:

$$h_{Z_\beta}(t,y|s,f) = \frac{1}{\beta}\log\int_{\mathbb{R}} e^{\beta f(x)}Z_\beta(t,y|s,x)\,dx. \tag{1.8}$$

Corollary 1.6. *Let $\beta>0$, $\alpha,s\in\mathbb{R}$, and $\lambda_1<\cdots<\lambda_k$. Then, as processes in $C(\mathbb{R}_{>s}\times\mathbb{R},\mathbb{R}^k)$ equipped with the uniform-on-compacts topology,*

$$\left\{2^{1/3}T^{-1/3}\left[\beta h_{Z_\beta}\Big(\frac{Tt}{\beta^4},\frac{2^{1/3}T^{2/3}y}{\beta^2}\,\Big|\,\frac{Ts}{\beta^4},F_\beta^{\alpha+2^{1/3}T^{-1/3}\lambda_i}(\bullet)-\alpha\bullet\Big)+\frac{T(t-s)}{24}-\frac{2}{3}\log(\sqrt{2}T)\right] : (t,y)\in\mathbb{R}_{>s}\times\mathbb{R}\right\}_{1\le i\le k}$$
$$\overset{T\to\infty}{\Longrightarrow}\{h_{\mathcal{L}}(t,y|s,G^{\lambda_i}) : (t,y)\in\mathbb{R}_{>s}\times\mathbb{R}\}_{1\le i\le k}.$$

Furthermore, let $\{\mathcal{B}^{\lambda\square} : \lambda\in\mathbb{R},\square\in\{-,+\}\}$ be the Busemann process for the DL discussed in Appendix B. Then, for any $\beta>0$ and $\lambda_1<\cdots<\lambda_k$, as processes in $C(\mathbb{R}^4,\mathbb{R}^k)$ equipped with the uniform-on-compacts topology,

$$\left\{2^{1/3}T^{-1/3}\Big[b_\beta^{\beta 2^{1/3}T^{-1/3}\lambda_i}\Big(\frac{Ts}{\beta^4},\frac{2^{1/3}T^{2/3}x}{\beta^2},\frac{Tt}{\beta^4},\frac{2^{1/3}T^{2/3}y}{\beta^2}\Big)+\frac{T(t-s)}{24}\Big] : (x,s;y,t)\in\mathbb{R}^4\right\}_{1\le i\le k}$$
$$\overset{T\to\infty}{\Longrightarrow}\{\mathcal{B}^{\lambda_i}(y,-t;x,-s) : (x,s;y,t)\in\mathbb{R}^4\}_{1\le i\le k}.$$

The temporal reflection in the process $\{\mathcal{B}^{\lambda_i}(y,-t;x,-s) : (x,s;y,t)\in\mathbb{R}^4\}_{1\le i\le k}$ is a manifestation of the fact that in [JRS22], the infinite paths travel south, while the infinite geodesics in [RV21] and [BSS22b] travel north.

1.3.1. *Jointly invariant measures for the Edwards-Wilkinson fixed point.* In contrast with the $\gamma\to\infty$ limit to the SH and in light of the $\gamma\searrow 0$ limit in Theorem 1.5, it is natural to ask whether $\{B(\bullet)+\lambda_i\bullet\}_{1\le i\le k}$ is a jointly invariant measure for the Edwards-Wilkinson fixed point [EW82, Cor12]. The Edwards-Wilkinson fixed point is governed by the 1-dimensional additive stochastic heat equation $\partial_t u = \frac{1}{2}u_{xx}+W$. It is well-known that this equation, started from initial data f at time 0, is solved as

$$u(t,x|f) = \int_{\mathbb{R}}\rho(t,x-y)f(y)\,dy + \int_0^t\int_{\mathbb{R}}\rho(t-s,x-y)W(ds\,dy). \tag{1.9}$$

It is also well-known that the increments of two-sided Brownian motion B is invariant in time for u. That is, $u(t,\bullet;B)-u(t,0;B)\overset{d}{=}B$. From (1.9) it follows that, for any appropriate function $f:\mathbb{R}\to\mathbb{R}$ and $\lambda\in\mathbb{R}$,

$$u(t,x|f(\bullet)+\lambda\bullet) = u(t,x|f)+\lambda x.$$

Hence, in the sense of (1.3), $\{B(\bullet)+\lambda_1\bullet,\ldots,B(\bullet)+\lambda_k\bullet\}$ is a jointly invariant measure for the SHE with additive noise, where the common noise W drives the equation from the different initial conditions. Indeed, this can be expected from Theorems 1.1 and 1.5, as the $\beta\searrow 0$ limit of (1.1) is precisely the additive SHE.

1.4. **Methods and related literature.**

1.4.1. *Convergence of the O'Connell-Yor polymer to the SHE.* The proof of Theorem 1.1 comes from first showing that the KPZH$_\beta$ describes jointly invariant measures for the semi-discrete O'Connell-Yor (OCY) polymer introduced in [OY01]. We show that the KPZH$_\beta$ satisfies certain distributional invariances under scaling to initial data for the SHE. Then, we show that the KPZH$_\beta$ is jointly invariant for the SHE and use a uniqueness result from [JRS22] to conclude the proof.

The convergence step requires a substantial amount of nontrivial work. Convergence of the OCY polymer (with the initial point fixed) to narrow wedge solutions of the SHE was established in the sense of finite-dimensional distributions by Nica [Nic21]. In Section 3.2 of the present paper, we prove in full detail, using different methods than those in [Nic21], the convergence of the four-parameter field of the OCY polymer to the Green's function of the SHE (in the sense of finite-dimensional distributions) and prove convergence of solutions from appropriate initial data.

Similar items to Lemmas 3.5, 3.8, 3.10, and Theorem 3.9 in Section 3 appeared in an unfinished manuscript of Moreno Flores, Quastel, and Remenik [MFQR]. As no proofs for the precise results we need appears in the literature, we provide them in Section 3. We develop several new ideas to allow us to complete the technical details of these results. We believe that the results of Section 3 will provide valuable statements that were not previously fully accessible to the community.

1.4.2. *Stationary horizon and KPZ universality.* SH was first constructed by Busani [Bus21] as the scaling limit of the Busemann process of the exponential corner growth model. [Bus21] conjectured SH to be the universal scaling limit of Busemann processes of models in the KPZ universality class. Shortly afterwards, SH was independently discovered in the context of Brownian last-passage percolation by the third and fourth authors [SS23b]. A brief introduction to the SH is given in Appendix B.

In [BSS22b, BSS22a, BSS23], the third and fourth authors, together with Busani, studied the role of SH in the KPZ class and established further evidence of its universality:

(i) Given appropriate conditions on the asymptotic slope of the initial data, the SH is the unique multi-type stationary distribution of the KPZ fixed point that evolves in the environment given by the directed landscape.

(ii) As a consequence, the SH gives the distribution of the fixed-time-level Busemann process of the directed landscape. In this representation, the parameter λ corresponds to the space-time slope of semi-infinite geodesics.

(iii) The suitably scaled TASEP speed process introduced by [AAV11] converges to the SH. In the limit, λ represents the scaled and centered values of the speed process. This suggests that SH is a general scaling limit of multitype invariant distributions, beyond the Busemann functions of stochastic growth models.

(iv) A framework is given in the forthcoming work [BSS23] to show convergence to the SH under conditions that are widely expected to hold in great generality. the conditions are convergence of the LPP model to the DL, marginal convergence of a single Busemann function to Brownian motion with drift, and tightness of exit point bounds from stationary initial conditions on the scale $N^{2/3}$. As a corollary, it is shown that the Busemann process for six solvable LPP models converge to the SH in the sense of finite-dimensional distributions.

The high-level analogy between KPZH_β and SH is that they both describe unique jointly invariant distributions, KPZH_β for the KPZ equation and SH for the KPZ fixed point. Additionally, KPZH_β and SH share certain properties. Both are couplings of Brownian motions with drift whose increments are ordered. Both are translation-invariant and have a reflection symmetry (Theorem 2.10(i) and (iv)). However, the two processes are not the same in law. One way to see this is Theorem 1.3). While the full SH process $\lambda \mapsto G^\lambda \in C(\mathbb{R})$ has a dense set of discontinuities $\lambda \in \mathbb{R}$, for given $x < y$ the points of discontinuity of the restricted process $\lambda \mapsto G^\lambda(y) - G^\lambda(x)$ are isolated. In contrast, for *any* $x < y$, the process $\lambda \mapsto F_\beta^\lambda(y) - F_\beta^\lambda(x)$ contains the full countable dense set of discontinuities of the process $\lambda \mapsto F_\beta^\lambda \in C(\mathbb{R})$.

There has been much recent work on the convergence of the KPZ equation to the KPZ fixed point. This was first accomplished in two independent works of Quastel and Sarkar [QS23] and Virág [Vir20]. Recently, Wu [Wu23] proved that the Green's function of the KPZ equation converges to the directed landscape. Combined with the previous work of Das and Zhu [DZ22a, DZ22b], who showed localization of polymer path measures in the CDRP, this establishes that the annealed polymer measures of the CDRP converge in distribution to the geodesics of the DL (See [DZ22b, Theorem 1.9]).

1.4.3. *One force–one solution.* Most of the previous work on the one force–one solution principle was focused on a fixed, nonrandom direction of space. Busemann functions and the one force–one solution principle have been studied for the Burgers' equation with discrete random forcing, both in compact and noncompact settings [Sin91, GIKP05, IK03, Bak07, Kif97, DS05, Kif97, Bak13, BCK14, Bak16a, Bak16b, BK18, BL18, BL19, HK03, DDG+22]. Specifically, in the works of Bakhtin and coauthors [Bak16b, BCK14, Bak16a, Bak13, BL18, BL19], one sees analogous results for Busemann functions and semi-infinite geodesics–the zero temperature analogue of semi-infinite polymer measures.

The first observation of random discontinuities of the Busemann process was completed by Fan and the third author [FS20] for the exactly solvable exponential corner growth model. Across a single horizontal edge, they showed that the Busemann process, indexed by the direction, can be described by a compound Poisson process. Across all edges, the union of the discontinuities is countably infinite and dense. This result was used in [JRAS23] to characterize the set of directions with non-unique semi-infinite geodesics as the same as the set of discontinuities of the Busemann process.

Similar studies were carried out for Brownian last-passage percolation (BLPP) by the third and fourth authors [SS23b] and for the directed landscape (DL) [BSS22a] by Busani and the third and fourth authors. Here, the characterization of exceptional directions of semi-infinite geodesics is exactly analogous to that of [JRAS23], but additional non-uniqueness of initial segments of geodesics appears due to the continuum setting in these models. As a result, new methods of proof were developed to achieve these results. The studies [SS23b, BSS22b] used a description of the Busemann process for Brownian LPP developed in [SS23a] and the remarkable fact that the Busemann process along a horizontal line for BLPP agrees with that of the DL. Unlike the exponential corner growth model, the Busemann process for BLPP along a single horizontal interval is not a compound Poisson process, nor does it have independent increments. Thus, obtaining an explicit description of this process remains out of reach. However, the description of the process in terms of coupled Brownian motions allows one to make some distributional calculations, which are enough to show that the Busemann process along an interval, and indexed by the direction, is a step function.

The recent work [BFS23] studies the Busemann process for the inverse-gamma polymer and discovers a similar explicit description as in [FS20]. However, the jumps of the Busemann process across a single horizontal interval are now dense, unlike in the zero temperature case where they are isolated. Likewise, the paper [JRS22] showed that, for the KPZ equation, if the set (1.6) is nonempty, the jumps are present along each horizontal interval and are therefore dense. In the present work, we obtain a description of the Busemann process for the SHE in terms of coupled Brownian motions with drift. Just as in the zero temperature cases of BLPP and the DL, the process along a horizontal interval does not have an explicit description that we know. However, we can compute the distribution of an increment of this process, and in Corollary 2.12, we apply a condition that is developed in Lemma 2.11 to show the existence of jumps. Our work demonstrates that the corresponding phenomenon in the work of [BFS23] is *not* simply a manifestation of discrete lattice effects.

1.5. **Organization of the paper.** Section 2 constructs the KPZ horizon. The theory of the mappings that define the projections of this process onto $C(\mathbb{R}, \mathbb{R}^k)$ is developed in Section 2.1. In Section 2.3, we construct the KPZH as a process of Brownian motions indexed by the drift $\lambda \in \mathbb{R}$, and the remaining subsections of Section 2 state properties of this process, including the proof of Theorem 1.2 in Section 2.5. In Section 2.6, we show the existence of discontinuities in the λ parameter. In Section 3, we begin by discussing the necessary background on the stochastic heat equation from [AKQ14a, AJRS22, JRS22]. Then, we prove the details needed for convergence of the O'Connell-Yor polymer to the stochastic heat equation, using input tools from [MFQR] and [Nic21]. The paper culminates in the proofs of the main theorems in Section 4, except for Theorem 1.2, which is proved earlier. The appendices contain some standard facts and inputs from the literature.

1.6. **Notation and conventions.**

- $C_{\text{pin}}(\mathbb{R})$ denotes the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$.
- Increments of a single-variable function F are denoted by $F(x, y) = F(y) - F(x)$. Increment ordering between functions $f, g : \mathbb{R} \to \mathbb{R}$: $f \le_{\text{inc}} g$ if $f(x, y) \le g(x, y)$ for all $x < y$, and $f <_{\text{inc}} g$ if $f(x, y) < g(x, y)$ for all $x < y$.
- For random variables X and Y and probability measures μ, $X \overset{d}{=} Y$ and $X \sim Y$ both mean that X and Y are equal in distribution, and $X \sim \mu$ means that X has probability distribution μ.
- Random variable X has the gamma distribution with shape parameter $\alpha > 0$ and rate $\beta > 0$, abbreviated $X \sim \text{Gamma}(\alpha, \beta)$, if X has density function $f(x) = \Gamma(\alpha)^{-1}\beta^\alpha x^{\alpha-1} e^{-\beta x}$ on $\mathbb{R}_+$.
- A two-sided standard Brownian motion is a continuous random process $\{B(x) : x \in \mathbb{R}\}$ such that $B(0) = 0$ almost surely and $\{B(x) : x \ge 0\}$ and $\{B(-x) : x \ge 0\}$ are two independent standard Brownian motions on $[0, \infty)$.
- If B is a two-sided standard Brownian motion, then $\{cB(x) + \mu x : x \in \mathbb{R}\}$ is a two-sided Brownian motion with diffusivity $c > 0$ and drift $\mu \in \mathbb{R}$.
- The complementary error function erfc is defined as $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du$.
- The heat kernel is $\rho(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \mathbf{1}_{t>0}$ for $(t, x) \in \mathbb{R}^2$.
- Ranges of indices in vectors and sequences are abbreviated as in $x_{m:n} = (x_m, x_{m+1}, \dots, x_n)$.
- The domain of pairs of space-time points with strictly ordered times is $\mathbb{R}^4_\uparrow = \{(s, x, t, y) \in \mathbb{R}^4 : s < t\}$.
- In a $C(\mathbb{R})$-valued stochastic process $\lambda \mapsto Y^\lambda(\bullet)$, the bullet marks the missing real variable: $Y^\lambda(\bullet) = (x \mapsto Y^\lambda(x)) \in C(\mathbb{R})$.
- Coordinatewise order on $\mathbb{R}^2$: $(x, y) \le (a, b)$ means that $x \le a$ and $y \le b$.

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2. Construction and properties of the KPZ horizon

2.1. **Mappings defining finite-dimensional distributions.** Let $C_{\text{pin}}(\mathbb{R})$ denote the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 0$. For $Y, B \in C_{\text{pin}}(\mathbb{R})$ satisfying

$$\limsup_{x \to -\infty} \frac{Y(x) - B(x)}{x} > 0, \tag{2.1}$$

and for $\beta > 0$, define the following transformations:

$$\begin{aligned} Q_\beta(B,Y)(y) &= \beta^{-1} \log \int_{-\infty}^{y} \exp\left(\beta(B(x,y) - Y(x,y))\right) dx \\ D_\beta(B,Y)(y) &= Y(y) + Q_\beta(B,Y)(y) - Q_\beta(B,Y)(0), \\ R_\beta(B,Y)(y) &= B(y) + Q_\beta(B,Y)(0) - Q_\beta(B,Y)(y). \end{aligned} \tag{2.2}$$

Iterate the mapping D_β as follows:

$$D_\beta^{(1)}(Y) = Y, \quad \text{and} \quad D_\beta^{(n)}(Y^1, Y^2, \dots, Y^n) = D_\beta(Y^1, D_\beta^{(n-1)}(Y^2, \dots, Y^n)) \quad \text{for } n \ge 2. \tag{2.3}$$

Given a Borel subset $A \subseteq \mathbb{R}$, we define three state spaces of n-tuples of functions.

$$\mathcal{A}_n^A := \Big\{ \mathbf{Y} = (Y^1, \dots, Y^n) \in C_{\text{pin}}(\mathbb{R})^n : \text{ for } 1 \le i \le n, \ \lim_{x \to -\infty} \frac{Y^i(x)}{x} \text{ exists and lies in } A \Big\}. \tag{2.4}$$

Note that if the components of $\mathbf{Z} \in C_{\text{pin}}(\mathbb{R})^n$ are Brownian motions with drifts in A, then $\mathbf{Z} \in \mathcal{A}_n^A$ almost surely. Next, set

$$\begin{aligned} \mathcal{Y}_n^A := \Bigg\{ &\mathbf{Y} = (Y^1, \dots, Y^n) \in C_{\text{pin}}(\mathbb{R})^n : \text{ for } 1 \le i \le n, \lim_{x \to -\infty} \frac{Y^i(x)}{x} \text{ exists and lies in } A, \\ &\qquad \text{and for } 2 \le i \le n, \lim_{x \to -\infty} \frac{Y^i(x)}{x} > \lim_{x \to -\infty} \frac{Y^{i-1}(x)}{x} \Bigg\} \end{aligned} \tag{2.5}$$

and

$$\mathcal{X}_n^A := \Big\{ \boldsymbol{\eta} = (\eta^1, \dots, \eta^n) \in \mathcal{Y}_n^A : \eta^i >_{\text{inc}} \eta^{i-1} \text{ for } 2 \le i \le n \Big\}. \tag{2.6}$$

The most common choices for A will be $\mathbb{R}_{>0}$ (to be used for the state space of invariant measures in the O'Connell-Yor polymer) and $\mathbb{R}$ (to be used as the state space of invariant measures in the KPZ equation). Section 7 of [SS23b] shows that these state spaces are Borel measurable subsets of the space $C(\mathbb{R}, \mathbb{R}^n)$.

Next, define a transformation $\mathbf{D}_\beta^{(n)}$ on n-tuples of functions as follows. Let $A \subseteq \mathbb{R}$. For $\mathbf{Y} = (Y^1, \dots, Y^n) \in \mathcal{Y}_n^A$, the image $\boldsymbol{\eta} = (\eta^1, \dots, \eta^n) = \mathbf{D}_\beta^{(n)}(\mathbf{Y}) \in \mathcal{X}_n^A$ is defined by

$$\eta^i = D_\beta^{(i)}(Y^1, \dots, Y^i) \quad \text{for } 1 \le i \le n. \tag{2.7}$$

Lemma 2.4 below proves that $\mathbf{D}_\beta^{(n)} : \mathcal{Y}_n^A \to \mathcal{X}_n^A$.

For a finite increasing real vector $\bar\lambda = (\lambda_1 < \lambda_2 < \cdots < \lambda_n)$, define the measure $\nu^{\bar\lambda}$ on $\mathcal{Y}_n^{\mathbb{R}}$ as follows: $(Y^1, \dots, Y^n) \sim \nu^{\bar\lambda}$ if $Y^1, \dots, Y^n$ are mutually independent and Y^i is a Brownian motion with drift λ_i. Define the measure $\mu_\beta^{\bar\lambda}$ on $\mathcal{X}_n^{\mathbb{R}}$ as

$$\mu_\beta^{\bar\lambda} = \nu^{\bar\lambda} \circ (\mathbf{D}_\beta^{(n)})^{-1}. \tag{2.8}$$

This is the key definition of the section. In each application of (2.7) the drifts satisfy $\lambda_1 < \cdots < \lambda_i$ and so the mappings are well-defined.

We prove a series of lemmas about these measures. The first derives a formula for $D_\beta^{(n)}(Y^1,\dots,Y^n)$. Once the first properties of the mappings and measures are established, some proofs go through just as they do for zero temperature in [SS23b]. For such results, we provide the full details in Appendix A.

Lemma 2.1. *Let $Y^1,\dots,Y^n \in C_{pin}(\mathbb{R})$ be such that all the following integrals are finite. Then, for $n \ge 2$ and $\beta > 0$,*

$$\exp\bigl[\beta D_\beta^{(n)}(Y^1,\dots,Y^n)(y)\bigr] = e^{\beta Y^1(y)} \cdot \frac{\displaystyle\int_{-\infty<x_{n-1}<\cdots<x_1<y} \prod_{i=1}^{n-1} e^{\beta(Y^{i+1}(x_i)-Y^i(x_i))}dx_i}{\displaystyle\int_{-\infty<x_{n-1}<\cdots<x_1<0} \prod_{i=1}^{n-1} e^{\beta(Y^{i+1}(x_i)-Y^i(x_i))}dx_i}. \tag{2.9}$$

Furthermore,

$$\exp\bigl(\beta R_\beta(Y^1,Y^2)(y)\bigr) = \frac{e^{\beta Y^2(y)}\int_{-\infty}^0 \exp\bigl[\beta(Y^2(x)-Y^1(x)\bigr]\,dx}{\int_{-\infty}^y \exp\bigl[\beta(Y^2(x)-Y^1(x)\bigr]\,dx}. \tag{2.10}$$

Proof. We prove this by induction on n. We start with the base case $n=2$. From (2.2),

$$\begin{aligned}\exp\bigl(\beta D_\beta(Y^1,Y^2)(y)\bigr) &= \frac{e^{\beta Y^2(y)}\int_{-\infty}^y \exp\bigl(\beta(Y^1(x,y)-Y^2(x,y))\bigr)\,dx}{\int_{-\infty}^0 \exp\bigl(\beta(Y^1(x,0)-Y^2(x,0))\bigr)\,dx} \\ &= \frac{e^{\beta Y^1(y)}\int_{-\infty}^y \exp\bigl(\beta(Y^2(x)-Y^1(x))\bigr)\,dx}{\int_{-\infty}^0 \exp\bigl(\beta(Y^2(x)-Y^1(x))\bigr)\,dx}.\end{aligned} \tag{2.11}$$

The proof of (2.10) is analogous. Now, assume that (2.9) holds for $n \ge 2$. Then,

$$\begin{aligned}&\exp\bigl(\beta D_\beta^{(n)}(Y^1,\dots,Y^n)(y)\bigr) \\ &= \exp\bigl(\beta D_\beta(Y^1,D_\beta^{(n-1)}(Y^2,\dots,Y^n))(y)\bigr) \\ &= \frac{e^{\beta Y^1(y)}\int_{-\infty}^y \exp\bigl(\beta(D_\beta^{(n-1)}(Y^2,\dots,Y^n)(x_1)-Y^1(x_1)\bigr)dx_1}{\int_{-\infty}^0 \exp\bigl(\beta(D_\beta^{(n-1)}(Y^2,\dots,Y^n)(x_1)-Y^1(x_1)\bigr)dx_1} \\ &= \frac{e^{\beta Y^1(y)}\int_{-\infty}^y \bigl(e^{\beta Y^2(x_1)}\int_{-\infty<x_{n-1}<\cdots<x_2<x_1}\prod_{i=2}^{n-1}\exp\bigl[\beta(Y^{i+1}(x_i)-Y^i(x_i))\bigr]\,dx_i\bigr)e^{-\beta Y^1(x_1)}dx_1}{\int_{-\infty}^0 \bigl(e^{\beta Y^2(x_1)}\int_{-\infty<x_{n-1}<\cdots<x_2<x_1}\prod_{i=2}^{n-1}\exp\bigl[\beta(Y^{i+1}(x_i)-Y^i(x_i))\bigr]\,dx_i\bigr)e^{-\beta Y^1(x_1)}dx_1} \\ &= \frac{e^{\beta Y^1(y)}\int_{-\infty<x_{n-1}<\cdots<x_1<y}\prod_{i=1}^{n-1}\exp\bigl[\beta(Y^{i+1}(x_i)-Y^i(x_i))\bigr]dx_i}{\int_{-\infty<x_{n-1}<\cdots<x_1<0}\prod_{i=1}^{n-1}\exp\bigl[\beta(Y^{i+1}(x_i)-Y^i(x_i))\bigr]dx_i}.\end{aligned}$$

The first equality used the definition of $D^{(n)}$, the second the $n=2$ case, and in the third the induction assumption. In the third equality, an integral over the set $\{-\infty < x_{n-1} < \cdots < x_2 < 0\}$ was cancelled from the numerator and the denominator. ☐

Lemma 2.2. *Assume that $(B,Y) \in \mathcal{Y}_2^{\mathbb{R}}$ with*

$$\lim_{x\to-\infty}\frac{B(x)}{x} = a < b = \lim_{x\to-\infty}\frac{Y(x)}{x}.$$

Then,

$$\lim_{x\to-\infty}\frac{R_\beta(B,Y)(x)}{x} = a, \qquad \textit{and} \qquad \lim_{x\to-\infty}\frac{D_\beta(B,Y)(x)}{x} = b.$$

Proof. By Lemma 2.1, it suffices to show that

$$\lim_{y\to-\infty}\frac{1}{y}\log\int_{-\infty}^y e^{\beta(Y(x)-B(x))}\,dx = \beta(b-a).$$

Fix $\varepsilon > 0$, and let $y < 0$ be such that $x\varepsilon < B(x) - ax < -x\varepsilon$ and $x\varepsilon < Y(x) - bx < -x\varepsilon$ for all $x < y$. Then, for such y,

$$\frac{e^{\beta(b-a+2\varepsilon)y}}{\beta(b-a+2\varepsilon)} = \int_{-\infty}^y e^{\beta(b-a+2\varepsilon)x}\,dx \le \int_{-\infty}^y e^{\beta(Y(x)-B(x))}\,dx \le \int_{-\infty}^y e^{\beta(b-a-2\varepsilon)x}\,dx \le \frac{e^{\beta(b-a-2\varepsilon)y}}{\beta(b-a-2\varepsilon)}.$$

Taking the log of all sides and dividing by y yields

$$\beta(b-a-2\varepsilon) \le \liminf_{y\to-\infty} \frac{1}{y}\log\int_{-\infty}^{y} e^{\beta(Y(x)-B(x))}\,dx \le \limsup_{y\to-\infty}\frac{1}{y}\log\int_{-\infty}^{y} e^{\beta(Y(x)-B(x))}\,dx \le \beta(b-a+2\varepsilon).$$

Sending $\varepsilon \searrow 0$ completes the proof. ☐

Lemma 2.3. *Let $(B,Y),(B,Y') \in \mathcal{Y}_2^{\mathbb{R}}$ be such that $Y \le_{\mathrm{inc}} Y'$. Then, $B <_{\mathrm{inc}} D_\beta(B,Y) \le_{\mathrm{inc}} D_\beta(B,Y')$. If $Y <_{\mathrm{inc}} Y'$, then $D_\beta(B,Y) <_{\mathrm{inc}} D_\beta(B,Y')$ as well.*

Proof. Let $x<y$. We use Lemma 2.1 to write

$$\begin{aligned}\exp[\beta D_\beta(B,Y)(x,y)] &= e^{\beta B(x,y)}\frac{\int_{-\infty}^{y} e^{\beta(Y(z)-B(z))}\,dz}{\int_{-\infty}^{x} e^{\beta(Y(z)-B(z))}\,dz}\\ = e^{\beta B(x,y)}\left(1+\frac{\int_x^y e^{\beta(Y(z)-B(z))}\,dz}{\int_{-\infty}^{x} e^{\beta(Y(z)-B(z))}\,dz}\right) &= e^{\beta B(x,y)}\left(1+\frac{\int_x^y e^{\beta(Y(x,z)-B(x,z))}\,dz}{\int_{-\infty}^{x} e^{\beta(Y(x,z)-B(x,z))}\,dz}\right).\end{aligned}$$

All statements of the lemma now follow from the last equality. ☐

Lemma 2.4. *For $\beta>0$ and $A\subseteq\mathbb{R}$, $\mathbf{D}_\beta^{(n)}:\mathcal{Y}_n^A\to\mathcal{X}_n^A$.*

Proof. Let $\mathbf{Y}=(Y^1,\dots,Y^n)$ and $\boldsymbol{\eta}=\mathbf{D}_\beta^{(n)}(\mathbf{Y})$. From Lemma 2.2 and induction, it follows that for $1\le i\le n$,

$$\lim_{x\to-\infty}\frac{Y^i(x)}{x} = \lim_{x\to-\infty}\frac{\eta^i(x)}{x}.$$

The fact that $\eta_{i-1}\le_{\mathrm{inc}}\eta_i$ follows by induction. By Lemma 2.3, $\eta^2 = D_\beta(Y^1,Y^2)\ge_{\mathrm{inc}} Y^1=\eta^1$. Now, we assume that

$$\eta^i = D_\beta^{(i)}(Y^1,\dots,Y^i)\ge_{\mathrm{inc}} D_\beta^{(i-1)}(Y^1,\dots,Y^{i-1})=\eta^{i-1}.$$

We apply this assumption, replacing $Y^1,\dots,Y^i$ with $Y^2,\dots,Y^{i+1}$ along with Lemma 2.3 to get

$$\begin{aligned}\eta^{i+1} &= D_\beta^{(i+1)}(Y^1,\dots,Y^{i+1}) = D_\beta(Y^1,D_\beta^{(i)}(Y^2,\dots,Y^{i+1}))\\ &\ge_{\mathrm{inc}} D_\beta(Y^1,D_\beta^{(i-1)}(Y^2,\dots,Y^{i})) = D_\beta^{(i)}(Y^1,\dots,Y^i)=\eta^i.\end{aligned}$$

☐

Lemma 2.5. *Let $\beta>0$ and $\lambda_1<\cdots<\lambda_k$. Let λ_i^N,β^N be sequences such that $\lambda_i^N\to\lambda_i$ and $\beta^N\to\beta$ as $N\to\infty$. Set $\bar\lambda^N=(\lambda_1^N,\dots,\lambda_k^N)$ and $\bar\lambda=(\lambda_1,\dots,\lambda_k)$. Then, $\mu_{\beta_N}^{\bar\lambda^N}\to\mu_\beta^{\bar\lambda}$ weakly as measures on $C(\mathbb{R},\mathbb{R}^k)$.*

Proof. Realize the distributions in terms of $(\eta_N^1,\dots,\eta_N^k)\sim\mu_{\beta_N}^{\bar\lambda^N}$ and $(\eta^1,\dots,\eta^k)\sim\mu_\beta^{\bar\lambda}$, where $(\eta_N^1,\dots,\eta_N^k)=\mathbf{D}_{\beta^N}^{(k)}(Z_N^1,\dots,Z_N^k)$ and $(\eta^1,\dots,\eta^k)=\mathbf{D}_\beta^{(k)}(Y^1,\dots,Y^k)$, and the Brownian motions $(Z_N^1,\dots,Z_N^k)\sim\nu^{\bar\lambda^N}$ and $(Y^1,\dots,Y^k)\sim\nu^{\bar\lambda}$ are coupled so that $Z_N^i(x)=Y^i(x)+(\lambda_i^N-\lambda_i)x$. By (2.9),

$$\begin{aligned}\eta_N^j(y) = Z_N^1(y) &+ \frac{1}{\beta_N}\log\int_{-\infty<x_{j-1}<\cdots<x_1<y}\prod_{i=1}^{j-1}\exp\Big\{\beta_N(Z_N^{i+1}(x_i)-Z_N^i(x_i))\Big\}dx_i\\ &-\frac{1}{\beta_N}\log\int_{-\infty<x_{j-1}<\cdots<x_1<0}\prod_{i=1}^{j-1}\exp\Big\{\beta_N(Z_N^{i+1}(x_i)-Z_N^i(x_i))\Big\}dx_i.\end{aligned}$$

Dominated convergence applied to the integrals gives $(\eta_N^1,\dots,\eta_N^k)\Rightarrow(\eta^1,\dots,\eta^k)$ in the sense of finite-dimensional distributions. Each η_N^i is a Brownian motion with drift λ_i^N, so each marginal is tight in $C(\mathbb{R})$. Hence, the process $(\eta_N^1,\dots,\eta_N^k)$ is tight in $C(\mathbb{R},\mathbb{R}^k)$. ☐

For $\gamma>0$ and $\alpha\in\mathbb{R}$ define the mapping $T_{\gamma,\alpha}:C(\mathbb{R})\to C(\mathbb{R})$ as

$$T_{\gamma,\alpha}f(x)=\gamma^{-1}f(\gamma^2x)+\alpha x.$$

Extend it to a mapping $T_{\gamma,\alpha}^n:C(\mathbb{R},\mathbb{R}^n)\to C(\mathbb{R},\mathbb{R}^n)$ of n-tuples componentwise:

$$T_{\gamma,\alpha}^n(f_1,\dots,f_n)=(T_{\gamma,\alpha}f_1,\dots,T_{\gamma,\alpha}f_n).$$

For $\alpha=0$ use the shorthand notation $T_\gamma=T_{\gamma,0}$ and $T_\gamma^n=T_{\gamma,0}^n$.

Lemma 2.6. *For $\beta,\gamma > 0$, $\alpha \in \mathbb{R}$, and $Y^1,\dots,Y^n$ such that the following are all finite, we have*

$$T_{\gamma,\alpha} D_\beta^{(n)}(Y^1,\dots,Y^n) = D_{\beta\gamma}^{(n)}(T_{\gamma,\alpha}^n(Y^1,\dots,Y^n)), \tag{2.12}$$

and

$$T_{\gamma,\alpha} R_\beta(Y^1,Y^2) = R_{\beta\gamma}(T_{\gamma,\alpha}^2(Y^1,Y^2)). \tag{2.13}$$

Consequently, for $\bar\lambda = (\lambda_1 < \cdots < \lambda_n)$,

$$\mu_\beta^{\bar\lambda} \circ (T_{\gamma,\alpha}^n)^{-1} = \mu_{\beta\gamma}^{\gamma\bar\lambda + (\alpha,\dots,\alpha)}. \tag{2.14}$$

In particular,

$$\mu_\beta^{\bar\lambda} = \mu_1^{\beta^{-1}\bar\lambda} \circ (T_\beta^n)^{-1} \tag{2.15}$$

Remark. Equation (2.15) allows us to perform computations for $\beta = 1$ and extend to general β.

Proof. Equation (2.14) follows from (2.12) because, if $(Y^1,\dots,Y^n) \sim \nu^{\bar\lambda}$, then $T_{\gamma,\alpha}^n(Y^1,\dots,Y^n) \sim \nu^{\gamma\bar\lambda+(\alpha,\dots,\alpha)}$.

We turn our attention to proving (2.12). To do so, we use Lemma 2.1. For $y \in \mathbb{R}$,

$$\begin{aligned}
&T_{\gamma,\alpha}^n D_\beta^{(n)}(Y^1,\dots,Y^n)(y)\\
&= \gamma^{-1}Y^1(\gamma^2 y) + \alpha y + \frac{1}{\beta\gamma}\log \int_{-\infty < x_{n-1} < \cdots < x_1 < \gamma^2 y} \prod_{i=1}^{n-1} \exp\bigl(\beta(Y^{i+1}(x_i) - Y^i(x_i))\bigr)dx_i \\
&\qquad\qquad - \frac{1}{\beta\gamma}\log \int_{-\infty < x_{n-1} < \cdots < x_1 < 0} \prod_{i=1}^{n-1} \exp\bigl(\beta(Y^{i+1}(x_i) - Y^i(x_i))\bigr)dx_i \\
&= T_{\gamma,\alpha}Y^1(y) + \frac{1}{\beta\gamma}\log \int_{-\infty < w_{n-1} < \cdots < w_1 < y} \prod_{i=1}^{n-1} \exp\bigl(\beta\gamma(T_{\gamma,\alpha}Y^{i+1}(w_i) - T_{\gamma,\alpha}Y^i(w_i))\bigr)dw_i \\
&\qquad\qquad - \frac{1}{\beta\gamma}\log \int_{-\infty < w_{n-1} < \cdots < w_1 < 0} \prod_{i=1}^{n-1} \exp\bigl(\beta\gamma(T_{\gamma,\alpha}Y^{i+1}(w_i) - T_{\gamma,\alpha}Y^i(w_i))\bigr)dw_i \\
&= D_{\beta\gamma}^{(n)}(T_{\gamma,\alpha}^n(Y^1,\dots,Y^n)),
\end{aligned}$$

where in the second equality, we made the change of variables $x_i = \gamma^2 w_i$, with the Jacobian term cancelling in the difference of the logs of the two integrals. The proof of (2.13) is analogous. ☐

2.2. Consistency and invariance.

Lemma 2.7. *Let $\bar\lambda = (\lambda_1 < \lambda_2 < \cdots < \lambda_n) \in \mathbb{R}^n$. If $(\eta^1,\dots,\eta^n) \sim \mu_\beta^{\bar\lambda}$, then for any subsequence $\lambda_{i_1} < \cdots < \lambda_{i_k}$, $(\eta^{i_1},\dots,\eta^{i_k}) \sim \mu_\beta^{(\lambda_{i_1},\dots,\lambda_{i_k})}$.*

Proof. The proof follows just as the proof of Lemma 3.6(ii) in [SS23b]. The details are found in Appendix A. ☐

Theorem 2.8. *For an increasing vector $\bar\lambda = (\lambda_1,\dots,\lambda_n)$ of strictly positive drifts $0 < \lambda_1 < \cdots < \lambda_n$, the measure $\mu_\beta^{\bar\lambda}$ is an invariant measure for the Markov chain on $\mathcal{X}_n^{\mathbb{R}_{>0}}$ whose time $m-1$ to time m transition is defined as follows. Let $(\eta_{m-1}^1,\dots,\eta_{m-1}^n)$ be the state at time $m-1$, and let B_m be a standard two-sided Brownian motion, independent of the Markov chain in the past. Then the state at time m is*

$$(\eta_m^1,\dots,\eta_m^n) = (D_\beta(B_m,\eta_{m-1}^1),\dots,D_\beta(B_m,\eta_{m-1}^n)). \tag{2.16}$$

Remark. The strictly positive drifts ensure that condition (2.1) is satisfied, and the transformations above are well-defined almost surely.

Proof. This follows by an intertwining argument originating from [FM07] and completed in zero-temperature LPP models in [FS20] and [SS23b]. The details may be found in Appendix A. ☐

2.3. **Construction of the KPZ horizon.** The Skorokhod space $D(\mathbb{R}, C(\mathbb{R}))$ consists of functions $\mathbb{R} \to C(\mathbb{R})$ that are right-continuous with left limits. $C(\mathbb{R})$ is endowed with the topology of uniform convergence on compact sets. A generic element of $D(\mathbb{R}, C(\mathbb{R}))$ is denoted by $F = \{F^\lambda\}_{\lambda\in\mathbb{R}}$, where $F^\lambda \in C(\mathbb{R})$ for each λ. The standard σ-algebra $\mathcal{B}_D$ on $D(\mathbb{R}, C(\mathbb{R}))$ is generated by the projections $\pi^{\lambda_1,\dots,\lambda_k} : D(\mathbb{R}, C(\mathbb{R})) \to C(\mathbb{R},\mathbb{R}^k)$ defined by $\pi^{\lambda_1,\dots,\lambda_k}(F) = (F^{\lambda_1},\dots,F^{\lambda_k})$ (See, for example, [Bil99, Sections 12-13] and [Sch73, Page 101].) Recall the measures $\mu_\beta^{\bar\lambda}$ defined in (2.8).

Proposition 2.9. *On the space $(D(\mathbb{R}, C(\mathbb{R})), \mathcal{B}_D)$, there exists a family of probability measures $\mathbb{P}_\beta$ indexed by the inverse temperature $\beta > 0$, satisfying the following properties. Let $F_\beta = \{F_\beta^\lambda\}_{\lambda\in\mathbb{R}}$ denote the random element of $D(\mathbb{R}, C(\mathbb{R}))$ under the measure $\mathbb{P}_\beta$.*

(i) *For $\beta > 0$ and $\lambda \in \mathbb{R}$, F_β^λ is a two-sided Brownian motion with diffusivity 1 and drift λ. In particular, $\mathbb{P}_\beta$-almost surely, $F_\beta^\lambda(0) = 0$ for each $\lambda \in \mathbb{R}$.*

(ii) *For $\beta > 0$ and an increasing vector $\bar\lambda = (\lambda_1 < \cdots < \lambda_k) \in \mathbb{R}^k$ of drifts, the $C(\mathbb{R}, \mathbb{R}^k)$-valued k-tuple $(F_\beta^{\lambda_1}, \dots, F_\beta^{\lambda_k})$ has distribution $\mu_\beta^{\bar\lambda}$. Equivalently, in terms of projections, $\mathbb{P}_\beta \circ (\pi^{\bar\lambda})^{-1} = \mu_\beta^{\bar\lambda}$. In terms of the mapping $\mathbf{D}_\beta^{(k)}$ and independent Brownian motions $Y^1, \dots, Y^k$ with drifts $\lambda_1 < \cdots < \lambda_k$,*
$$(F_\beta^{\lambda_1}, \dots, F_\beta^{\lambda_k}) \overset{d}{=} (Y^1, D_\beta^{(2)}(Y^1, Y^2), \dots, D_\beta^{(k)}(Y^1, \dots, Y^k)). \tag{2.17}$$
The measure $\mathbb{P}_\beta$ is the unique probability measure on $D(\mathbb{R}, C(\mathbb{R}))$ satisfying this finite-dimensional marginal condition.

(iii) *For $\beta > 0$, $\mathbb{P}_\beta$-almost surely, for all $\lambda_1 < \lambda_2$, $F_\beta^{\lambda_1} <_{\mathrm{inc}} F_\beta^{\lambda_2}$.*

Remark. With a nod to the stationary horizon (SH) discussed above in Section 1.3, we call the process $\{F_\beta^\lambda\}_{\lambda\in\mathbb{R}}$ the *KPZ horizon at inverse temperature* β, abbreviated KPZH_β.

Proof. The construction follows a similar procedure as the construction of the SH in [Sor23] (we note here that the SH was originally constructed as a limit of the Busemann process in exponential LPP in [Bus21]). We start by recalling Lemma 2.7, which states that the measures $\mu_\beta^{\bar\lambda}$ are consistent. Thus, for $\bar\lambda = (\lambda_1 < \cdots < \lambda_k)$, if $(\eta^1, \dots, \eta^k) \sim \mu^{\bar\lambda}$, each η^i has distribution $\mu_\beta^{\lambda_i}$, which is the law of a two-sided Brownian motion with diffusion coefficient 1 and drift λ_i. By Kolmogorov's extension theorem, there exists a unique measure $\mu_\beta^{\mathbb{Q}}$ on $C(\mathbb{R})^{\mathbb{Q}} = \prod_{\mathbb{Q}} C(\mathbb{R})$ under which, for $\{\widetilde F^\alpha\}_{\alpha\in\mathbb{Q}} \in C(\mathbb{R})^{\mathbb{Q}}$ and any choice of $\bar\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Q}^k$ with $\alpha_1 < \cdots < \alpha_k$, $(\widetilde F^{\alpha_1}, \dots, \widetilde F^{\alpha_k}) \sim \mu_\beta^{\bar\alpha}$. In particular, under $\mu_\beta^{\mathbb{Q}}$ each $\widetilde F^\alpha$ is a Brownian motion with drift α.

Because the measures $\mu_\beta^{\bar\lambda}$ are supported on the sets $\mathcal{X}_k^{\mathbb{R}}$ of (2.6), we have that
$$\mu_\beta^{\mathbb{Q}}\big(\widetilde F^{\alpha_1} <_{\mathrm{inc}} \widetilde F^{\alpha_2} \quad \forall \alpha_1 < \alpha_2 \in \mathbb{Q}\big) = 1. \tag{2.18}$$
Hence, there is a full probability event for $\mu_\beta^{\mathbb{Q}}$, on which, for each $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$, the limits
$$F^\lambda(x) := \lim_{\mathbb{Q}\ni\alpha\searrow\lambda} \widetilde F^\alpha(x) \qquad \text{and} \qquad F^{\lambda-} \lim_{\mathbb{Q}\ni\alpha\nearrow\lambda} \widetilde F^\alpha(x) \tag{2.19}$$
exist. By construction,
$$\mu_\beta^{\mathbb{Q}}(F^\lambda \le_{\mathrm{inc}} \widetilde F^\alpha \quad \forall \lambda \in \mathbb{R}, \alpha \in \mathbb{Q} \text{ with } \alpha > \lambda) = 1. \tag{2.20}$$
Then, on the event of (2.20), for $A < a < b < B$,
$$F^\lambda(A, a) + F^\lambda(b, B) \le \widetilde F^\alpha(A, a) + \widetilde F^\alpha(b, B),$$
or equivalently,
$$0 \le \widetilde F^\alpha(a, b) - F^\lambda(a, b) \le \widetilde F^\alpha(A, B) - F^\lambda(A, B),$$
implying that the convergence is uniform on compact sets. The same holds for limits from the left. By monotonicity, $\mu_\beta^{\mathbb{Q}}(F^{\lambda-} \le_{\mathrm{inc}} \widetilde F^\lambda \le_{\mathrm{inc}} F^\lambda \;\; \forall \lambda \in \mathbb{Q}) = 1$. Additionally, uniform convergence ensures that, for each $\lambda \in \mathbb{R}$, $F^{\lambda-}$ and F^λ are both Brownian motions with drift λ. Hence, for each $\lambda \in \mathbb{Q}$,
$$\mu_\beta^{\mathbb{Q}}(F^{\lambda-} = \widetilde F^\lambda = F^\lambda) = 1.$$

In summary, we have defined a stochastic process $\{F^\lambda\}_{\lambda\in\mathbb{R}}$ whose projection to the rationals agrees with $\{\widetilde F^\lambda\}_{\lambda\in\mathbb{Q}}$ under the measure $\mu_\beta^{\mathbb{Q}}$. Let $\lambda_1 < \lambda_2 < \lambda_3$ be real, and choose rational values $\alpha_1 < \lambda_1 < \alpha_2 < \lambda_2 < \alpha_3 < \lambda_3 < \alpha_4$. Then,
$$F^{\alpha_1} \le_{\mathrm{inc}} F^{\lambda_1} \le_{\mathrm{inc}} F^{\alpha_2} \le_{\mathrm{inc}} F^{\lambda_2} \le_{\mathrm{inc}} F^{\alpha_3} \le_{\mathrm{inc}} F^{\lambda_3} \le_{\mathrm{inc}} F^{\alpha_4}. \tag{2.21}$$
This implies that, $\mu_\beta^{\mathbb{Q}}$-almost surely, simultaneously for every $\lambda \in \mathbb{R}$, the following limits exist uniformly on compact sets, and they agree with the limits along rational directions.
$$F^\lambda = \lim_{\alpha\searrow\lambda} F^\alpha \qquad \text{and} \qquad F^{\lambda-} = \lim_{\alpha\nearrow\lambda} F^\alpha.$$

Therefore, the process $\{F^\lambda\}_{\lambda\in\mathbb{R}}$ lies in the space $D(\mathbb{R}, C(\mathbb{R}))$. Let $\mathbb{P}_\beta$ be the pushforward of the measure $\mu_\beta^{\mathbb{Q}}$ to $D(\mathbb{R}, C(\mathbb{R}))$ under the map defined by (2.19). Without reference to the measure, we use $\{F_\beta^\lambda\}_{\lambda\in\mathbb{R}}$ to denote the process.

We check that $\{F^\lambda\}_{\lambda\in\mathbb{R}}$ satisfies the claims of the theorem. Item (i) follows from the uniform convergence along rational directions. Item (ii) follows because for rational directions the finite-dimensional distributions were defined to be $\mu_\beta^{\bar\lambda}$. The limits in (2.19) and the weak convergence of Lemma 2.5 extend this property to all real directions. Since the σ-algebra on $D(\mathbb{R},C(\mathbb{R}))$ is generated by the projections, uniqueness of this process on $D(\mathbb{R},C(\mathbb{R}))$ follows. To verify Item (iii) for real $\lambda_1<\lambda_2$, pick rational α_1,α_2 such that $\lambda_1<\alpha_1<\alpha_2<\lambda_2$. Then (2.21) and (2.18) give $F_\beta^{\lambda_1}\le_{\text{inc}} F_\beta^{\alpha_1} <_{\text{inc}} F_\beta^{\alpha_2}\le_{\text{inc}} F_\beta^{\lambda_2}$. $\square$

2.4. **Distributional invariances of the** KPZH. We prove the following distributional invariances of KPZH_β. Item (iv) below is not needed elsewhere in this paper, but it is included here for future use.

Theorem 2.10. *For $\beta>0$, let F_β be the* KPZH_β.

(i) *Translation invariance: for each $x\in\mathbb{R}$, $\{F_\beta^\lambda(x,x+\bullet)\}_{\lambda\in\mathbb{R}}\overset{d}{=}F_\beta$.*
(ii) *Scaling invariance: for each $\beta>0$, $\gamma>0$, and $\alpha\in\mathbb{R}$, $\{\gamma^{-1}F_\beta^\lambda(\gamma^2\bullet)+\alpha\bullet\}_{\lambda\in\mathbb{R}}\overset{d}{=}\{F_{\gamma\beta}^{\gamma\lambda+\alpha}\}_{\lambda\in\mathbb{R}}$.*
(iii) *Stationarity of increments: for $\lambda_1<\lambda_2<\cdots<\lambda_n$ and $\lambda^\star\in\mathbb{R}$,*
$$(F_\beta^{\lambda_2}-F_\beta^{\lambda_1},\dots,F_\beta^{\lambda_n}-F_\beta^{\lambda_{n-1}})\overset{d}{=}(F_\beta^{\lambda_2+\lambda^\star}-F_\beta^{\lambda_1+\lambda^\star},\dots,F_\beta^{\lambda_n+\lambda^\star}-F_\beta^{\lambda_{n-1}+\lambda^\star}).$$
(iv) *Reflection invariance: $\{F_\beta^{(-\lambda)-}(-\bullet)\}_{\lambda\in\mathbb{R}}\overset{d}{=}F_\beta$.*

Proof. For $\lambda_1<\cdots<\lambda_k$, by definition of the KPZH_β (Proposition 2.9(ii)),
$$(F_\beta^{\lambda_1},\dots,F_\beta^{\lambda_k})\overset{d}{=}(Y^1,D_\beta(Y^1,Y^2),\dots,D_\beta^{(k)}(Y^1,\dots,Y^k)),\tag{2.22}$$
where $Y^1,\dots,Y^n$ are independent Brownian motions with drifts $\lambda_1,\dots,\lambda_n$. It follows from Lemma 2.1 that $D_\beta^{(k)}(Y^1,\dots,Y^k)(x,x+y)=D_\beta^{(k)}(Y^1(x,x+\bullet),\dots,Y^k(x,x+\bullet))(y)$, from which Item (i) follows. Item (ii) follows from Lemma 2.6.

Now we note that Item (iii) follows from Item (ii): Setting $\gamma=1$, we obtain
$$(F_\beta^{\lambda_1},\dots,F_\beta^{\lambda_n})\overset{d}{=}(F_\beta^{\lambda_1+\lambda^\star}(\bullet)-\lambda^\star\bullet,\dots,F_\beta^{\lambda_n+\lambda^\star}-\lambda^\star\bullet),$$
so
$$(F_\beta^{\lambda_2}-F_\beta^{\lambda_1},\dots,F_\beta^{\lambda_n}-F_\beta^{\lambda_{n-1}})\overset{d}{=}(F_\beta^{\lambda_2+\lambda^\star}-F_\beta^{\lambda_1+\lambda^\star},\dots,F_\beta^{\lambda_n+\lambda^\star}-F_\beta^{\lambda_{n-1}+\lambda^\star}).$$
Item (iv) follows from Theorem 3.2(iii) and Corollary 4.3, the first of which is proved in [JRS22] and the second of which we prove later in this paper. One may notice that Item (i) also follows from Theorems 3.2(ii) and Corollary 4.3, but we have proved this item here to avoid circular logic because it is used to prove Corollary 4.3. $\square$

2.5. **Difference of two functions (Proof of Theorem 1.2).**

Proof of Theorem 1.2. Let Y^1,Y^2 be two independent Brownian motions with drifts $\lambda_1<\lambda_2$. By (2.17) and (2.9), as processes indexed by $y\ge 0$,
$$e^{\beta(F_\beta^{\lambda_2}(y)-F_\beta^{\lambda_1}(y))}\overset{d}{=}\frac{\int_{-\infty}^y e^{\beta(Y^2(x)-Y^1(x))}\,dx}{\int_{-\infty}^0 e^{\beta(Y^2(x)-Y^1(x))}\,dx}=1+\frac{\int_0^y e^{\beta(Y^2(x)-Y^1(x))}\,dx}{\int_{-\infty}^0 e^{\beta(Y^2(x)-Y^1(x))}\,dx}.\tag{2.23}$$
By the independence of Brownian increments, the numerator and the denominator of the last ratio in (2.23) are independent. The process $\beta(Y^2(\bullet)-Y^1(\bullet))$ has the distribution of $\sqrt{2}\beta B(\bullet)+\beta\lambda\,\bullet$, where B is a standard Brownian motion. The distribution of the denominator is computed in Lemma A.2 using results from [Duf90]. $\square$

2.6. **Discontinuities of** KPZH **in the drift parameter.**

Lemma 2.11. *On a probability space $(\Omega,\mathcal{F},\mathbb{P})$, let $\lambda\mapsto X(\lambda)$ be an increment-stationary, nondecreasing, almost surely continuous process with $\mathbb{E}[X(1)-X(0)]<\infty$. Then, for every $\varepsilon>0$,*
$$\lim_{\mathbb{Z}\ni n\to\infty} n\mathbb{P}(X(n^{-1})-X(0)>\varepsilon)=0.\tag{2.24}$$

Proof. Partition $[0,1]$ into disjoint intervals of length n^{-1}, and let
$$J_n^\varepsilon=\sum_{i=1}^n \mathbf{1}\{X(i/n)-X((i-1)/n)>\varepsilon\}.$$
By increment-stationarity, $\mathbb{E}[J_n^\varepsilon]=n\mathbb{P}(X(n^{-1})-X(0)>\varepsilon)$. By pathwise uniform continuity of X, $J_n^\varepsilon=0$ for large enough n. The bound $\varepsilon J_n^\varepsilon\le X(1)-X(0)$ and dominated convergence complete the proof. $\square$

Remark. The condition (2.24) appears in Chapter 12 of [Bre68] as a defining condition of Brownian motion. The process we will apply this to is of a different nature, as it is nondecreasing and does not have independent increments.

Corollary 2.12. *For $y \in \mathbb{R}$ and $\beta > 0$, the process $\lambda \mapsto F_\beta^\lambda(y)$ is not almost surely continuous.*

Proof. We may take $y > 0$ because for $y < 0$, Theorem 2.10(i) and $F_\beta^\lambda(0) = 0$ (Proposition 2.9(i)) implies

$$\{F_\beta^\lambda(y)\}_{\lambda\in\mathbb{R}} = \{-F_\beta^\lambda(y,0)\}_{\lambda\in\mathbb{R}} \overset{d}{=} \{-F_\beta^\lambda(0,-y)\}_{\lambda\in\mathbb{R}} = \{-F_\beta^\lambda(-y)\}_{\lambda\in\mathbb{R}}.$$

By the scaling relations of Theorem 2.10(ii), it suffices to take $\beta = 1$. We apply Lemma 2.11 to the process $\lambda \mapsto F_1^\lambda(y)$, which has stationary increments by Theorem 2.10(iii) and is strictly increasing by Proposition 2.9(iii). Since F_1^λ is a Brownian motion with drift λ (Proposition 2.9(i)), we have $\mathbb{E}[F_1^1(y))] = y < \infty$. Lemma 2.11 reduces the problem to showing that for some $\varepsilon > 0$,

$$\liminf_{\lambda\searrow 0} \lambda^{-1}\mathbb{P}(F_1^\lambda(y) - F_1^0(y) > \varepsilon) > 0.$$

In fact, we show that this is true for all $\varepsilon > 0$. For each $\lambda > 0$, let $X_\lambda, Y_\lambda(y)$ be the independent random variables of Theorem 1.2 with $\beta = 1$ so that $F_1^\lambda(y) - F_1^0(y) \overset{d}{=} \log(1 + X_\lambda Y_\lambda(y))$. Observe that for a standard Brownian motion B,

$$Y_\lambda(y) \overset{d}{=} \int_0^y \exp\bigl(\sqrt{2}B(x) + \lambda x\bigr)\,dx > \int_0^y \exp\bigl(\sqrt{2}B(x)\bigr)\,dx =: Y,$$

where Y is taken as a new random variable independent of X_λ. By formula 1.8.4 on page 612 of [BS02], Y has a density function f_Y that is strictly positive on $(0,\infty)$. For $\varepsilon > 0$, let $\varepsilon' = e^\varepsilon - 1 > 0$. Since $X_\lambda \sim \mathrm{Gamma}(\lambda, 1)$,

$$\begin{aligned}&\mathbb{P}(F_1^\lambda(y) - F_1^0(y) > \varepsilon) = \mathbb{P}(X_\lambda Y_\lambda(y) > \varepsilon') \ge \mathbb{P}(X_\lambda Y > \varepsilon') = \mathbb{P}(X_\lambda > \varepsilon'/Y)\\ &= \int_0^\infty \int_{\varepsilon'/w} f_Y(w) \frac{1}{\Gamma(\lambda)} x^{\lambda-1} e^{-x}\,dx\,dw \ge \frac{1}{\Gamma(\lambda)} \int_0^\infty \int_{1\vee(\varepsilon'/w)} f_Y(w) x^{-1} e^{-x}\,dx\,dw = \frac{C_\varepsilon}{\Gamma(\lambda)},\end{aligned}$$

where C_ε is a positive constant. Thus, $\liminf_{\lambda\searrow 0} \lambda^{-1}\mathbb{P}(F_1^\lambda(y) > \varepsilon) \ge C_\varepsilon > 0$ because $\lim_{\lambda\searrow 0} \lambda\Gamma(\lambda) = 1$. ☐

3. The stochastic heat equation

This section collects the necessary background on the SHE and KPZ equation. Section 3.1 mainly summarizes results from [AJRS22, JRS22]. Section 3.2 deals with convergence of the OCY polymer to the SHE.

3.1. Green's function and Busemann process of the SHE.

We briefly describe the construction of the four-parameter field $Z_\beta(\bullet,\bullet|\bullet,\bullet)$ from [AKQ14a, AKQ14b, AJRS22]. We primarily adopt the notation of [AJRS22] and [JRS22].

On an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a space-time white noise W is a mean-zero Gaussian process whose index set is $L^2(\mathbb{R}^2)$, with Lebesgue measure. It satisfies the almost sure linearity $W(af + bg) = aW(f) + bW(g)$ as well as the L^2 isometry property:

$$\mathbb{E}[W(f)W(g)] = \int_{\mathbb{R}^2} f(t,x)g(t,x)\,dt\,dx.$$

One immediate consequence is that, whenever A and B are disjoint, or more generally, their intersection has Lebesgue measure 0, $W(\mathbf{1}_A)$ and $W(\mathbf{1}_B)$ are independent. As a point of notation, we often write

$$W(f) = \int_{\mathbb{R}^2} f(t,x) W(dt\,dx) = \int_{\mathbb{R}^2} f(t,x) W(t,x)\,dt\,dx,$$

where the second equality is formal because W is a random distribution and not defined pointwise.

We define $\mathcal{Z}_\beta$ as the following chaos expansion, where convergence holds in $L^2(\mathbb{P})$.

$$Z_\beta(t,y|s,x) = \sum_{k=0}^\infty \beta^k \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \prod_{i=0}^k \rho(t_{i+1} - t_i, x_{i+1} - x_i) \prod_{i=1}^k W(dt_i, dx_i). \tag{3.1}$$

Here, $\rho(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \mathbf{1}(t > 0)$ is the heat kernel, and the conventions in the integrals are $t_0 = s, x_0 = x, t_{k+1} = t$, and $x_{k+1} = y$. For $f : \mathbb{R} \to \mathbb{R}_{>0}$ with sufficient decay at $\pm\infty$, and $t > s$, define

$$Z_\beta(t,y|s,f) = \int_{\mathbb{R}} Z_\beta(t,y|s,x) f(x)\,dx. \tag{3.2}$$

When the value of s is unspecified in (3.2), we take $s = 0$. Theorem 2.2 and Lemma A.5 of [AJRS22] prove that, in the rigorous sense of solutions in [BC95, BG97, CD14, CD15], the process (3.2) solves the SHE defined in (1.2), for strictly positive functions $f = e^{h_s}$ satisfying

$$\int_{\mathbb{R}} e^{-\alpha x^2} f(x)\, dx < \infty$$

for all $\alpha > 0$. In fact, solutions can be defined for a class of measures which are not necessarily absolutely continuous with respect to Lebesgue measure, but in all applications of this paper, $f(x) = e^{B(x)+\lambda x}$, where B is a Brownian motion, and $\lambda \in \mathbb{R}$, so the necessary conditions are satisfied. We refer the reader to [AJRS22, Appendix A] and the references therein for a more technical discussion on the solution of the SHE from measure-valued initial data.

Theorem 3.1. [AJRS22, Proposition 2.3],[AKQ14a, Equation (18)] *Let $\beta > 0$. Then the following distributional equalities hold between random elements of $C(\mathbb{R}^4_\uparrow, \mathbb{R})$.*

(i) (Shift invariance) For given $u, z \in \mathbb{R}$, $Z_\beta(t, y\,|\,s, x) \overset{d}{=} Z_\beta(t + u, y + z\,|\,s + u, x + z)$.

(ii) (Reflection invariance) $Z_\beta(t, y\,|\,s, x) \overset{d}{=} Z_\beta(t, -y|, s, -x)$.

(iii) (Rescaling) For given $\lambda > 0$, $Z_\beta(t, y\,|\,s, x) \overset{d}{=} \lambda Z_{\beta/\sqrt{\lambda}}(\lambda^2 t, \lambda y\,|\,\lambda^2 s, \lambda x)$.

Furthermore,

(iv) *There exists a constant $C = C_\beta$ so that for all $t > s$ and $x, y \in \mathbb{R}$,*

$$\mathbb{E}[Z_\beta^2(t, y\,|\,s, x)] \le C\rho^2(t - s, y - x).$$

We are particularly interested in the $\lambda = \beta^2$ case of Theorem 3.1(iii), in which the distributional equality becomes

$$Z_\beta(t, y\,|\,s, x) \overset{d}{=} \beta^2 Z_1(\beta^4 t, \beta^2 y\,|\,\beta^4 s, \beta^2 x). \tag{3.3}$$

Jointly with the Busemann functions, by appeal to the Busemann limits (1.5), we have this distributional equality:

$$\begin{aligned}\big\{b_\beta^{\lambda\square}(s, x, t, y), Z_\beta(t', y'\,|\,s', x') &: (s, x, t, y) \in \mathbb{R}^4,\ (s', x', t', y') \in \mathbb{R}^4_\uparrow,\ \lambda \in \mathbb{R},\ \square \in \{-, +\}\big\} \\ \overset{d}{=} \big\{b_1^{\lambda\beta^{-2}\square}(\beta^4 s, \beta^2 x, \beta^4 t, \beta^2 y), &\beta^2 Z_1(\beta^4 t', \beta^2 y'\,|\,\beta^4 s', \beta^2 x') : \\ &(s, x, t, y) \in \mathbb{R}^4,\ (s', x', t', y') \in \mathbb{R}^4_\uparrow,\ \lambda \in \mathbb{R},\ \square \in \{-, +\}\big\}.\end{aligned} \tag{3.4}$$

In the following theorems, we use (3.4) to transfer the statements for $\beta = 1$ from [JRS22] to general $\beta > 0$. We introduce the following class of functions, named $\mathbb{F}_\lambda$. Let $f : \mathbb{R} \to (0, \infty)$ be a Borel function that is locally bounded. Then, for $\lambda \in \mathbb{R}$, we say that $f \in \mathbb{F}_\lambda$ if

$$\begin{aligned} -\infty \le \limsup_{x \to -\infty} \frac{\log f(x)}{|x|} < \lambda = \lim_{x \to \infty} \frac{\log f(x)}{x} &\qquad \text{if } \lambda > 0 \\ \lim_{x \to -\infty} \frac{\log f(x)}{|x|} = |\lambda| > \limsup_{x \to \infty} \frac{\log f(x)}{x} \ge -\infty &\qquad \text{if } \lambda < 0 \\ -\infty \le \limsup_{|x| \to \infty} \frac{\log f(x)}{|x|} \le 0 &\qquad \text{if } \lambda = 0. \end{aligned} \tag{3.5}$$

Theorem 3.2. [JRS22, Theorems 3.1, 3.3, 3.5, 3.23, and Corollary 3.4] *Let $\beta > 0$. Then, there exists a stochastic process $\{b_\beta^{\lambda\square}(s, x, t, y) : s, x, t, y, \lambda \in \mathbb{R}, \square \in \{-, +\}\}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and satisfying the following properties. For this process, we define*

$$\Lambda_{b_\beta} = \{\lambda \in \mathbb{R} : b_\beta^{\lambda-}(s, x, t, y) \ne b_\beta^{\lambda+}(s, x, t, y) \text{ for some } (s, x, t, y) \in \mathbb{R}^4\}.$$

When $\lambda \notin \Lambda_{b_\beta}$, we write $b_\beta^\lambda = b_\beta^{\lambda-} = b_\beta^{\lambda+}$.

(i) *For each $t, \lambda \in \mathbb{R}$, under $\mathbb{P}$, the process $y \mapsto b_\beta^\lambda(t, 0, t, y)$ is a two-sided Brownian motion with diffusivity β and drift λ.*

(ii) (Shift) For $r, z \in \mathbb{R}$, as processes in $s, x, t, y, \lambda \in \mathbb{R}$, $\square \in \{-, +\}$,

$$b_\beta^{\lambda\square}(s, x, t, y) \overset{d}{=} b_\beta^{\lambda\square}(s + r, x + z, t + r, y + z).$$

(iii) (Reflection) As processes in $s, x, t, y, \lambda \in \mathbb{R}$, $\square \in \{-, +\}$,

$$b_\beta^{\lambda\square}(s, x, t, y) \overset{d}{=} b_\beta^{(-\lambda)(-\square)}(s, -x, t, -y).$$

(iv) For each $\lambda \in \mathbb{R}$, $\mathbb{P}(\lambda \in \Lambda_{b_\beta}) = 0$.

(v) Either $\mathbb{P}(\Lambda_{b_\beta} = \varnothing) = 1$ or $\mathbb{P}(\Lambda_{b_\beta}$ is countable and dense in $\mathbb{R}) = 1$.

Furthermore, there exists an event of full probability on which the following hold:

(vi) *For each* $\lambda \in \mathbb{R}$ *and* $\Box \in \{-,+\}$, $b_\beta^{\lambda\Box} \in C(\mathbb{R}^4, \mathbb{R})$.

(vii) *For all* $x < y, t,$ *and* $\alpha < \lambda$,

$$b_\beta^{\alpha-}(t,x,t,y) \le b_\beta^{\alpha+}(t,x,t,y) < b_\beta^{\lambda-}(t,x,t,y) \le b_\beta^{\lambda+}(t,x,t,y).$$

More specifically, whenever $\lambda \in \Lambda_{b_\beta}$, $b_\beta^{\lambda-}(t,x,t,y) < b_\beta^{\lambda+}(t,x,t,y)$, *and consequently, for each* $a \neq 0$,

$$\Lambda_{b_\beta} = \{\lambda \in \mathbb{R} : b_\beta^{\lambda-}(0,0,0,a) \neq b_\beta^{\lambda+}(0,0,0,a)\}.$$

(viii) *For all* r,x,s,y,t,z,λ *and all* $\Box \in \{-,+\}$,

$$b_\beta^{\lambda\Box}(r,x,s,y) + b_\beta^{\lambda\Box}(s,y,t,z) = b_\beta^{\lambda\Box}(r,x,t,z).$$

(ix) *For all* s,x,t,y,λ *and all* $\Box \in \{-,+\}$,

$$b_\beta^{\lambda-}(s,x,t,y) = \lim_{\alpha \nearrow \lambda} b_\beta^{\alpha\Box}(s,x,t,y), \quad \textit{and} \quad b_\beta^{\lambda+}(s,x,t,y) = \lim_{\alpha \searrow \lambda} b_\beta^{\alpha\Box}(s,x,t,y).$$

(x) *For all* $t > r$, *all* s,x,y,λ, *and all* $\Box \in \{-,+\}$,

$$e^{b_\beta^{\lambda\Box}(s,x,t,y)} = \int_{\mathbb{R}} e^{b_\beta^{\lambda\Box}(s,x,r,z)} Z_\beta(t,y|r,z)\,dz.$$

(xi) *For all* $\lambda \notin \Lambda_{b_\beta}$ *and* $f \in \mathbb{F}_\lambda$, *the following limit holds uniformly on compact sets of* $(s,x,t,y) \in \mathbb{R}^4$*:*

$$\lim_{r\to-\infty} \frac{\int_{\mathbb{R}} f(z) Z_\beta(t,y|r,z)\,dz}{\int_{\mathbb{R}} f(z) Z_\beta(s,x|r,z)\,dz} = e^{b_\beta^\lambda(s,x,t,y)}.$$

For later use, we derive the following uniqueness result from Theorem 3.2.

Theorem 3.3. *Let* $(f^1,\dots,f^k)$ *be a coupling of initial data with* $f_i \in \mathbb{F}_{\lambda_i}$ *almost surely for* $i \in \{1,\dots,k\}$. *If, for all* $t > 0$,

$$\left\{\frac{Z_\beta(t,\bullet|e^{f^i})}{Z_\beta(t,0|e^{f^i})}\right\}_{1\le i\le k} \stackrel{d}{=} \{\exp(f^i)\}_{1\le i\le k},$$

then

$$\{\exp(f^i)\}_{1\le i\le k} \stackrel{d}{=} \{\exp(b_\beta^{\lambda_i}(0,0,0,\bullet))\}_{1\le i\le k}.$$

Remark. A stronger uniqueness property is true. The joint Busemann process is the unique stationary and ergodic jointly invariant distribution for the KPZ equation under more general conditions on the asymptotic slopes at $\pm\infty$. We refer the reader to Section 3.4 of [JRS22] for a more detailed discussion.

Proof. The $r = s$ case of Theorem 3.2(x) along with the additivity of Theorem 3.2(viii) implies that

$$e^{b_\beta^{\lambda\Box}(t,0,t,y)} = \frac{\int_{\mathbb{R}} e^{b_\beta^{\lambda\Box}(s,0,s,z)} Z_\beta(t,y|s,z)\,dz}{\int_{\mathbb{R}} e^{b_\beta^{\lambda\Box}(s,0,s,z)} Z_\beta(t,0|s,z)\,dz}, \tag{3.6}$$

and Theorem 3.2(ii) implies that for all $s < t$,

$$\exp(b_\beta^{\lambda\Box}(t,0,t,\bullet)) \stackrel{d}{=} \exp(b_\beta^{\lambda\Box}(s,0,s,\bullet)). \tag{3.7}$$

The $s = t$ case of Theorem 3.2(xi) states that for $\lambda \notin \Lambda_{b_\beta}$ and $f \in \mathbb{F}_\lambda$,

$$\lim_{r\to-\infty} \frac{\int_{\mathbb{R}} f(z) Z_\beta(t,y|r,z)\,dz}{\int_{\mathbb{R}} f(z) Z_\beta(t,0|r,z)\,dz} = e^{b_\beta^\lambda(t,0,t,y)},$$

uniformly on compact subsets of $y \in \mathbb{R}$. Then, using (3.6), Theorem 3.2(iv), and the shift invariance of Theorem 3.1(i), for any (deterministic or random) k-tuple of functions $(f^1,\dots,f^k)$, so that, with probability one, each $f^i \in \mathbb{F}_{\lambda_i}$, as $t \to \infty$, we have the following distributional convergence on $C(\mathbb{R}^k,\mathbb{R})$:

$$\left\{\frac{Z_\beta(t,\bullet|e^{f^i})}{Z_\beta(t,0|e^{f^i})}\right\}_{1\le i\le k} \Longrightarrow \{\exp(b_\beta^{\lambda_i}(0,0,0,\bullet))\}_{1\le i\le k}$$

In particular, if for all $t > 0$,

$$\left\{\frac{Z_\beta(t,\bullet|e^{f^i})}{Z_\beta(t,0|e^{f^i})}\right\}_{1\le i\le k} \stackrel{d}{=} \{\exp(f^i)\}_{1\le i\le k},$$

then $\{\exp(f^i)\}_{1\le i\le k} \stackrel{d}{=} \{\exp(b_\beta^{\lambda_i}(0,0,0,\bullet))\}_{1\le i\le k}$. $\square$

3.2. **Convergence of the O'Connell-Yor polymer to SHE.** In his section, we show convergence of the O'Connell-Yor polymer to the Green's function of the SHE (Theorem 3.9) and prove a convergence result for the model started from initial data (Theorem 3.10). The O'Connell-Yor polymer (alternatively, the Brownian polymer), first introduced in [OY01], is defined as follows. On a probability space $(\Omega, \mathcal{F}, P)$, let $\mathbf{B} = (B_r)_{r\in\mathbb{Z}}$ be a sequence of independent, two-sided standard Brownian motions. For $(m,x) \le (n,y) \in \mathbb{Z}\times\mathbb{R}$, define the path space

$$\mathbb{X}_{(m,x),(n,y)} := \{(x_{m-1}, x_m, \ldots, x_n) \in \mathbb{R}^{n-m+2} : x = x_{m-1} \le x_m \le \cdots \le x_n = y\}.$$

For $(x,m),(y,n) \in \mathbb{R}\times\mathbb{Z}$ with $m < n$ and $x \le y$, the point-to-point partition function is defined as

$$Z_\beta^{\mathrm{sd}}(n,y\,|\,m,x)(\mathbf{B}) = \int_{\mathbb{X}_{(m,x),(n,y)}} \exp\Big\{\beta \sum_{r=m}^{n} B_r(x_{r-1},x_r)\Big\} dx_{m:n-1}. \tag{3.8}$$

Throughout the paper, $\beta > 0$ is a positive inverse-temperature parameter. The superscript in Z_β^{sd} stands for *semi-discrete*. For $m = n$, define

$$Z_\beta^{\mathrm{sd}}(m,y\,|\,m,x)(\mathbf{B}) = e^{\beta B_m(x,y)}.$$

The argument $\mathbf{B}$ will often be omitted from the notation. From the definition, the Chapman-Kolmogorov equation: namely that, for $m < r \le n$ and $x < y$,

$$Z_\beta^{\mathrm{sd}}(n,y\,|\,m,x) = \int_x^y Z_\beta^{\mathrm{sd}}(n,y\,|\,r,w) Z_\beta^{\mathrm{sd}}(r-1,w\,|\,m,x)\,dw. \tag{3.9}$$

We also define a partition function with a boundary at level $m = -1$. For a random or deterministic initial function $f : \mathbb{R}\to\mathbb{R}$ with $f(x) \to 0$ sufficiently fast as $x \to -\infty$, define, for $n \ge 0$ and $y \in \mathbb{R}$,

$$Z_\beta^{\mathrm{sd}}(n,y\,|\,f) = \int_{-\infty}^y f(x)\, Z_\beta^{\mathrm{sd}}(n,y\,|\,0,x)\,dx. \tag{3.10}$$

For $n = -1$, define $Z_\beta^{\mathrm{sd}}(-1,y\,|\,f) = f(y)$.

Abbreviate the Poisson distribution as

$$q(n,y) = e^{-y}\frac{y^n}{n!}\mathbf{1}((n,y) \in \mathbb{Z}_{\ge 0}\times\mathbb{R}_{\ge 0}). \tag{3.11}$$

For integers $n \ge m$, real numbers $y \ge x$, and $\gamma \in \mathbb{R}_{>0}$, set

$$Y_\gamma(n,y\,|\,m,x) = e^{-(y-x)-\frac{\gamma^2}{2}(y-x)} Z_\gamma^{\mathrm{sd}}(n,y\,|\,m,x). \tag{3.12}$$

Next, define

$$\begin{aligned} \delta_k(n\,|\,m) &= \{m = n_0 \le n_1 \le \cdots \le n_k \le n_{k+1} = n : n_i \in \mathbb{Z}\}, \quad \text{and} \\ \Delta_k(y\,|\,x) &= \{x = y_0 < y_1 < \cdots < y_k < y_{k+1} = y : y_i \in \mathbb{R}\}. \end{aligned}$$

Lemma 3.4. *There exists a constant $C > 0$ so that, for all integers $k > 0$, $n \ge m$, and all real numbers $y > x$, whenever $\mathbf{n} \in \delta_k(n\,|\,m)$ and $\mathbf{y} \in \Delta_k(y\,|\,x)$,*

$$\begin{aligned} &\prod_{i=0}^k q^2(n_{i+1}-n_i, y_{i+1}-y_i) \le C^k g(\mathbf{n},\mathbf{y}) \\ &\qquad := C^k \frac{e^{-2(y-x)} 2^{2(n-m)}}{\pi^{(k+1)/2}} \prod_{i=0}^k \frac{(y_{i+1}-y_i)^{2(n_{i+1}-n_i)}}{[2(n_{i+1}-n_i)]!\sqrt{(n_{i+1}-n_i)\vee 1}}. \end{aligned} \tag{3.13}$$

Furthermore, for each $\mathbf{n} \in \delta_k(n\,|\,m)$,

$$\int_{\Delta_k(y\,|\,x)} g(\mathbf{n},\mathbf{y}) \prod_{i=1}^k dy_i = \frac{2^{2(n-m)}[(n-m)!]^2 (y-x)^k q^2(n-m,y-x)}{\pi^{(k+1)/2}[2(n-m)+k]!} \prod_{i=0}^k \frac{1}{\sqrt{(n_{i+1}-n_i)\vee 1}}. \tag{3.14}$$

Proof. From the definition, it follows that

$$\prod_{i=0}^k q^2(n_{i+1}-n_i, y_{i+1}-y_i) = e^{-2(y-x)} \prod_{i=0}^k \frac{(y_{i+1}-y_i)^{2(n_{i+1}-n_i)}}{[(n_{i+1}-n_i)!]^2}, \tag{3.15}$$

and Stirling's approximation implies that for large n,

$$[n!]^2 \sim \frac{[2n]!\sqrt{\pi n}}{2^{2n}}$$

In particular, there exists a constant C so that $\frac{1}{[n!]^2} \le C\frac{2^{2n}}{[2n]!\sqrt{\pi n}}$ for $n \ge 1$. Inserting this bound into (3.15) proves (3.13). The integral (3.14) is the computation of a Dirichlet integral after the change of variable $w_i = \frac{y_i - y_{i-1}}{y-x}$. □

Lemma 3.5. *For $\varepsilon \ge 0$, for all integers $n \ge m$ and real numbers $y \ge x+\varepsilon$, the field $Y_\gamma(n,y|m,x)$ satisfies the following Itô integral equation:*

$$Y_\gamma(n,y|m,x) = \sum_{m\le k\le n} q(n-k, y-(x+\varepsilon))Y_\gamma(k, x+\varepsilon|m,x) + \gamma\int_{x+\varepsilon}^{y} \sum_{m\le k\le n} q(n-k, y-w)\, Y_\gamma(k,w|m,x)\, dB_k(w), \tag{3.16}$$

where $\{B_r\}_{r\in\mathbb{Z}}$ are the i.i.d. Brownian motions that define Z^{sd}_γ.

Remark. We observe that in the $\varepsilon = 0$ case, since $Y_\gamma(k,x|m,x) = \mathbf{1}(k=m)$, we obtain

$$Y_\gamma(n,y|m,x) = q(n-m, y-x) + \gamma\int_x^y \sum_{m\le k\le n} q(n-k,y-w)\,Y_\gamma(k,w|m,x)\,dB_k(w).$$

This further implies that

$$\mathbb{E}[Y_\gamma(n,y|m,x)] = q(n-m,y-x). \tag{3.17}$$

Proof. With $Y_\gamma(n,y|m,x)$ defined, let $\widetilde{Y}_\gamma(n,y|m,x)$ denote the RHS of (3.16). We prove that $\widetilde{Y}_\gamma(n,y|m,x) = Y_\gamma(n,y|m,x)$ by induction on $n \ge m$. First, note that

$$\begin{aligned}
\widetilde{Y}_\gamma(m,y|m,x) &= q(0,y-x-\varepsilon)Y_\gamma(m,x+\varepsilon|m,x) + \gamma\int_{x+\varepsilon}^{y} q(0,y-w)Y_\gamma(m,w|m,x)\,dB_m(w)\\
&= e^{-(y-x)-\frac{\gamma^2}{2}\varepsilon+\gamma B_m(x,x+\varepsilon)} + \gamma\int_{x+\varepsilon}^{y} e^{-(y-x)-\frac{\gamma^2}{2}(w-x)+\gamma B_m(x,w)}\,dB_m(w)\\
&= e^{-(y-x)+\frac{\gamma^2}{2}x-\gamma B_m(x)}\Big(e^{-\frac{\gamma^2}{2}(x+\varepsilon)+\gamma B_m(x+\varepsilon)} + \gamma\int_{x+\varepsilon}^{y} e^{-\frac{\gamma^2}{2}w+\gamma B_m(w)}\,dB_m(w)\Big),
\end{aligned}$$

so the equality $Y_\gamma(m,y|m,x) = \widetilde{Y}_\gamma(m,y|m,x)$ reduces to

$$e^{-\frac{\gamma^2}{2}y+\gamma B_m(y)} = e^{-\frac{\gamma^2}{2}(x+\varepsilon)+\gamma B_m(x+\varepsilon)} + \gamma\int_{x+\varepsilon}^{y} e^{-\frac{\gamma^2}{2}w+\gamma B_m(w)}\,dB_m(w),$$

which follows from Itô's formula. Now, assume that for some $n > m$, $\widetilde{Y}_\gamma(n-1,w|m,x) = Y_\gamma(n-1,w|m,x)$ for all $w \ge x+\varepsilon$.

From the definition (3.12), the Chapman-Kolmogorov equation (3.9), and definition (3.12) again,

$$\begin{aligned}
Y_\gamma(n,y|m,x) &= e^{-(y-x)-\frac{\gamma^2}{2}(y-x)}\int_x^y Z^{\mathrm{sd}}_\gamma(n-1,w|m,x)e^{\gamma B_n(w,y)}\,dw\\
&= e^{-(y-x)-\frac{\gamma^2}{2}(y-x)+\gamma B_n(y)}\int_x^y Z^{\mathrm{sd}}_\gamma(n-1,w|m,x)e^{-\gamma B_n(w)}\,dw\\
&= e^{\gamma B_n(y)}\cdot e^{-(1+\frac{\gamma^2}{2})y}\int_x^y e^{(1+\frac{\gamma^2}{2})w}\,Y_\gamma(n-1,w|m,x)e^{-\gamma B_n(w)}\,dw.
\end{aligned}$$

Let d denote differentiation in the real variable y, with x fixed. An application of Itô's formula to the last line above gives

$$dY_\gamma(n,y|m,x) = [Y_\gamma(n-1,y|m,x) - Y_\gamma(n,y|m,x)]\,dy + \gamma Y_\gamma(n,y|m,x)\,dB_n(y). \tag{3.18}$$

Additionally, a simple computation shows

$$dq(n-k,y-x) = [q(n-1-k,y-x) - q(n-k,y-x)]\,dy \qquad \text{for } n\in\mathbb{Z}_{\ge k} \text{ and } y \ge x, \tag{3.19}$$

where we set $q(-1,w) = 0$ by convention. Differentiate the right-hand side of (3.16) and apply (3.19) and $q(n,0) = \mathbf{1}(n=0)$ to obtain

$$\begin{aligned}
d\widetilde{Y}_\gamma(n,y|m,x) &= \sum_{m\le k\le n} dq(n-k,y-(x+\varepsilon))Y_\gamma(k,x+\varepsilon|m,x) + \gamma\sum_{m\le k\le n} q(n-k,0)Y_\gamma(k,y|m,x)\,dB_k(y)\\
&\qquad + \gamma\int_{x+\varepsilon}^{y}\sum_{m\le k\le n} dq(n-k,y-w)Y_\gamma(k,w|m,x)\,dB_k(w) \qquad (3.20)\\
&= [\widetilde{Y}_\gamma(n-1,y|m,x) - \widetilde{Y}_\gamma(n,y|m,x)]\,dy + \gamma Y_\gamma(n,y|m,x)\,dB_n(y). \qquad (3.21)
\end{aligned}$$

In the first equality above, we have used the stochastic Leibniz rule (see for example, [Øks10, Equation (6.2.25)]). Then, comparing (3.18) and (3.21), the induction hypothesis implies that the process

$$X(y) := Y_\gamma(n,y|m,x) - \widetilde{Y}_\gamma(n,y|m,x)$$

satisfies $dX = -X dy$. Since $q(n,0) = \mathbf{1}(n=0)$, we observe further that $\widetilde{Y}_\gamma(n,x+\varepsilon|m,x) = Y_\gamma(n,x+\varepsilon|m,x)$ for $n \ge m$, so X has the initial condition $X(x+\varepsilon) = 0$. Thus, $X(y) = 0$ for all $y \ge x+\varepsilon$, and $Y_\gamma(n,y|m,x) = \widetilde{Y}_\gamma(n,y|m,x)$, as desired. ☐

We now use Lemma 3.5 to write Y_γ as an infinite series of iterated stochastic integrals.

Lemma 3.6. *Let q and Y_γ be defined as in* (3.11) *and* (3.12)*. For every $n \ge m$ and $y \ge x$, $Y_\gamma(n,y|m,x)$ can be written as the following $L^2(\mathbb{P})$-convergent infinite sum*

$$\begin{aligned} Y_\gamma(n,y|m,x) &= \sum_{k=0}^\infty \gamma^k I_k(n,y|m,x), \\ I_k(n,y|m,x) &= \sum_{\delta_k(n|m)} \int_{\Delta_k(y|x)} \prod_{i=0}^k q(n_{i+1}-n_i, y_{i+1}-y_i) \prod_{i=1}^k dB_{n_i}(y_i), \end{aligned} \tag{3.22}$$

where, in the $k=0$ case, we use this notation to mean $I_0(n,y|m,x) = q(n-m,y-x)$. Furthermore,

$$\mathbb{E}[Y_\gamma(n,y|m,x)^2] = \sum_{k=0}^\infty \gamma^{2k} \mathbb{E}[I_k(n,y|m,x)^2], \tag{3.23}$$

and there exists a universal constant $C>0$ so that for all integers $n \ge m$ and $k \ge 0$, and real numbers $y \ge x$ and $\gamma > 0$,

$$\mathbb{E}[I_k(n,y|m,x)^2] \le C^k q^2(n-m,y-x)(y-x)^k \frac{(n-m)^{k/2}}{(2(n-m)+k)^k \Gamma((k+1)/2)}. \tag{3.24}$$

Proof. Picard iteration of (3.16) in Lemma 3.5 in the case $\varepsilon = 0$ gives the expansion (3.22), assuming that the series is convergent. By independence of the B_k, the fact that Itô integrals have mean 0, and the Itô isometry, we have that

$$\mathbb{E}[I_k(n,y|m,x) I_j(n,y|m,x)] = \delta_{j=k} \sum_{\delta_k(n|m)} \int_{\Delta_k(y|x)} \prod_{i=0}^k q^2(n_{i+1}-n_i, y_{i+1}-y_i) \prod_{i=1}^k dy_i,$$

Hence, as long as the sum on the right-hand side of (3.23) is convergent, the expansion (3.22) is $L^2(\mathbb{P})$ convergent, and (3.23) holds. For this, it suffices to show (3.24), and it further suffices to show the $m=x=0$ case by translation invariance. For shorthand notation, set $Y_\gamma(n,y) = Y_\gamma(n,y|0,0)$. Then, by Lemma 3.4 and Stirling's approximation, there exists a constant $C>0$ (possibly changing from line to line) so that

$$\begin{aligned} \frac{\mathbb{E}[(I_k(n,y))^2]}{q^2(n,y)} &= \frac{1}{q^2(n,y)} \sum_{\delta_k(n)} \int_{\Delta_k(y)} \prod_{i=0}^k q^2(n_{i+1}-n_i, y_{i+1}-y_i) \prod_{i=1}^k dy_i \\ &\le C^k \frac{2^{2n}[n!]^2 y^k}{[2n+k]!} \sum_{\delta_k(n)} \prod_{i=0}^k \frac{1}{\sqrt{(n_{i+1}-n_i) \vee 1}} \\ &\le C^k y^k \frac{n^{2n+1} 2^{2n}}{(2n+k)^{2n+k+1/2}} \sum_{\delta_k(n)} \prod_{i=0}^k \frac{1}{\sqrt{(n_{i+1}-n_i) \vee 1}} \\ &\le C^k y^k \frac{n^{2n+1} 2^{2n}}{(2n+k)^{2n+k+1/2}} \int_{\Delta_k(n)} \prod_{i=0}^k \frac{1}{\sqrt{s_{i+1}-s_i}} \prod_{i=1}^k ds_i \\ &= C^k y^k \frac{n^{2n+k/2+1/2} 2^{2n}}{(2n+k)^{2n+k+1/2}} \int_{t_i>0, \sum_{i=1}^k t_i<1} \prod_{i=1}^{k+1} \frac{1}{\sqrt{t_i}} \prod_{i=1}^k dt_i \\ &\le C^k y^k \frac{n^{k/2}}{(2n+k)^k \Gamma((k+1)/2)}. \end{aligned}$$

Similar as before, the second-to-last equality is the change of variables $t_i = \frac{s_i - s_{i-1}}{n}$, and the last equality is the computation of a Dirichlet integral and the observation that $\frac{n^{2n+1/2} 2^{2n}}{(2n+k)^{2n+1/2}} \le 1$. ☐

Lemma 3.7. *Given a space-time white noise W, one can couple the field of i.i.d. Brownian motions $\{B_r\}_{r\in\mathbb{Z}}$ with W so that*

$$Y_\gamma(n,y|m,x) = \sum_{k=0}^{\infty}\gamma^k\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\prod_{i=0}^{k} q(\lfloor t_{i+1}\rfloor - \lfloor t_i\rfloor, y_{i+1}-y_i)\prod_{i=1}^{k} W(dt_i,dy_i), \tag{3.25}$$

where we define $t_0 = m, t_{k+1} = n, y_0 = x$, and $y_{k+1} = y$.

Proof. Given a space-time white noise W, we can define a field of i.i.d. two-sided Brownian motions $\{B_r\}_{r\in\mathbb{Z}}$ by

$$B_r(y) = \begin{cases} W(\mathbf{1}([r,r+1]\times[0,y])) & y\ge 0\\ -W(\mathbf{1}([r,r+1]\times[y,0])) & y<0\end{cases}$$

Alternatively, we can use a single definition using the formal equality

$$B_r(y) = \int_0^y dx\int_r^{r+1} dt\, W(t,x). \tag{3.26}$$

From the definition of B_r, we have the formal equality

$$dB_r(x) = dx\int_r^{r+1} dt\, W(t,x). \tag{3.27}$$

Now, with the B_r defined in terms of W, we show that we can write Y_γ as (3.25). Using (3.27), the $k=1$ term in (3.22) can be written as

$$\begin{aligned}
&\gamma\int_x^y\sum_{r=m}^{n} q(n-r,y-z)q(r-m,z-x)\,dB_r(z)\\
&=\gamma\int_x^y\sum_{r=m}^{n} q(n-r,y-z)q(r-m,z-x)\,dz\int_r^{r+1} dt\,W(t,z)\\
&=\gamma\int_x^y dz\sum_{r=m}^{n}\int_r^{r+1} dt\,q(n-\lfloor t\rfloor,y-z)q(\lfloor t\rfloor-m,z-x)W(t,z)\\
&=\gamma\int_x^y dz\int_m^{n+1} dt\,q(n-\lfloor t\rfloor,y-z)q(\lfloor t\rfloor-m,z-x)W(t,z)\\
&=\gamma\int_{\mathbb{R}} dz\int_{\mathbb{R}} dt\,q(n-\lfloor t\rfloor,y-z)q(\lfloor t\rfloor-m,z-x)W(t,z),
\end{aligned}$$

and this matches the $k=1$ term of (3.25). The last line follows because the integrand is 0 outside the original bounds of integration. The general case follows using the same reasoning and induction. □

We prove an intermediate lemma for a scaled transition function. With q as in (3.11), set

$$p_N(t,y|s,x) = \sqrt{N}q\big(\lfloor tN\rfloor - \lfloor sN\rfloor, (t-s)N+\sqrt{N}(y-x)\big) \qquad \text{and} \qquad p_N(t,y) = p_N(t,y|0,0). \tag{3.28}$$

Lemma 3.8. *The following hold.*

(i) *As $N\to\infty$, $p_N(t,y|s,x)\to\rho(t-s,y-x)$, pointwise, for $x,y\in\mathbb{R}$ and $t>s$.*

(ii) *For each $t>0$, $y\in\mathbb{R}$, $\alpha>0$, and integer $M\ge 1$,*

$$\lim_{N\to\infty}\int_{-\infty}^{t\sqrt{N}+y} e^{\alpha|x|}p_N^M(t,y|0,x)\,dx = \int_{\mathbb{R}} e^{\alpha|x|}\rho^M(t,y-x)\,dx<\infty. \tag{3.29}$$

Proof. **Item** (i)**:** The pointwise convergence $p_N(t,y|s,x)\to\rho(t-s,y-x)$ is a simple application of Stirling's approximation. We prove the $x=s=0$ case to avoid clutter, but the general case is entirely similar.

$$\begin{aligned}
p_N(t,y|s,x) &= \sqrt{N}e^{-N(t-s)+\sqrt{N}(y-x)}\Big(\frac{(N(t-s)+\sqrt{N}(y-x))^{\lfloor tN\rfloor}}{(\lfloor tN\rfloor)!}\Big)\\
&\sim\frac{\sqrt{N}}{\sqrt{2\pi\lfloor tN\rfloor}}e^{-tN+\lfloor tN\rfloor+\sqrt{N}y}\Big(\frac{tN+\sqrt{N}y}{\lfloor tN\rfloor}\Big)^{\lfloor tN\rfloor}\sim p(t,y),
\end{aligned} \tag{3.30}$$

where the last step follows from the Taylor expansion

$$\lfloor tN\rfloor\log\Big(\frac{tN+\sqrt{N}y}{\lfloor tN\rfloor}\Big) = \lfloor tN\rfloor\log\Big(1+\frac{tN+\sqrt{N}y-\lfloor tN\rfloor}{\lfloor tN\rfloor}\Big)$$

$$= tN - \sqrt{N}y - \lfloor tN \rfloor - \frac{\lfloor tN \rfloor}{2}\Big(\frac{tN + \sqrt{N}y - \lfloor tN \rfloor}{\lfloor tN \rfloor}\Big)^2 + O(N^{-1/2})$$
$$= tN - \sqrt{N}y - \lfloor tN \rfloor - \frac{y^2}{2t} + O(N^{-1/2})$$

Item (ii)**:** Recall the convention $p_N(t,y) = p_N(t,y|0,0)$. Changing variables, (3.29) is equivalent to

$$\lim_{N\to\infty}\int_{-\infty}^{t\sqrt{N}} e^{\alpha|x+y|}p_N^2(t,-x)\,dx = \int_{\mathbb{R}} e^{\alpha|x+y|}\rho^2(t,-x)\,dx$$

We prove this by showing separately that

$$\int_{-y}^{t\sqrt{N}} e^{\alpha(x+y)}p_N^M(t,-x)\,dx \to \int_{-y}^{\infty} e^{\alpha(x+y)}\rho^M(t,-x)\,dx, \tag{3.31}$$

and

$$\int_{-\infty}^{-y} e^{-\alpha(x+y)}p_N^M(t,-x)\,dx \to \int_{-\infty}^{-y} e^{-\alpha(x+y)}\rho^M(t,-x)\,dx. \tag{3.32}$$

First, by completing the square and changing variables, we obtain

$$\int_{-y}^{\infty} e^{\alpha x}\rho^M(t,-x)\,dx = \frac{\sqrt{2t}e^{\alpha^2 t/(2M)}}{\sqrt{M}(2\pi t)^{M/2}}\int_{\sqrt{\frac{M}{2t}}(-y-\frac{\alpha t}{M})}^{\infty} e^{-u^2}\,du$$
$$= \frac{e^{\alpha^2 t/(2M)}}{2\sqrt{M}(2\pi t)^{(M-1)/2}}\operatorname{erfc}\Big(\sqrt{\frac{M}{2t}}(-y-\alpha t/M)\Big).$$

On the other hand,

$$\int_{-y}^{t\sqrt{N}} e^{\alpha x}p_N^M(t,-x)\,dx = N^{M/2}\int_{-y}^{t\sqrt{N}} e^{\alpha x}e^{-M(tN - x\sqrt{N})}\frac{(tN - x\sqrt{N})^{M\lfloor tN \rfloor}}{(\lfloor tN \rfloor !)^M}\,dx,$$

which, upon the transformation $w = (tN - x\sqrt{N})(\alpha/\sqrt{N} + M)$, we obtain

$$\frac{N^{(M-1)/2}e^{\alpha t\sqrt{N}}}{(\lfloor tN \rfloor !)^M(\alpha/\sqrt{N}+M)^{M\lfloor tN \rfloor + 1}}\int_0^{(tN+y\sqrt{N})(\alpha/\sqrt{N}+M)} e^{-w}w^{M\lfloor tN \rfloor}\,dw \tag{3.33}$$

$$= \frac{N^{(M-1)/2}e^{\alpha t\sqrt{N}}(M\lfloor tN \rfloor)!}{(\lfloor tN \rfloor !)^M(\alpha/\sqrt{N}+M)^{M\lfloor tN \rfloor + 1}}\frac{\gamma(M\lfloor tN \rfloor + 1, (tN+y\sqrt{N})(\alpha/\sqrt{N}+M))}{(M\lfloor tN \rfloor)!} \tag{3.34}$$

where $\gamma(s,x) = \int_0^x e^{-u}u^{s-1}\,du$ is the lower incomplete gamma function. Tricomi [Tri50] showed that as $a \to \infty$, the function γ has the following asymptotic expansion that holds uniformly on compact subsets of z (see also [Tem75]):

$$\frac{\gamma(a+1, a + z(2a)^{1/2})}{\Gamma(a+1)} \sim \frac{1}{2}\operatorname{erfc}(-z) + o(1). \tag{3.35}$$

Inserting this asymptotic into (3.33) and using Stirling's approximation, we obtain

$$\int_0^{t\sqrt{N}} e^{\alpha x}p_N^M(t,-x)\,dx \sim \frac{N^{(M-1)/2}e^{\alpha t\sqrt{N}}(M\lfloor tN \rfloor)!}{2(\lfloor tN \rfloor !)^M(\alpha/\sqrt{N}+M)^{M\lfloor tN \rfloor + 1}}\operatorname{erfc}\Big(\sqrt{\frac{M}{2t}}(-y-\alpha t/M)\Big)$$
$$\sim \frac{N^{(M-1)/2}e^{\alpha t\sqrt{N}}\sqrt{2\pi M\lfloor tN \rfloor}(M\lfloor tN \rfloor/e)^{M\lfloor tN \rfloor}}{2(2\pi\lfloor tN \rfloor)^{M/2}(\lfloor tN \rfloor/e)^{M\lfloor tN \rfloor}M(\alpha/\sqrt{N}+M)^{M\lfloor tN \rfloor}}\operatorname{erfc}\Big(\sqrt{\frac{M}{2t}}(-y-\alpha t/M)\Big)$$
$$\sim \frac{e^{\alpha t\sqrt{N}}}{2\sqrt{M}(2\pi t)^{(M-1)/2}}\Big(1 + \frac{\alpha}{M\sqrt{N}}\Big)^{M\lfloor tN \rfloor} \sim \frac{e^{\alpha^2 t/(2M)}}{2\sqrt{M}(2\pi t)^{(M-1)/2}}\operatorname{erfc}\Big(\sqrt{\frac{M}{2t}}(-y-\alpha t/M)\Big).$$

The last step comes from

$$t\alpha\sqrt{N} - M\lfloor tN \rfloor\log\Big(1+\frac{\alpha}{M\sqrt{N}}\Big) = t\alpha\sqrt{N} - M\lfloor tN \rfloor\Big(\frac{\alpha}{M\sqrt{N}} - \frac{\alpha^2}{2M^2N} + o(N^{-1})\Big) = \frac{\alpha^2 t}{2M} + o(1).$$

This proves (3.31). The proof of (3.32) is similar: the left-hand side is transformed into an incomplete gamma function via the transformation $w = (M - \alpha/\sqrt{N})(Nt - x\sqrt{N})$. In this case, we are left with a gamma function minus an incomplete gamma function, and the asymptotic expansion (3.35) gives us the needed asymptotics. $\square$

We introduce the scaled O'Connell-Yor polymer partition function, whose convergence to the fundamental solution of SHE is proved next. For $\beta > 0$ and a sequence β_N such that $N^{1/4}\beta_N \to \beta$, define a scaling factor

$$\psi_N(s,t,x,y;\beta_N) = \sqrt{N}\exp\Big(-N\big(1+\frac{\beta_N^2}{2}\big)(t-s) - \sqrt{N}\big(1+\frac{\beta_N^2}{2}\big)(y-x)\Big) \tag{3.36}$$

and the scaled partition function

$$\begin{aligned} Z_N(t,y\,|\,s,x) &= \psi_N(s,t,x,y;\beta_N) Z^{\mathrm{sd}}_{\beta_N}\big(\lfloor tN\rfloor, tN+y\sqrt{N}\,\big|\,\lfloor sN\rfloor, sN+x\sqrt{N}\,\big)\mathbf{1}\{x \le (t-s)\sqrt{N}+y\} \\ &= \sqrt{N}\,Y_{\beta_N}\big(\lfloor tN\rfloor, tN+y\sqrt{N}\,\big|\,\lfloor sN\rfloor, sN+y\sqrt{N}\big). \end{aligned} \tag{3.37}$$

We use representation (3.25) in terms of white noise for $Z_N(t,y\,|\,s,x)$ and then scale the white noise suitably to relate $Z_N(t,y\,|\,s,x)$ to $Z_\beta(t,y\,|\,s,x)$. This produces for each N a coupling of $Z_N(t,y\,|\,s,x)$ and $Z_\beta(t,y\,|\,s,x)$ on the probability space of the white noise. We show that in this coupling, their L^2 distance converges to zero.

For the next proofs, recall the standard fact from analysis known as the generalized dominated convergence theorem: if $f_n \to f$ a.e., $|f_n| \le g_n \to g$ a.e., and $\int g_n \to \int g < \infty$, then $\int f_n \to \int f$.

Theorem 3.9. *Fix $\beta > 0$ and a sequence β_N such that $N^{1/4}\beta_N \to \beta$. For each N we have a coupling of Z_N and Z_β on the probability space of the white noise so that this limit holds:*

$$\lim_{N\to\infty} \mathbb{E}\big[\,\big|Z_N(t,y\,|\,s,x) \;-\; Z_\beta(t,y\,|\,s,x)\big|^2\,\big] = 0 \qquad \text{for each } s<t \text{ and } x,y\in\mathbb{R}.$$

In particular, the weak convergence $Z_N(t,y\,|\,s,x) \Longrightarrow Z_\beta(t,y\,|\,s,x)$ holds for each $s<t$ and $x,y\in\mathbb{R}$.

Remark. We sketch here how our result is consistent with that in [Nic21, Theorem 1.2]. An independent proof follows. Ours gives the result for the four-parameter field, while [Nic21] handles the two-parameter case. A change of coordinates is required to transfer between the two results, as shown in the discussion below. We note that [Nic21, Theorem 1.2] is a more general result about partition functions for d nonintersecting paths, while we only handle the $d=1$ case. There, the semi-discrete partition function is rescaled by the Lebesgue volume of the path space $\mathbb{X}_{(0,0),(n,x)}$:

$$\widetilde{Z}^{\mathrm{sd}}_\beta(n,x) = \frac{n!}{x^n} Z^{\mathrm{sd}}_\beta(n,x\,|\,0,0).$$

Theorem 1.2 of [Nic21] states that for any sequence β_N with $N^{1/4}\beta_N \to \beta$ as $N\to\infty$, we have the following convergence in distribution as $N\to\infty$:

$$\widetilde{Z}^{\mathrm{sd}}_{\beta_N}(\lfloor tN + x\sqrt{N}\rfloor, tN)\exp\Big(-\frac{\beta_N^2}{2}tN\Big) \Longrightarrow \frac{Z_\beta(t,x\,|\,0,0)}{\rho(t,x)}.$$

Furthermore, as a process indexed by $(t,x)\in(0,\infty)\times\mathbb{R}$, the convergence holds in the sense of finite-dimensional distributions, and there exists a coupling in which the convergence is in L^p for any $p\ge 1$. Using Stirling's approximation, one can directly apply this result to show that, in the $s=x=0$ case, the following convergence holds in $L^2(\mathbb{P})$:

$$\sqrt{N}\exp\Big(-N\big(1+\frac{\beta_N^2}{2}\big)(t-s)\Big) Z^{\mathrm{sd}}_{\beta_N}\big(\lfloor tN + y\sqrt{N}\rfloor, tN\,\big|\,\lfloor sN + x\sqrt{N}\rfloor, sN\big) \Longrightarrow Z_\beta(t,y\,|\,s,x). \tag{3.38}$$

It is reasonable to think this would extend to general $s<t$ and $x,y\in\mathbb{R}$, but some additional justification would be needed since the shift invariance does not immediately work through the floor functions. To see how the result in Theorem 3.9 appears from (3.38), replace t with $t-\frac{y}{\sqrt{N}}$ and s with $s-\frac{x}{\sqrt{N}}$, then replace x with $-x$ and y with $-y$ and use the reflection invariance of Z_β (Theorem 3.1(ii)). To make this argument directly rigorous, one would need to show uniform convergence on compact sets, or change the parameterization and show that the chaos series still converges. We emphasize here that our proof below is self-contained, uses different methods, and does not rely on the result of [Nic21], although the white noise coupling is the same.

Proof of Theorem 3.9. With $Z_N(t,y\,|\,s,x)$ from (3.37) and for a sequence β_N with $N^{1/4}\beta_N\to\beta$, Lemma 3.7 implies

$$\begin{aligned} Z_N(t,y\,|\,s,x) &= \sqrt{N}\,Y_{\beta_N}\big(\lfloor tN\rfloor, tN+y\sqrt{N}\,\big|\,\lfloor sN\rfloor, sN+y\sqrt{N}\big) \\ &= \sqrt{N}\sum_{k=0}^\infty \beta_N^k \int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\prod_{i=0}^k q(\lfloor t_{i+1}\rfloor - \lfloor t_i\rfloor, y_{i+1}-y_i)\,W(dt_i,dy_i), \end{aligned}$$

where $t_0=\lfloor sN\rfloor$, $t_{k+1}=\lfloor tN\rfloor$, $y_0 = sN+x\sqrt{N}$, and $y_k = tN+y\sqrt{N}$. Now, consider the transformation

$$(t_i,y_i)_{1\le i\le k} \mapsto \Phi_k((t_i,y_i)_{1\le i\le k}) = \Big(\frac{t_i}{N}, \frac{y_i-t_i}{\sqrt{N}}\Big)_{1\le i\le k}. \tag{3.39}$$

This transformation alters the white noise, but multiplying by the square-root Jacobian term $N^{3k/4}$, we have the following distributional equality on the level of processes in $(s,x,t,y) \in \mathbb{R}^4_\uparrow$ (note that the transformation does not depend on the choice of s,x,t,y):

$$\begin{aligned}
&Z_N(t,y\,|\,s,x) \overset{d}{=} \sum_{k=0}^{\infty} (N^{1/4}\beta_N)^k J_k^N(t,y\,|\,s,x) \\
&:= \sum_{k=0}^{\infty} (N^{1/4}\beta_N)^k N^{(k+1)/2} \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \prod_{i=0}^{k} q\big(\lfloor Nt_{i+1}\rfloor - \lfloor Nt_i\rfloor, N(t_{i+1}-t_i) + \sqrt{N}(y_{i+1}-y_i)\big) \prod_{i=1}^{k} W(dt_i, dy_i) \\
&= \sum_{k=0}^{\infty} (N^{1/4}\beta_N)^k \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \prod_{i=0}^{k} p_N(t_{i+1}, y_{i+1}\,|\,t_i, y_i) \prod_{i=1}^{k} W(dt_i\, dy_i)
\end{aligned} \tag{3.40}$$

where p_N is defined in (3.28), and we define $t_0 = s, t_{k+1} = t, y_0 = x$, and $y_{k+1} = y$. We recall that $N^{1/4}\beta_N \to \beta$. Since $q(n,y) = 0$ for $n < 0$ or $y < 0$, the integrand of the kth term in (3.40) is supported on the set

$$\begin{aligned}
A_k(N,s,t,y,x) := \Big\{ &sN + x\sqrt{N} \le t_i N + y_i\sqrt{N} \le t_{i+1}N + y_{i+1}\sqrt{N}, \\
&\text{and } \frac{\lfloor sN\rfloor}{N} \le t_i \le \frac{\lfloor t_{i+1}N\rfloor + 1}{N},\ 1 \le i \le k \Big\}.
\end{aligned}$$

The chaos series (3.40) is the version of $Z_N(t,y\,|\,s,x)$ that we couple with the SHE through the common white noise. It is compared with the chaos series (3.1) of Z_β:

$$Z_\beta(t,y\,|\,s,x) = \sum_{k=0}^{\infty} \beta^k J_k(t,y\,|\,s,x) := \sum_{k=0}^{\infty} \beta^k \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \prod_{i=0}^{k} \rho(t_{i+1}-t_i, y_{i+1}-y_i) \prod_{i=1}^{k} W(dt_i, dy_i), \tag{3.41}$$

where the integrand of the kth term is supported on the set $\Delta_k(t|s)$ where $s < t_i < t_{i+1}$ for $1 \le i \le k$. We seek to show that for fixed $(s,x,t,y) \in \mathbb{R}^4_\uparrow$, $\lim_{N\to\infty} \|Z_N(t,y\,|\,s,x) - Z_\beta(t,y\,|\,s,x)\|_{L^2(\mathbb{P})} = 0$. We note that for any integer $K_0 \ge 0$,

$$\begin{aligned}
&\Big\| Z_N(t,y\,|\,s,x) - Z_\beta(t,y\,|\,s,x) \Big\|_{L^2(\mathbb{P})} \\
&\le \sum_{k=0}^{K_0} \Big\| (N^{1/4}\beta_N)^k J_k^N(t,y\,|\,s,x) - \beta^k J_k(t,y\,|\,s,x) \Big\|_{L^2(\mathbb{P})} \\
&\qquad + \Big\| \sum_{k=K_0+1}^{\infty} (N^{1/4}\beta_N)^k J_k^N(t,y\,|\,s,x) \Big\|_{L^2(\mathbb{P})} + \Big\| \sum_{k=K_0+1}^{\infty} \beta^k J_k(t,y\,|\,s,x) \Big\|_{L^2(\mathbb{P})}.
\end{aligned} \tag{3.42}$$

Since the series for Z_β is almost surely convergent in $L^2(\mathbb{P})$, for any $\varepsilon > 0$, there exists $K_0 \ge 0$ so that

$$\Big\| \sum_{k=K_0+1}^{\infty} \beta^k J_k(t,y\,|\,s,x) \Big\|_{L^2(\mathbb{P})} < \varepsilon. \tag{3.43}$$

Then, by the $L^2(\mathbb{P})$ isometry property for white noise, reversing the transformation (3.39), and dividing by the $N^{3k/2}$ Jacobian term, we get

$$\begin{aligned}
&\Big\| J_k^N(t,y\,|\,s,x) \Big\|_{L^2(\mathbb{P})}^2 = \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \prod_{i=0}^{k} [p_N(t_{i+1}, y_{i+1}\,|\,t_i, y_i)]^2 \prod_{i=1}^{k} dt_i dy_i \\
&= N^{-k/2+1} \sum_{\delta_k(\lfloor tN\rfloor\,|\,\lfloor sN\rfloor)} \int_{\Delta_k(tN+y\sqrt{N}\,|\,sN+x\sqrt{N})} \prod_{i=0}^{k} q^2(n_{i+1}-n_i, y_{i+1}-y_i) \prod_{i=1}^{k} dy_i \\
&\overset{(3.24)}{\le} C^k N^{-k/2+1} \frac{q^2(\lfloor tN\rfloor - \lfloor sN\rfloor\,|\,(t-s)N + (y-x)\sqrt{N})((t-s)N + (y-x)\sqrt{N})^k (\lfloor tN\rfloor - \lfloor sN\rfloor)^{k/2}}{(2(\lfloor tN\rfloor - \lfloor sN\rfloor) + k)^k \Gamma((k+1)/2)} \\
&\le \frac{C^k [p_N(t,y\,|\,s,x)]^2}{\Gamma((k+1)/2)} \le \frac{C^k}{\Gamma((k+1)/2)}
\end{aligned} \tag{3.44}$$

where the constant C changes from line to line and depends on the fixed parameters x, y and $s < t$, but not on N. The last inequality follows from the pointwise convergence $p_N(t,y\,|\,s,x) \to \rho(t-s, y-x)$ (Lemma

3.8(i)). Then, using orthogonality of each chaos for different values of k, there exists K_0 sufficiently large so that for all $N \geq 1$,

$$\begin{aligned}\Big\|\sum_{k=K_0+1}^{\infty}(N^{1/4}\beta_N)^k J_k^N(t,y\,|\,s,x)\Big\|_{L^2(\mathbb{P})}^2 &= \sum_{k=K_0+1}^{\infty}(N^{1/4}\beta_N)^{2k}\|J_k^N(t,y\,|\,s,x)\|_{L^2(\mathbb{P})}^2 \\ &\leq \sum_{k=K_0+1}^{\infty}\frac{C^k}{\Gamma((k+1)/2)} < \varepsilon.\end{aligned} \tag{3.45}$$

Then, combining (3.42),(3.43), and (3.45), and recalling that $N^{1/4}\beta_N \to \beta$, the proof is complete once we show that, for each $k \geq 0$,

$$\limsup_{N\to\infty}\Big\|J_k^N(t,y\,|\,s,x) - J_k(t,y\,|\,s,x)\Big\|_{L^2(\mathbb{P})} = 0.$$

When $k = 0$, this is simply the convergence of the nonrandom quantity $[p_N(t,y\,|\,s,x)]^2$ to $\rho^2(t-s,y-x)$, which is Lemma 3.8(i). Thus, we take $k \geq 1$ in the sequel. We again use the $L^2(\mathbb{P})$ isometry property of white noise. That is,

$$\begin{aligned}&\Big\|J_k^N(t,y\,|\,s,x) - J_k(t,y\,|\,s,x)\Big\|_{L^2(\mathbb{P})}^2 \\ &= \int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\left(\prod_{i=0}^k p_N(t_{i+1},y_{i+1}\,|\,t_i,y_i) - \prod_{i=0}^k \rho(t_{i+1}-t_i,y_{i+1}-y_i)\right)^2\prod_{i=1}^k dt_i dy_i.\end{aligned} \tag{3.46}$$

Recall that $\prod_{i=0}^k p_N(t_{i+1},y_{i+1}\,|\,t_i,y_i)$ is supported on the set $A_k(N,s,t,y,x)$, while $\prod_{i=0}^k \rho(t_{i+1}-t_i\,|\,y_{i+1}-y_i)$ is supported on the set where $t_{i+1} > t_i$ for all i. By Lemma 3.8(i), the integrand in (3.46) converges to 0 Lebesgue-a.e. Expand the square, drop the cross term, and use (3.13) of Lemma 3.4 to conclude that the integrand in (3.46) is bounded by a k-dependent constant times

$$\frac{N^{k+1}}{\pi^{(k+1)/2}}e^{-2[(t-s)N+(y-x)\sqrt{N}]}2^{2(\lfloor tN\rfloor - \lfloor sN\rfloor)}\prod_{i=0}^k\frac{((t_{i+1}-t_i)N+(y_{i+1}-y_i)\sqrt{N})^{2(\lfloor t_{i+1}N\rfloor - \lfloor t_iN\rfloor)}}{[2(\lfloor t_{i+1}N\rfloor - \lfloor t_iN\rfloor)]!\sqrt{(\lfloor t_{i+1}N\rfloor - \lfloor t_iN\rfloor)\vee 1}} \tag{3.47}$$

$$+\prod_{i=0}^k\rho^2(t_{i+1}-t_i,y_{i+1}-y_i) \tag{3.48}$$

A Stirling's approximation computation nearly identical to that in the proof of Lemma 3.8(i) shows that the term in (3.47) converges pointwise to the term in (3.48). By the generalized dominated convergence theorem, it then suffices to show that the integral over $A_k(N,s,t,y,x)$ of the term in (3.47) converges as $N\to\infty$ to

$$\begin{aligned}&\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\prod_{i=0}^k\rho^2(t_{i+1}-t_i,y_{i+1}-y_i)\prod_{i=1}^k dt_i dy_i \\ &= \frac{\sqrt{t-s}}{2^k\pi^{k/2}}\rho^2(t-s,y-x)\int_{\mathbb{R}^k}\mathbf{1}_{B_k}\left(\prod_{i=1}^k dt_i\frac{1}{\sqrt{t_i}}\right)\frac{1}{\sqrt{t-s-\sum_{i=1}^k t_i}} < \infty,\end{aligned} \tag{3.49}$$

where

$$B_k = \{t_i > 0, 1 \leq i \leq k, \sum_{i=1}^k t_i < t-s\}.$$

The equality above comes as follows. To compute the integral on the left in (3.49), write the integrand as

$$\frac{1}{2^{k+1}\pi^{(k+1)/2}}\prod_{i=0}^k\frac{1}{\sqrt{t_{i+1}-t_i}}\prod_{i=0}^k\frac{1}{\sqrt{\pi(t_{i+1}-t_i)}}e^{-\frac{(y_{i+1}-y_i)^2}{t_{i+1}-t_i}},$$

and recognize the second product as a product of transition probabilities for a diffusivity $\frac{1}{\sqrt{2}}$ Brownian motion. Hence,

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\prod_{i=0}^k\rho^2(t_{i+1}-t_i,y_{i+1}-y_i)\prod_{i=1}^k dt_i dy_i = \frac{e^{-(y-x)^2/(t-s)}}{2^{k+1}\pi^{(k+1)/2}\sqrt{\pi(t-s)}}\int_{\mathbb{R}^k}\prod_{i=0}^k\frac{\mathbf{1}(t_{i+1}>t_i)}{\sqrt{t_{i+1}-t_i}}\prod_{i=1}^k dt_i,$$

and one readily verifies that this agrees with (3.49). Next, reversing the transformation (3.39) just as in (3.44) (and dividing by the $N^{3k/2}$ Jacobian term), the integral over $A_k(N,s,t,y,x)$ of the term in (3.47) is equal to

$$
\begin{aligned}
&\frac{N^{-k/2+1}}{\pi^{(k+1)/2}} e^{-2[(t-s)N+(y-x)\sqrt{N}]} 2^{2(\lfloor tN\rfloor-\lfloor sN\rfloor)} \\
&\qquad\times \sum_{\delta_k(\lfloor tN\rfloor\,|\,\lfloor sN\rfloor)} \int_{\Delta_k(tN+y\sqrt{N}\,|\,sN+x\sqrt{N})} \prod_{i=0}^{k} \frac{(y_{i+1}-y_i)^{2(n_i-n_{i-1})}}{[2(n_{i+1}-n_i)]!\sqrt{(n_{i+1}-n_i)\vee 1}} \prod_{i=1}^{k} dt_i \\
&\overset{(3.14)}{=} \frac{N^{-k/2}}{\pi^{(k+1)/2}} \frac{2^{2(\lfloor tN\rfloor-\lfloor sN\rfloor)}[(\lfloor tN\rfloor-\lfloor sN\rfloor)!]^2((t-s)N+(y-x)\sqrt{N})^k[p_N(t,y\,|\,s,x)]^2}{[2(\lfloor tN\rfloor-\lfloor sN\rfloor)+k]!} \\
&\qquad\times \sum_{\delta_k(\lfloor tN\rfloor\,|\,\lfloor sN\rfloor)} \prod_{i=0}^{k} \frac{1}{\sqrt{(n_{i+1}-n_i)\vee 1}}
\end{aligned}
$$

(next by Stirling's approximation and $p_N \to \rho$)

$$
\begin{aligned}
&\sim \frac{N^{-(k-1)/2}\sqrt{t-s}}{2^k\pi^{k/2}}\rho^2(t-s,y-x) \sum_{\delta_k(\lfloor tN\rfloor\,|\,\lfloor sN\rfloor)} \prod_{i=0}^{k} \frac{1}{\sqrt{(n_{i+1}-n_i)\vee 1}} \\
&= \frac{N^{-(k-1)/2}\sqrt{t-s}}{2^k\pi^{k/2}}\rho^2(t-s,y-x) \sum_{\substack{m_i\ge 0 \\ \sum_{i=1}^{k+1} m_i=\lfloor tN\rfloor-\lfloor sN\rfloor}} \prod_{i=1}^{k+1} \frac{1}{\sqrt{m_i\vee 1}} \\
&= \frac{N^{-(k-1)/2}\sqrt{t-s}}{2^k\pi^{k/2}}\rho^2(t-s,y-x) \\
&\qquad\times \int_{\mathbb{R}^k} \mathbf{1}\Big(t_i>0, 1\le i\le k, t_{k+1}=\lfloor tN\rfloor-\lfloor sN\rfloor-\sum_{i=1}^{k}\lfloor t_i\rfloor\Big) \prod_{i=1}^{k+1}\frac{1}{\sqrt{\lfloor t_i\rfloor\vee 1}} \prod_{i=1}^{k} dt_i \\
&= \frac{\sqrt{t-s}}{2^k\pi^{k/2}}\rho^2(t-s,y-x) \int_{B_k(N)} \left(\prod_{i=1}^{k} dt_i \sqrt{\frac{N}{\lfloor t_iN\rfloor\vee 1}}\right)\sqrt{\frac{N}{(\lfloor tN\rfloor-\lfloor sN\rfloor-\sum_{i=1}^{k}\lfloor t_i\rfloor)\vee 1}},
\end{aligned}
$$

where the last integration is over the set

$$
B_k(N) = \Big\{t_i>0,\ 1\le i\le k,\ \sum_{i=1}^{k}\lfloor t_iN\rfloor \le \lfloor tN\rfloor-\lfloor sN\rfloor\Big\}.
$$

Comparing to (3.49), the proof is complete once we show that

$$
\lim_{N\to\infty}\int_{\mathbb{R}^k} \mathbf{1}_{B_k(N)}\left(\prod_{i=1}^{k} dt_i\sqrt{\frac{N}{\lfloor t_iN\rfloor\vee 1}}\right)\sqrt{\frac{N}{(\lfloor tN\rfloor-\lfloor sN\rfloor-\sum_{i=1}^{k}\lfloor t_i\rfloor)\vee 1}} \tag{3.50}
$$

$$
= \int_{\mathbb{R}^k} \mathbf{1}_{B_k}\left(\prod_{i=1}^{k} dt_i\frac{1}{\sqrt{t_i}}\right)\frac{1}{\sqrt{t-s-\sum_{i=1}^{k} t_i}}. \tag{3.51}
$$

The proof is technical and lengthy, and is handled in Lemma 3.11 at the end of the section. □

Lemma 3.10. *Let $f:\Omega\times\mathbb{R}\to\mathbb{R}_{>0}$ be a jointly measurable function, independent of $Z^{\mathrm{sd}}_{\beta_N}$ and Z_β such that, for some $\alpha>0$,*

$$
\mathbb{E}[f(x)] \le e^{\alpha|x|} \quad \forall x\in\mathbb{R}.
$$

For a sequence β_N with $N^{1/4}\beta_N \to \beta > 0$ and $Z_N(t,y\,|\,s,x)$ defined as in (3.37), under the coupling of Theorem 3.9, the following convergence holds for each choice of $y\in\mathbb{R}$ and $t>0$:

$$
\lim_{N\to\infty}\mathbb{E}\left[\,\left|\int_{-\infty}^{t\sqrt{N}+y} f(x)Z_N(t,y\,|\,0,x)\,dx \;-\; \int_{\mathbb{R}} f(x)Z_\beta(t,y\,|\,0,x)\,dx\right|\,\right] = 0. \tag{3.52}
$$

In particular, as $N\to\infty$, the following weak convergence holds for finite-dimensional distributions of these processes indexed by $y\in\mathbb{R}$:

$$
\left\{\int_{-\infty}^{t\sqrt{N}+y} f(x)Z_N(t,y\,|\,0,x)\,dx : y\in\mathbb{R}\right\} \Longrightarrow \left\{\int f(x)Z_\beta(t,y\,|\,0,x)\,dx : y\in\mathbb{R}\right\}.
$$

Proof. We can integrate over the whole space $\mathbb{R}$ in the left integral in (3.52) because $Z_N(t,y|0,x)$ is defined to be zero for $x > t\sqrt{N}+y$. Then, by independence and the growth assumption on f, we have

$$\mathbb{E}\Big[\int_{\mathbb{R}} f(x)|Z_N(t,y|0,x) - Z_\beta(t,y|0,x)|\,dx\Big] = \int_{\mathbb{R}} e^{\alpha|x|}\mathbb{E}[|Z_N(t,y|0,x) - Z_\beta(t,y|0,x)|]\,dx. \tag{3.53}$$

By Theorem 3.9, we have, for each $t>0$ and $x,y\in\mathbb{R}$,

$$\mathbb{E}[|Z_N(t,y|0,x) - Z_\beta(t,y|0,x)|] \le \mathbb{E}[|Z_N(t,y|0,x) - Z_\beta(t,y|0,x)|^2]^{1/2} \overset{N\to\infty}{\longrightarrow} 0.$$

A consequence of (3.1), (3.17), and the choice of scaling is that $\mathbb{E}[Z_N(t,y|0,x)] = p_N(t,y|0,x)$ and $\mathbb{E}[Z_\beta(t,y|0,x)] = \rho(t,y-x)$. Thus, the integrand on the right-hand side of (3.53) is bounded above by

$$e^{\alpha|x|}(p_N(t,y-x) + \rho(t,y-x)).$$

Lemma 3.8(i)–(ii) implies that $p_N(t,y-x)\to\rho(t,y-x)$ pointwise and that

$$\int_{\mathbb{R}} e^{\alpha|x|}(p_N(t,y-x)+\rho(t,y-x))\,dx \overset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} 2e^{\alpha|x|}\rho(t,y-x)\,dx < \infty,$$

so the generalized dominated convergence theorem completes the proof. The finite-dimensional weak convergence holds because finite linear combinations also satisfy the limit in (3.52), so the Cramér-Wold theorem completes the proof. $\square$

We conclude this section by completing the unfinished business of the proof of Theorem 3.9.

Lemma 3.11. *The convergence of* (3.50) *to* (3.51) *holds.*

Proof. For this, we break the set $B_k(N)$ into two disjoint pieces,

$$B_k^1(N) = \Big\{t_i > \frac{k+1}{N}\ \ \forall\, 1\le i\le k \text{ and } t-s-\sum_{i=1}^k t_i > \frac{2}{N}\Big\} \quad\text{and}\quad B_k^2(N) = B_k(N)\setminus B_k^1(N).$$

We use the dominated convergence theorem to show that the integral over $B_k^1(N)$ converges to the desired limit, and we argue separately that the integral over $B_k^2(N)$ goes to 0. First, observe that, Lebesgue a.e.,

$$\mathbf{1}_{B_k^1(N)}\Bigg(\prod_{i=1}^k\sqrt{\frac{N}{\lfloor t_iN\rfloor\vee 1}}\Bigg)\sqrt{\frac{N}{(\lfloor tN\rfloor - \lfloor sN\rfloor - \sum_{i=1}^k\lfloor t_iN\rfloor)\vee 1}} \to \mathbf{1}_{B_k}\Bigg(\prod_{i=1}^k\frac{1}{\sqrt{t_i}}\Bigg)\frac{1}{\sqrt{t-s-\sum_{i=1}^k t_i}}. \tag{3.54}$$

Observe that, since $x-1\le\lfloor x\rfloor\le x$,

$$\frac{\lfloor t_iN\rfloor\vee 1}{N} \ge t_i - \frac{1}{N}\quad\text{and}\quad \frac{\lfloor tN\rfloor - \lfloor sN\rfloor - \sum_{i=1}^k\lfloor t_iN\rfloor}{N} \ge t-s-\sum_{i=1}^k t_i - \frac{1}{N},$$

and thus, for N large,

$$t_i > \frac{k+1}{N} \Longrightarrow t_i > \frac{2}{N} \Longrightarrow t_i - \frac{1}{N} > \frac{t_i}{2} \Longrightarrow \sqrt{\frac{\lfloor t_iN\rfloor\vee 1}{N}} \le \sqrt{\frac{2}{t_i}} \tag{3.55}$$

and

$$t-s-\sum_{i=1}^k t_i > \frac{2}{N} \Longrightarrow \sqrt{\frac{N}{(\lfloor tN\rfloor - \lfloor sN\rfloor - \sum_{i=1}^k\lfloor t_iN\rfloor)\vee 1}} \le \sqrt{\frac{2}{t-s-\sum_{i=1}^k t_i}}. \tag{3.56}$$

Therefore,

$$\mathbf{1}_{B_k^1(N)}\Bigg(\prod_{i=1}^k\sqrt{\frac{N}{\lfloor t_iN\rfloor\vee 1}}\Bigg)\sqrt{\frac{N}{(\lfloor tN\rfloor - \lfloor sN\rfloor - \sum_{i=1}^k\lfloor t_iN\rfloor)\vee 1}} \le \mathbf{1}_{B_k}\Bigg(\prod_{i=1}^k\sqrt{\frac{2}{t_i}}\Bigg)\sqrt{\frac{2}{t-s-\sum_{i=1}^k t_i}},$$

and the right-hand side is integrable over $\mathbb{R}^k$ (it is a constant multiple of the Dirichlet density). The dominated convergence theorem now implies the convergence of integrals of the functions in (3.54).

We turn to showing the integral over $B_k^2(N)$ converges to 0. Observe first that on the set $B_k(N)$,

$$t-s-\sum_{i=1}^k t_i \ge \frac{\lfloor tN\rfloor - \lfloor sN\rfloor - \sum_{i=1}^k\lfloor t_iN\rfloor}{N} - \frac{k+1}{N} \ge -\frac{k+1}{N}. \tag{3.57}$$

From the first inequality of (3.57), we observe that, for all N, sufficiently large (depending on t,s),

$$t_i \le \frac{k+1}{N}\ \ \forall\ 1\le i\le k \Longrightarrow t-s-\sum_{i=1}^k t_i \ge \frac{\lfloor tN\rfloor - \lfloor sN\rfloor}{N} - \frac{k(k+1)}{N} - \frac{k+1}{N} > \frac{2}{N}. \tag{3.58}$$

Next, we break up the set $B_k^2(N)$ into $2^{k+1}-2$ disjoint sets determined by whether $t_i \le \frac{k+1}{N}$ for $1 \le i \le k$ and by whether $t - s - \sum_{i=1}^k t_i \le \frac{2}{N}$. The minus 2 comes because $B_k^1(N)$ is one of these possible sets, and (3.58) eliminates another possibility. Enumerate these sets as $\{B_k^{2,j}\}_{1\le j\le 2^{k+1}-2}$. We show that the integral over each $B_k^{2,j}$ converges to 0. We do this by considering four separate cases for $B_k^{2,j}$. To avoid messy calculations, we use the shorthand notation

$$I_k^j(N) := \int_{\mathbb{R}^k} \mathbf{1}_{B_k^{2,j}(N)} \left(\prod_{i=1}^k dt_i \sqrt{\frac{N}{\lfloor t_i N\rfloor \vee 1}}\right) \sqrt{\frac{N}{(\lfloor tN\rfloor - \lfloor sN\rfloor - \sum_{i=1}^k \lfloor t_i\rfloor)\vee 1}}.$$

Case 1: 2 or more of the t_i for $1 \le i \le k$ satisfy $t_i \le \frac{k+1}{N}$: Without loss of generality, we say that, for some $\ell \ge 2$, $t_i \le \frac{k+1}{N}$ for $1 \le i \le \ell$, and $t_i > \frac{k+1}{N}$ for $\ell+1 \le i \le k$. For $\ell+1 \le i \le k$, we use the bound in (3.55). We also make use of the following bounds which hold in general:

$$\sqrt{\frac{N}{\lfloor t_i N\rfloor \vee 1}} \le \sqrt{N} \quad \text{and} \quad \sqrt{\frac{N}{(\lfloor tN\rfloor - \lfloor sN\rfloor - \sum_{i=1}^k \lfloor t_i\rfloor)\vee 1}} \le \sqrt{N}. \tag{3.59}$$

Observe also that on $B_k(N)$, for $1 \le i \le k$, and all N sufficiently large,

$$0 < t_i \le \frac{\lfloor tN\rfloor - \lfloor sN\rfloor + 1}{N} \le t - s + 1.$$

Then,

$$I_k^j(N) \le \sqrt{N}\left(\int_0^{(k+1)/N} \sqrt{N}\,dt\right)^{\ell} \left(\int_0^{t-s+1} \frac{2}{\sqrt{u}}\,du\right)^{k-\ell} \le C(k,\ell) N^{-(\ell-1)/2} \to 0.$$

Case 2: Exactly one of the t_i for $1 \le i \le k$ satisfies $t_i \le \frac{k+1}{N}$ and $t - s - \sum_{i=1}^k t_i > \frac{2}{N}$: Without loss of generality, we will say $t_1 \le \frac{k+1}{N}$. We start similarly to the last case, but instead use the bound (3.56) for the last term. In the following, the constant $C > 0$ depends on $t-s$ and k and may change from line to line.

$$\begin{aligned}
I_k^j(N) &\le \sqrt{N}\int_{\mathbb{R}^k} dt_1 \left(\prod_{i=2}^k dt_i \sqrt{\frac{2}{t_i}}\right) \sqrt{\frac{2}{t-s-\sum_{i=1}^k t_i}}\mathbf{1}\Big(0 < t_1 < \frac{k+1}{N},\ t_i > 0, 2 \le i \le k,\ \sum_{i=1}^k t_i < t-s\Big) \\
&\le C\sqrt{N}\int_{\mathbb{R}^k} dt_1 \left(\prod_{i=2}^k dt_i \sqrt{\frac{1}{t_i}}\right) \sqrt{\frac{1}{1-\sum_{i=1}^k t_i}}\mathbf{1}\Big(0 < t_1 < \frac{k+1}{(t-s)N},\ t_i > 0, 2 \le i \le k,\ \sum_{i=1}^k t_i < 1\Big) \\
&\le C\sqrt{N} P\Big(0 < X_1 < \frac{k+1}{(t-s)N}\Big).
\end{aligned}$$

where P is the distribution of a random vector $(X_1,\dots,X_{k+1})$ that is distributed according to the Dirichlet distribution with parameter vector $(1, \frac{1}{2},\dots,\frac{1}{2})$. The next step follows from X_1 having a Beta distribution with parameters $(1, \frac{k}{2})$. Thus, for constants $C_1, C_2 > 0$ changing from term to term,

$$\sqrt{N}P\Big(0 < X_1 < \frac{k+1}{(t-s)N}\Big) = C_1\sqrt{N}\int_0^{C_2/N} (1-t)^{k/2-1}\,dt = C_1\sqrt{N}(1-(1-C_2/N)^{k/2}) \le CN^{-1/2}.$$

Case 3: $t_i > \frac{k+1}{N}$ for $1 \le i \le k$ and $t - s - \sum_{i=1}^k t_i \le \frac{2}{N}$: We use (3.55) and (3.59) to get the estimate

$$I_k^j(N) \le C\sqrt{N}\int_{\mathbb{R}^k} \mathbf{1}_{B_k^{2,j}(N)} \prod_{i=1}^k dt_i \sqrt{\frac{1}{t_i}}$$

for a constant C depending on k. Next, consider the following change of variable:

$$\widetilde{t}_1 = \frac{k+1}{N} + t - s - \sum_{i=1}^k t_i, \quad \widetilde{t}_i = t_i,\ \ 2 \le i \le k. \tag{3.60}$$

On the set $B_k^{2,j}$, the assumption $t - s - \sum_{i=1}^k t_i \le \frac{2}{N}$ and (3.57) imply that $0 \le \widetilde{t}_1 \le \frac{k+3}{N}$. Furthermore,

$$t - s - \sum_{i=1}^k \widetilde{t}_i = t_1 - \frac{k+1}{N} > 0.$$

In summary, the transformed vector lies in the set

$$\widetilde{B}_k^{2,j}(N) := \Big\{0 < \widetilde{t}_1 \le \frac{k+3}{N}, \quad \widetilde{t}_i > 0, 2 \le i \le k, \quad \sum_{i=1}^k \widetilde{t}_i < t - s\Big\}$$

Putting this all together,

$$\begin{aligned} I_k^j(N) &\le C\sqrt{N}\int_{\mathbb{R}^k} \mathbf{1}_{\widetilde{B}_k^{2,j}}\, d\widetilde{t}_1 \left(\prod_{i=2}^k d\widetilde{t}_i \frac{1}{\sqrt{\widetilde{t}_i}}\right)\sqrt{\frac{1}{t-s-\sum_{i=1}^k \widetilde{t}_i + \frac{k+1}{N}}} \\ &\le C\sqrt{N}\int_{\mathbb{R}^k} \mathbf{1}_{\widetilde{B}_k^{2,j}}\, d\widetilde{t}_1 \left(\prod_{i=2}^k d\widetilde{t}_i \frac{1}{\sqrt{\widetilde{t}_i}}\right)\sqrt{\frac{1}{t-s-\sum_{i=1}^k \widetilde{t}_i}}. \end{aligned}$$

The asymptotics of the integral can now be reduced to the computation of a beta probability, just as in the previous case.

Case 4: Exactly one of the t_i for $1 \le i \le k$ satisfies $t_i \le \frac{k+1}{N}$ and $t - s - \sum_{i=1}^k t_i \le \frac{2}{N}$. Without loss of generality, we will say that $t_2 \le \frac{k+1}{N}$. Then, using the bounds (3.59) for $i = 1, 2$ and the last factor, then (3.55) for $3 \le i \le k$,

$$I_k^j(N) \le CN^{3/2}\int_{\mathbb{R}^k} \mathbf{1}_{B_k^{j,2}(N)} dt_1 dt_2 \prod_{i=3}^k \frac{dt_i}{\sqrt{t_i}}.$$

Making the same change of variables (3.60) as in the previous case,

$$I_k^j(N) \le CN^{3/2}\int_0^{(k+3)/N} dt_1 \int_0^{(k+1)/N} dt_2 \left(\int_0^{t-s+1} \frac{1}{\sqrt{u}}\, du\right)^{k-2} \le CN^{-1/2} \to 0,$$

where the last $k-2$ integrals may be taken from 0 to $t-s+1$ for sufficiently large N by the same reasoning as in Case 1. This concludes all cases. ☐

4. Proofs of the main theorems

4.1. **Characterization and regularity of the Busemann process.** We now turn to proving Theorem 1.4. To do this, we prove invariance of the KPZH$_\beta$ for the SHE. Then, we use the uniqueness result from [JRS22] (recorded as Theorem 3.3) to show that KPZH$_\beta$ describes the Busemann process. Corollary 2.12 gives the existence of discontinuities. We first prove an intermediate invariance result for the O'Connell-Yor polymer.

Proposition 4.1. *Let $\beta > 0$, and let $F_\beta = \{F_\beta^\lambda\}_{\lambda\in\mathbb{R}}$ be the* KPZH$_\beta$. *Let $B_0, B_1, \dots$ be a sequence of i.i.d. Brownian motions, independent of F_β and defining the partition function* (3.8) *for $m, n \ge 0$. Then, for $0 < \lambda_1 < \cdots < \lambda_k$, we have this distributional equality on $C(\mathbb{R})^k$:*

$$\left\{\frac{Z_\beta^{\mathrm{sd}}(n,\bullet\,|\,e^{\beta F_\beta^{\lambda_i}})}{Z_\beta^{\mathrm{sd}}(n,0\,|\,e^{\beta F_\beta^{\lambda_i}})}\right\}_{1\le i\le k} \overset{d}{=} \{\exp\bigl(\beta F_\beta^{\lambda_i}(\bullet)\bigr)\}_{1\le i\le k}. \tag{4.1}$$

Proof. We prove this by induction. For $n = 0$, $\lambda > 0$, and $y \in \mathbb{R}$,

$$Z_\beta^{\mathrm{sd}}(0,y\,|\,e^{\beta F_\beta^\lambda}) = \int_{-\infty}^y e^{\beta F_\beta^\lambda(x)} Z_\beta(0,y\,|\,0,x)\,dx = \int_{-\infty}^y \exp\bigl(\beta(F_\beta^\lambda(x) + B_0(x,y))\bigr)\,dx,$$

and therefore, for $0 < \lambda_1 < \cdots < \lambda_k$,

$$\begin{aligned} \left\{\frac{Z_\beta^{\mathrm{sd}}(0,y\,|\,e^{\beta F_\beta^{\lambda_i}})}{Z_\beta^{\mathrm{sd}}(0,0\,|\,e^{\beta F_\beta^{\lambda_i}})} : y\in\mathbb{R}\right\}_{1\le i\le k} &= \left\{\frac{e^{\beta B_0(y)}\int_{-\infty}^y \exp\bigl(\beta(F_\beta^{\lambda_i}(x) - B_0(x))\bigr)\,dx}{\int_{-\infty}^0 \exp\bigl(\beta(F_\beta^{\lambda_i}(x) - B_0(x))\bigr)\,dx} : y\in\mathbb{R}\right\}_{1\le i\le k} \\ &\overset{(2.9)}{=} \bigl\{\exp\bigl(\beta D_\beta(B_0, F_\beta^{\lambda_i})(y)\bigr)\bigr\}_{1\le i\le k}. \end{aligned}$$

By Theorem 2.8, this has the same distribution as $\{\exp(\beta F_\beta^{\lambda_i})\}_{1\le i\le k}$.

Now, assume the invariance (4.1) holds for some $n \ge 0$. Then,

$$\begin{aligned} Z_\beta^{\mathrm{sd}}(n+1,y\,|\exp(\beta F_\beta^\lambda)) &= \int_{-\infty}^y e^{\beta F_\beta^\lambda(x)} Z_\beta^{\mathrm{sd}}(n+1,y\,|\,0,x)\,dx \\ &\overset{(3.9)}{=} \int_{-\infty}^y \int_x^y \exp\bigl(\beta(F_\beta^\lambda(x) + B_{n+1}(w,y))\bigr) Z_\beta^{\mathrm{sd}}(n,w|0,x)\,dw\,dx \\ &= \int_{-\infty}^y \int_{-\infty}^w \exp\bigl(\beta(F_\beta^\lambda(x) + B_{n+1}(w,y))\bigr) Z_\beta^{\mathrm{sd}}(n,w|0,x)\,dx\,dw \\ &\overset{(3.10)}{=} \int_{-\infty}^y \exp\bigl(\beta B_{n+1}(w,y)\bigr) Z_\beta^{\mathrm{sd}}(n,w\,|\exp(\beta F_\beta^\lambda))\,dw. \end{aligned}$$

Then,

$$\left\{\frac{Z^{\rm sd}_\beta(n+1,y\,|\,e^{\beta F^{\lambda_i}_\beta})}{Z^{\rm sd}_\beta(n+1,0\,|\,e^{\beta F^{\lambda_i}_\beta})}:y\in\mathbb{R}\right\}_{1\le i\le k}=\left\{\frac{e^{\beta B_{n+1}(y)}\displaystyle\int_{-\infty}^{y}\frac{Z^{\rm sd}_\beta(n,w\,|\,e^{\beta F^{\lambda_i}_\beta})}{Z^{\rm sd}_\beta(n,0\,|\,e^{\beta F^{\lambda_i}_\beta})}e^{-\beta B_{n+1}(w)}\,dw}{\displaystyle\int_{-\infty}^{0}\frac{Z^{\rm sd}_\beta(n,w\,|\,e^{\beta F^{\lambda_i}_\beta})}{Z^{\rm sd}_\beta(n,0\,|\,e^{\beta F^{\lambda_i}_\beta})}e^{-\beta B_{n+1}(w)}\,dw}:y\in\mathbb{R}\right\}_{1\le i\le k}$$

$$\overset{d}{=}\left\{\frac{e^{\beta B_{n+1}(y))}\int_{-\infty}^{y}e^{\beta F^{\lambda_i}_\beta(w)}e^{-\beta B_{n+1}(w)}\,dw}{\int_{-\infty}^{0}e^{\beta F^{\lambda_i}_\beta(w)}e^{-\beta B_{n+1}(w)}\,dw}:y\in\mathbb{R}\right\}_{1\le i\le k}\overset{(2.11)}{=}\{e^{\beta D(B_{n+1},F^{\lambda_i}_\beta)}\}_{1\le i\le k}\overset{d}{=}\{e^{\beta F^{\lambda_i}_\beta}\}_{1\le i\le k}.$$

The first distributional equality is the induction assumption and the second one Theorem 2.8. $\square$

Let the fundamental solution Z_β of SHE be defined as in (3.1), and recall the definition with initial data (3.2).

Theorem 4.2. *Let $\beta>0$, and let F_β be the* KPZH$_\beta$*, defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ and independent of the SHE Green's function Z_β. Let $t>0$ and real $\lambda_1<\cdots<\lambda_k$. Then,*

$$\left\{\frac{Z_\beta(t,\bullet\,|\,0,e^{\beta F^{\lambda_i}_\beta})}{Z_\beta(t,0\,|\,0,e^{\beta F^{\lambda_i}_\beta})}\right\}_{1\le i\le k}\overset{d}{=}\{\exp(\beta F^{\lambda_i}_\beta(\bullet))\}_{1\le i\le k}. \tag{4.2}$$

Proof. For $N\in\mathbb{N}$ and $1\le i\le k$, set $\mu_i^N=(\lambda_i+\frac{\beta}{2})N^{-1/4}+\beta^{-1}N^{1/4}$ and $\beta_N=N^{-1/4}\beta$. Let N be large enough so that $\mu_i^N>0$ for $i\in\{1,\dots,k\}$. By (4.1), for every $n\ge1$,

$$\left\{\frac{Z^{\rm sd}_{\beta_N}(n,\bullet\,|\exp(\beta_N F^{\mu_i^N}_{\beta_N}))}{Z^{\rm sd}_{\beta_N}(n,0\,|\exp(\beta_N F^{\mu_i^N}_{\beta_N}))}\right\}_{1\le i\le k}\overset{d}{=}\{\exp(\beta_N F^{\mu_i^N}_{\beta_N})\}_{1\le i\le k}.$$

Then, we have that

$$\begin{aligned}&\left\{\frac{\exp(-(\sqrt{N}+\beta^2/2)y)\int_{-\infty}^{tN+y\sqrt{N}}\exp(\beta_N F^{\mu_i^N}_{\beta_N}(x))Z^{\rm sd}_{\beta_N}(tN,tN+y\sqrt{N}\,|\,0,x)\,dx}{\int_{-\infty}^{tN}\exp(\beta_N F^{\mu_i^N}_{\beta_N}(x))Z^{\rm sd}_{\beta_N}(tN,tN\,|\,0,x)\,dx}:y\in\mathbb{R}\right\}_{1\le i\le k}\\&\overset{d}{=}\Big\{\exp\big(\beta_N F^{\mu_i^N}_{\beta_N}(tN,tN+y\sqrt{N})-(\sqrt{N}+\beta^2/2)y\big):y\in\mathbb{R}\Big\}_{1\le i\le k}\\&\overset{d}{=}\big\{\exp\big(\beta_N F^{\mu_i^N}_{\beta_N}(y\sqrt{N})-(\sqrt{N}+\beta^2/2)y\big):y\in\mathbb{R}\big\}_{1\le i\le k}\\&\overset{d}{=}\big\{\exp\big(\beta F^{\lambda_i}_\beta(y)\big):y\in\mathbb{R}\big\}_{1\le i\le k}.\end{aligned}\tag{4.3}$$

where the second equality follows from shift invariance (Theorem 2.10(i)), and the third equality follows from the scaling relations of Theorem 2.10(ii). For $t>0$ and $x,y\in\mathbb{R}$, set

$$\psi_N(t,x,y)=\sqrt{N}\exp\Big(-\big(N+\frac{\beta^2\sqrt{N}}{2}\big)t-\big(\sqrt{N}+\frac{\beta^2}{2}\big)(y-x)\Big)$$

so that, for our choice of β_N, $\psi_N(t,x,y)=\psi_N(0,t,x,y;\beta_N)$, where the latter was defined in (3.36). By a change of variable from x to $\sqrt{N}x$, the first line of (4.3) is equal to

$$\left\{\frac{\sqrt{N}\exp(-(\sqrt{N}+\beta^2/2)y)\int_{-\infty}^{t\sqrt{N}+y}\exp(\beta_N F^{\mu_i^N}_{\beta_N}(x\sqrt{N}))Z^{\rm sd}_{\beta_N}(tN,tN+y\sqrt{N}|0,x\sqrt{N})\,dx}{\sqrt{N}\int_{-\infty}^{t\sqrt{N}}\exp(\beta_N F^{\mu_i^N}_{\beta_N}(x\sqrt{N}))Z^{\rm sd}_{\beta_N}(tN,tN|0,x\sqrt{N})\,dx}:y\in\mathbb{R}\right\}_{1\le i\le k}$$

$$=\left\{\frac{\int_{-\infty}^{t\sqrt{N}+y}e^{\beta_N F^{\mu_i^N}_{\beta_N}(x\sqrt{N})-(\sqrt{N}+\beta^2/2)x}\psi_N(t,x,y)Z^{\rm sd}_{\beta_N}(tN,tN+y\sqrt{N}\,|\,0,x\sqrt{N})\,dx}{\int_{-\infty}^{t\sqrt{N}}e^{\beta_N F^{\mu_i^N}_{\beta_N}(x\sqrt{N})-(\sqrt{N}+\beta^2/2)x}\psi_N(t,x,0)Z^{\rm sd}_{\beta_N}(tN,tN\,|\,0,x\sqrt{N})\,dx}:y\in\mathbb{R}\right\}_{1\le i\le k}$$

$$\overset{d}{=}\left\{\frac{\int_{-\infty}^{t\sqrt{N}+y}\exp(\beta F^{\lambda_i}_\beta(x))\psi_N(t,x,y)Z^{\rm sd}_{\beta_N}(tN,tN+y\sqrt{N}|0,x\sqrt{N})\,dx}{\int_{-\infty}^{t\sqrt{N}}\exp(\beta F^{\lambda_i}_\beta(x))\psi_N(t,x,0)Z^{\rm sd}_{\beta_N}(tN,tN|0,x\sqrt{N})\,dx}:y\in\mathbb{R}\right\}_{1\le i\le k}.$$

where the distributional equality follows from the scaling of Theorem 2.10(ii), similarly as is done in (4.3). Lemma 3.10 implies that the above converges to, in the sense of finite dimensional distributions on $C(\mathbb{R},\mathbb{R}^k)$, to

$$\left\{\frac{\int_{\mathbb{R}}\exp(\beta F_\beta^{\lambda_i}(x))Z_\beta(t,y|0,x)\,dx}{\int_{\mathbb{R}}\exp(\beta F_\beta^{\lambda_i}(x))Z_\beta(t,0|0,x)\,dx}: y\in\mathbb{R}\right\}_{1\le i\le k} = \left\{\frac{Z_\beta(t,y|0,e^{\beta F_\beta^{\lambda_i}})}{Z_\beta(t,0|0,e^{\beta F_\beta^{\lambda_i}})}: y\in\mathbb{R}\right\}_{1\le i\le k}. \tag{4.4}$$

In the application of the Lemma, $f(x)=e^{\beta F_\beta^{\lambda_i}(x)}$ for which the condition $\mathbb{E}[f(x)]\le e^{\alpha|x|}$ follows immediately. Tightness follows because the distribution of the process does not depend on N by (4.3). Then, by comparing (4.3) and (4.4), for each $t\ge 0$,

$$\left\{\frac{Z_\beta(t,\bullet|0,e^{\beta F_\beta^{\lambda_i}})}{Z_\beta(t,0|0,e^{\beta F_\beta^{\lambda_i}})}\right\}_{1\le i\le k} \overset{d}{=} \{\exp(\beta F_\beta^{\lambda_i})\}_{1\le i\le k}. \qquad \square$$

Corollary 4.3. *Let $\beta>0$. Then, the following distributional equality holds as processes in $D(\mathbb{R},C(\mathbb{R}))$:*

$$\{b_\beta^{(\beta\lambda)+}(0,0,0,\bullet)\}_{\lambda\in\mathbb{R}} \overset{d}{=} \{\beta F_\beta^\lambda\}_{\lambda\in\mathbb{R}}.$$

Proof. The invariance of Theorem 4.2 and the uniqueness of Theorem 3.3 establish that for $\lambda_1<\cdots<\lambda_k$,

$$\{b_\beta^{(\beta\lambda_i)+}(0,0,0,\bullet)\}_{1\le i\le k} \overset{d}{=} \{\beta F_\beta^{\lambda_i}\}_{1\le i\le k}.$$

The choice of factors of β comes by comparing drifts, using Proposition 2.9(i) and Theorem 3.2(i). The equality of processes in the path space $D(\mathbb{R},C(\mathbb{R}))$ follows by the uniqueness of Proposition 2.9(ii). $\square$

Proof of Theorem 1.1. The description of the measures used in the theorem comes from Lemma 2.1 and the definition of the finite-dimensional marginals of the KPZH$_\beta$ in Proposition 2.9(ii). Uniqueness follows directly from Corollary 4.3 and the uniqueness in Theorem 3.3. In handling the factor of β, we recall that we define solutions to the KPZ equation as in (1.8) as

$$h_{Z_\beta}(t,y|s,f)=\frac{1}{\beta}\log\int_{\mathbb{R}} e^{\beta f(x)}Z_\beta(t,y|s,x)\,dx. \qquad \square$$

Proof of Theorem 1.3. By Corollary 2.12, the KPZH$_\beta$ is not almost surely continuous. Corollary 4.3 gives the equality of the Busemann process and the KPZH$_\beta$. By Theorem 3.2(v), the set of discontinuities Λ_β is countable and dense in $\mathbb{R}$ with probability 1. The presence of the discontinuities for the process $\lambda\mapsto F_\beta^\lambda(x,y)$ is Theorem 3.2(vii) (originally proved in [JRS22]). This completes the proof. $\square$

Proof of Theorem 1.4. This is a direct consequence of Corollary 4.3 and Theorem 1.3. $\square$

4.2. Limits as $\beta\nearrow\infty$ and $\beta\searrow 0$.

Proof of Theorem 1.5. We first prove the limit as $\beta\nearrow\infty$: Proposition 2.9(ii) implies that, for $\lambda_1<\cdots<\lambda_k$, $(F_\beta^{\lambda_1},\dots,F_\beta^{\lambda_k})\sim\mu_\beta^{\lambda_1,\dots,\lambda_k}$. By Lemma 2.1, we can describe this distribution as $(F_\beta^{\lambda_1},\dots,F_\beta^{\lambda_k})\overset{d}{=}(\eta_\beta^1,\dots,\eta_\beta^k)$, where for independent Brownian motions $Y^1,\dots,Y^k$ with drifts $\lambda_1,\dots,\lambda_k$, $\eta_\beta^1=Y^1$, and for $2\le n\le k$,

$$\begin{aligned}\eta_\beta^n(y)=Y^1(y)&+\beta^{-1}\log\int_{-\infty<x_{n-1}<\cdots<x_1<y}\prod_{i=1}^{n-1}\exp\bigl[\beta(Y^{i+1}(x_i)-Y^i(x_i))\bigr]\,dx_i\\ &-\beta^{-1}\log\int_{-\infty<x_{n-1}<\cdots<x_1<0}\prod_{i=1}^{n-1}\exp\bigl[\beta(Y^{i+1}(x_i)-Y^i(x_i))\bigr]\,dx_i.\end{aligned}$$

By the convergence of the L^β norm as $\beta\nearrow\infty$, the zero-temperature limit $\beta\to\infty$ converts the polymer free energy into last-passage percolation. Therefore, on a single event of full probability, simultaneously for each $y\in\mathbb{R}$ and $n\in\{2,\dots,k\}$,

$$\begin{aligned}\lim_{\beta\nearrow\infty}\eta_\beta^n(y)=Y^1(y)&+\sup_{-\infty<x_{n-1}\le\cdots\le x_1\le y}\left\{\sum_{i=1}^{n-1}(Y^{i+1}(x_i)-Y^i(x_i))\right\}\\ &-\sup_{-\infty<x_{n-1}\le\cdots\le x_1\le 0}\left\{\sum_{i=1}^{n-1}(Y^{i+1}(x_i)-Y^i(x_i))\right\}.\end{aligned}$$

Lemma B.3 and Definition B.1 imply that, in the sense of finite-dimensional distributions on $C(\mathbb{R}^k,\mathbb{R})$,

$$(\eta_\beta^1(2\,\bullet),\dots,\eta_\beta^k(2\,\bullet))\overset{\beta\nearrow\infty}{\Longrightarrow}(G^{\lambda_1},\dots,G^{\lambda_k}).$$

Tightness holds because each component in the prelimit is a Brownian motion with a fixed drift.

For each β, the process in Item (ii) has the same distribution as the process in Item (i) by the scaling relations of Theorem 2.10(ii).

Now, we prove the convergence as $\beta \searrow 0$. By Theorem 1.2, as processes in $y > 0$,

$$F_\beta^{\lambda_2}(y) - F_\beta^{\lambda_1}(y) \overset{d}{=} \beta^{-1}\log(1 + X_{\lambda,\beta}Y_{\lambda,\beta}(y)) = \log\left(1 + \frac{\beta^{-1}X_{\lambda,\beta}Y_{\lambda,\beta}}{\beta^{-1}}\right)^{\beta^{-1}}, \tag{4.5}$$

where $\lambda := \lambda_2 - \lambda_1$, $X_{\lambda,\beta}$ has the Gamma distribution with shape $\lambda\beta^{-1}$ and rate β^{-2}, and

$$Y_{\lambda,\beta}(y) = \int_0^y \exp(\sqrt{2}\beta B(x) + \lambda\beta x)\,dx \tag{4.6}$$

where B is a standard Brownian motion. For fixed $\lambda, y > 0$, couple the $Y_{\lambda,\beta}(y)$ together with a single Brownian motion B using (4.6). Note that for $\beta < 1$,

$$\int_0^y \exp(\sqrt{2}\beta B(x) + \lambda\beta x)\,dx \le \int_0^y \exp(\sqrt{2}|B(x)| + \lambda x)\,dx,$$

and the right-hand side is finite almost surely. By dominated convergence, $Y_{\lambda,\beta}(y)$ converges almost surely to y as $\beta \searrow 0$. Next, the random variable $X_{\lambda,\beta}/\beta$ has mean λ and variance $\lambda\beta$, so for any $\varepsilon > 0$, by Chebyshev's inequality,

$$\lim_{\beta\searrow 0}\mathbb{P}\Big(|\beta^{-1}X_{\lambda,\beta} - \lambda| > \varepsilon\Big) = 0. \tag{4.7}$$

Hence, there exists a coupling of copies of $X_{\lambda,\beta}$ (which we may keep independent of $Y_{\lambda,\beta}(y)$) so that $X_{\lambda,\beta} \to \lambda$ almost surely as $\beta \searrow 0$. In the product space, using (4.5), $\beta^{-1}\log(1 + X_{\lambda,\beta}Y_{\lambda,\beta}(y))$ converges almost surely to λy. Therefore, for each $\varepsilon > 0$ and $y > 0$,

$$\limsup_{\beta\searrow 0}\mathbb{P}(|F_\beta^{\lambda_2}(y) - F_\beta^{\lambda_1}(y) - (\lambda_2 - \lambda_1)y| > \varepsilon) = 0.$$

The result also holds for $y < 0$ by Theorem 2.10(ii) because $\{F_\beta^\lambda(y)\}_{\lambda\in\mathbb{R}} = \{-F_\beta^\lambda(y,0)\}_{\lambda\in\mathbb{R}} \overset{d}{=} \{-F_\beta^\lambda(0,-y)\}_{\lambda\in\mathbb{R}}$. Now, let $\lambda_1 < \ldots < \lambda_k$ and $\{y_{i,j} : 2 \le i \le k, 1 \le j \le J_i\}$ be a finite collection of points in $\mathbb{R}$. By a simple union bound, for each $\varepsilon > 0$, we have

$$\limsup_{\beta\searrow 0}\mathbb{P}\Big(\sup_{2\le i\le k, 1\le j\le J_i}|F_\beta^{\lambda_i}(y_{i,j}) - F_\beta^{\lambda_1}(y_{i,j}) - (\lambda_i - \lambda_1)y_{i,j}| > \varepsilon\Big) = 0.$$

Since the marginal distribution of $F_\beta^{\lambda_1}$ does not change as $\beta \searrow 0$ (a Brownian motion with drift λ_1), it follows by Slutsky's Theorem that, in the sense of finite-dimensional distributions,

$$(F_\beta^{\lambda_1}(2\bullet), \ldots, F_\beta^{\lambda_k}(2\bullet)) \overset{\beta\searrow 0}{\Longrightarrow} (B(2\bullet) + 2\lambda_1\bullet, B(2\bullet) + 2\lambda_2\bullet, \ldots, B(2\bullet) + 2\lambda_k\bullet),$$

where B is a standard Brownian motion. Convergence on $C(\mathbb{R}^k)$ follows because the marginal distribution of each component on the left-hand side is a Brownian motion with drift λ_i and therefore is tight. $\square$

Proof of Corollary 1.6. Given Theorem 1.5, we follow a similar procedure to the proof of [Wu23, Corollary 1.9]. The only needed change is that the joint distribution of the initial data changes with T. Let $\mathcal{H}_\beta = \log Z_\beta$. Recalling the definition (1.8) of solutions to the KPZ equation, we observe that

$$
\begin{aligned}
&2^{1/3}T^{-1/3}\left[\beta h_{Z_\beta}\Big(\frac{Tt}{\beta^4}, \frac{2^{1/3}T^{2/3}y}{\beta^2}; \frac{Ts}{\beta^4}, F_\beta^{\alpha+2^{1/3}T^{-1/3}\lambda_i}(\bullet) - \alpha\bullet\Big) + \frac{T(t-s)}{24} - \frac{2}{3}\log(\sqrt{2}T)\right] \\
&= 2^{1/3}T^{-1/3}\log\int_{\mathbb{R}}\frac{1}{2^{1/3}T^{2/3}}\exp\left(\beta F_\beta^{\alpha+\beta 2^{1/3}T^{-1/3}\lambda_i}(x) - \beta\alpha x + \mathcal{H}_\beta\Big(\frac{Tt}{\beta^4}, \frac{2^{1/3}T^{2/3}y}{\beta^2}\Big|\frac{Ts}{\beta^4}, x\Big) + \frac{T(t-s)}{24}\right)dx \\
&= 2^{1/3}T^{-1/3}\log\int_{\mathbb{R}}\beta^{-2}\exp\left(\beta F_\beta^{\alpha+\beta 2^{1/3}T^{-1/3}\lambda_i}\Big(\frac{2^{1/3}T^{2/3}}{\beta^2}x\Big) - \frac{2^{1/3}T^{2/3}\alpha x}{\beta}\right. \\
&\qquad\qquad \left. + \mathcal{H}_\beta\Big(\frac{Tt}{\beta^4}, \frac{2^{1/3}T^{2/3}y}{\beta^2}\Big|\frac{Ts}{\beta^4}, \frac{2^{1/3}T^{2/3}x}{\beta^2}\Big) + \frac{T(t-s)}{24}\right)dx \\
&\overset{d}{=} 2^{1/3}T^{-1/3}\log\int_{\mathbb{R}}\exp\left(\beta F_\beta^{\alpha+\beta 2^{1/3}T^{-1/3}\lambda_i}\Big(\frac{2^{1/3}T^{2/3}}{\beta^2}x\Big) - \frac{2^{1/3}T^{2/3}\alpha x}{\beta}\right. \\
&\qquad\qquad \left. + \mathcal{H}_1\Big(Tt, 2^{1/3}T^{2/3}y\Big|Ts, 2^{1/3}T^{2/3}x\Big) + \frac{T(t-s)}{24}\right)dx
\end{aligned}
$$

$$
\begin{aligned}
&= 2^{1/3}T^{-1/3}\log\int_{\mathbb{R}}\exp\biggl(2^{-1/3}T^{1/3}\biggl[\beta 2^{1/3}T^{-1/3}F_\beta^{\alpha+\beta 2^{1/3}T^{-1/3}\lambda_i}\Bigl(\frac{2^{1/3}T^{2/3}}{\beta^2}x\Bigr)-\frac{2^{2/3}T^{1/3}\alpha x}{\beta} \\
&\qquad\qquad + 2^{1/3}T^{-1/3}\mathcal{H}_1\Bigl(Tt,2^{1/3}T^{2/3}y\Bigm|Ts,2^{1/3}T^{2/3}x\Bigr)+\frac{2^{1/3}T^{2/3}(t-s)}{24}\biggr]\biggr)\,dx \\
&= 2^{1/3}T^{-1/3}\log\int_{\mathbb{R}}\exp\Bigl(2^{-1/3}T^{1/3}[F_T^i(x)+\mathcal{H}^T(t,y\,|\,s,x)]\Bigr)\,dx
\end{aligned}
$$

where the distributional equality is theorem 3.1(iii), and we define

$$
\begin{aligned}
F_i^T(x) &= \beta 2^{1/3}T^{-1/3}F_\beta^{\alpha+\beta 2^{1/3}T^{-1/3}\lambda_i}\Bigl(\frac{2^{1/3}T^{2/3}}{\beta^2}x\Bigr)-\frac{2^{2/3}T^{1/3}\alpha x}{\beta} \\
\mathcal{H}^T(t,y|s,x) &= 2^{1/3}T^{-1/3}\mathcal{H}_1\Bigl(Tt,2^{1/3}T^{2/3}y\Bigm|Ts,2^{1/3}T^{2/3}x\Bigr)+\frac{2^{1/3}T^{2/3}(t-s)}{24} \\
h_i^T(t,y) &= 2^{1/3}T^{-1/3}\log\int_{\mathbb{R}}\exp\Bigl(2^{-1/3}T^{1/3}[F_T^i(x)+\mathcal{H}^T(t,y\,|\,s,x)]\Bigr)\,dx.
\end{aligned}
$$

Note that $\{F_i^T\}_{1\le i\le k}$ and $\mathcal{H}^T := \{\mathcal{H}^T(t,y|s,x) : t > s, x, y \in \mathbb{R}\}$ are independent by assumption. We observe that F_T^i is a Brownian motion with diffusion $\sqrt{2}$ and drift $2\lambda_i$; hence its law does not depend on T. By [Wu23, Theorem 1.6], $\mathcal{H}^T$ converges to $\mathcal{L}$ in $C(\mathbb{R}^4_\uparrow,\mathbb{R})$. By [QS23, Vir20], for each i, $h_i^T := \{h_i^T(t,x;F_i^T) : t > s, x \in \mathbb{R}\}$ converges in distribution on $C(\mathbb{R}_{>s},\mathbb{R})$ to the KPZ fixed point $h_{\mathcal{L}}(t,y;s,G^{\lambda_i}) := \sup_{x\in\mathbb{R}}\{G^{\lambda_i}(x)+\mathcal{L}(x,s;y,t)\}$. Hence, this sequence is tight in $C(\mathbb{R}_{>s}\times\mathbb{R},\mathbb{R}^k)$. All together, the sequence

$$
\bigl(\{F_T^i\}_{1\le i\le k},\mathcal{H}^T,\{h_i^T\}_{1\le i\le k}\bigr) \tag{4.8}
$$

is tight on $C(\mathbb{R},\mathbb{R}^k)\times C(\mathbb{R}^4_\uparrow,\mathbb{R})\times C(\mathbb{R}_{>s}\times\mathbb{R},\mathbb{R})$. Let

$$
(\{G^{\lambda_i}\}_{1\le i\le k},\mathcal{L},\{g_i\}_{1\le i\le k}) \tag{4.9}
$$

be a subsequential limit. We may write the first component $\{G^{\lambda_i}\}_{1\le i\le k}$ and the second as $\mathcal{L}$ because we know the SH and DL are, respectively, the law of the limits of the first and second component. by the weak convergence of h_i^T, we also know that, marginally, for each i, $g_i \overset{d}{=} h_{\mathcal{L}}(\bullet,\bullet;s,G^{\lambda_i})$. By Skorokhod representation ([Dud89, Thm. 11.7.2], [EK86, Thm. 3.1.8]), there exists a coupling of (4.8) and (4.9) where, as $T\to\infty$, convergence holds in the sense of uniform convergence on compact sets. Now, we follow the procedure of [Wu23]. We observe that for fixed $t > s$ and $y\in\mathbb{R}$, with probability one,

$$
\begin{aligned}
h_{\mathcal{L}}(t,y;G^{\lambda_i}) &= \sup_{x\in\mathbb{R}}\{G^{\lambda_i}(x)+\mathcal{L}(x,0;y,t)\} \\
&= \lim_{M\to\infty}\sup_{|x|\le M}\{G^{\lambda_i}(x)+\mathcal{L}(x,0;y,t)\} \\
&= \lim_{M\to\infty}\lim_{T\to\infty}2^{1/3}T^{-1/3}\log\int_{-M}^{M}\exp\Bigl(2^{-1/3}T^{1/3}[F_T^i(x)+\mathcal{H}^T(t,y\,|\,s,x)]\Bigr)\,dx \\
&\le \lim_{M\to\infty}\lim_{T\to\infty}2^{1/3}T^{-1/3}\log\int_{\mathbb{R}}\exp\Bigl(2^{-1/3}T^{1/3}[F_T^i(x)+\mathcal{H}^T(t,y\,|\,s,x)]\Bigr)\,dx = g_i(t,y).
\end{aligned}
$$

Hence, since we already established $h_{\mathcal{L}}(t,y;G^{\lambda_i}) \overset{d}{=} g_i(t,y)$, there exists an event of probability one on which, for $1\le i\le k$, all $(t,y)\in\mathbb{Q}_{>s}\times\mathbb{Q}$, $h_{\mathcal{L}}(t,y;G^{\lambda_i}) \overset{d}{=} g_i(t,y)$. Equality on $\mathbb{R}_{>s}\times\mathbb{R}$ follows on this full probability event by continuity.

We next turn to the convergence of the Busemann process. It suffices to show that, for each $r\in\mathbb{R}$, the following distributional convergence holds, in the sense of uniform convergence on compact sets.

$$
\begin{aligned}
&\biggl\{2^{1/3}T^{-1/3}\Bigl[b_\beta^{\beta 2^{1/3}T^{-1/3}\lambda_i}\Bigl(\frac{Ts}{\beta^4},\frac{2^{1/3}T^{2/3}x}{\beta^2};\frac{Tt}{\beta^4},\frac{2^{1/3}T^{2/3}y}{\beta^2}\Bigr)+\frac{T(t-s)}{24}\Bigr] : (x,s;y,t)\in\mathbb{R}^4_r\biggr\} \\
&\qquad \overset{T\to\infty}{\Longrightarrow} \{\mathcal{B}^{\lambda_i}(y,-t;x,-s) : (x,s;y,t)\in\mathbb{R}^4_r\}_{1\le i\le k},
\end{aligned}
$$

where we use the shorthand notation $\mathbb{R}^4_r = \mathbb{R}\times\mathbb{R}_{>r}\times\mathbb{R}\times\mathbb{R}_{>r}$. By the dynamic programming principle and additivity of the Busemann process (Theorem 3.2(viii),(x)) as well as the relation between Busemann process and the KPZH (Corollary 4.3), for $s,t>r$,

$$\begin{aligned} b_\beta^{(\beta\lambda)+}(s,x,t,y) &= b_\beta^{(\beta\lambda)+}(r,0,t,y) - b_\beta^{(\beta\lambda)+}(r,0,s,x) \\ &= \log \int_{\mathbb{R}} e^{b_\beta^{(\beta\lambda)+}(r,0,r,z)} Z_\beta(t,y|r,z)\,dz - \log \int_{\mathbb{R}} e^{b_\beta^{(\beta\lambda)+}(r,0,r,z)} Z_\beta(s,x|r,z)\,dz \\ &\overset{d}{=} \beta h_{Z_\beta}(t,y|r,F_\beta^\lambda) - \beta h_{Z_\beta}(s,x;|r,F_\beta^\lambda), \end{aligned} \tag{4.10}$$

where the distributional equality holds as processes in $\lambda \times (x,s;y,t) \in \mathbb{R}\times\mathbb{R}^4_r$.

Similarly, by the additivity and evolution of the Busemann process from Theorem B.4, along with the distributional equalities $\mathcal{L}(x,s;y,t) \overset{d}{=} \mathcal{L}(y,-t;x,-s)$ (Lemma B.5) and the distributional equality between Busemann functions and the SH (Theorem B.4(v)), for $s,t>r$,

$$\begin{aligned} \mathcal{B}^{\lambda+}(y,-t;x,-s) &= \mathcal{B}^{\lambda+}(y,-t;0,-r) - \mathcal{B}^{\lambda+}(x,-s;0,-r) \\ &= \sup_{z\in\mathbb{R}}\{\mathcal{L}(y,-t;z,-r) + \mathcal{B}^{\lambda+}(z,-r;0,-r)\} \\ &\qquad - \sup_{z\in\mathbb{R}}\{\mathcal{L}(x,-s;z,-r) + \mathcal{B}^{\lambda+}(z,-r;0,-r)\} \\ &\overset{d}{=} \sup_{z\in\mathbb{R}}\{G^\lambda(z) + \mathcal{L}(z,r;y,t)\} - \sup_{z\in\mathbb{R}}\{G^\lambda(z) + \mathcal{L}(z,r;x,s)\} \\ &= h_{\mathcal{L}}(t,y|r,G^\lambda) - h_{\mathcal{L}}(s,x|r,G^\lambda), \end{aligned} \tag{4.11}$$

where, again, the distributional equality holds as processes in $\lambda \times (x,s;y,t) \in \mathbb{R}\times\mathbb{R}^4_r$. Here, we have also used the independence of the Busemann process at time $-r$ and the DL for times less than $-r$ (Theorem B.4(iv)). Comparing (4.10) to (4.11) and using the first part of the theorem in the $\alpha = 0$ case, we get

$$\begin{aligned} &\left\{2^{1/3}T^{-1/3}\Big[b_\beta^{\beta 2^{1/3}T^{-1/3}\lambda_i}\Big(\frac{Ts}{\beta^4}, \frac{2^{1/3}T^{2/3}x}{\beta^2}, \frac{Tt}{\beta^4}, \frac{2^{1/3}T^{2/3}y}{\beta^2}\Big) + \frac{T(t-s)}{24}\Big] : (x,s;y,t)\in\mathbb{R}^4_r\right\}_{1\le i\le k} \\ &\overset{d}{=} \Bigg\{2^{1/3}T^{-1/3}\Big[h_{Z_\beta}\Big(\frac{Tt}{\beta^4}, \frac{2^{1/3}T^{2/3x}}{\beta^2} | \frac{Tr}{\beta^4}, \beta F_\beta^{2^{1/3}T^{-1/3}\lambda_i}\Big) + \frac{T(t-r)}{24} - \frac{2}{3}\log(\sqrt{2}T) \\ &\qquad - h_{Z_\beta}\Big(\frac{Ts}{\beta^4}, \frac{2^{1/3}T^{2/3y}}{\beta^2} | \frac{Tr}{\beta^4}, \beta F_\beta^{2^{1/3}T^{-1/3}\lambda_i}\Big) - \frac{T(s-r)}{24} + \frac{2}{3}\log(\sqrt{2}T)\Big] : (x,s;y,t)\in\mathbb{R}^4_r\Bigg\}_{1\le i\le k} \\ &\overset{T\to\infty}{\Longrightarrow} \left\{h_{\mathcal{L}}(t,y|r,G^{\lambda_i}) - h_{\mathcal{L}}(s,x|r,G^{\lambda_i}) : (x,s;y,t)\in\mathbb{R}^4_r\right\}_{1\le i\le k} \\ &\overset{d}{=} \{\mathcal{B}^{\lambda_i}(y,-t;x,-s) : (x,s;y,t)\in\mathbb{R}^4_r\}_{1\le i\le k}. \end{aligned}$$

□

Appendix A. Queues and the O'Connell-Yor polymer

Recall the transformations $Q_\beta, D_\beta, R_\beta$ defined in (2.2). We state one of the main theorems from [OY01]. Their theorem is stated for $\beta=1$. The statement for general $\beta>0$ follows from Lemma 2.6 since

$$D_\beta(B,Y) = T_\beta D_1(T^2_{\beta^{-1}}(B,Y)), \qquad \text{and} \qquad R_\beta(B,Y) = T_\beta R_1(T^2_{\beta^{-1}}(B,Y)).$$

Theorem A.1. [OY01, Theorem 5] *Let B and Y be independent two-sided Brownian motions with drift so that the drift of Y is strictly larger than the drift of B. Let $\beta>0$, and let $Q_\beta, D_\beta, R_\beta$ be defined as in* (2.2)*. Then, $(R_\beta(B,Y), D_\beta(B,Y)) \overset{d}{=} (B,Y)$, and for each $y\in\mathbb{R}$, $\{D_\beta(Y,B)(x), R_\beta(Y,B)(x) : -\infty < x \le y\}$ is independent of $\{Q_\beta(x) : x \ge y\}$.*

Lemma A.2. *Let B be a standard two-sided Brownian motion, and let $\beta,\lambda>0$. Then,*

$$\Big(\int_{-\infty}^0 e^{\sqrt{2}\beta B(x) + \lambda\beta x}\Big)^{-1} \sim \text{Gamma}(\lambda\beta^{-1}, \beta^{-2}).$$

Proof. Theorem 4.4 in [Duf90] (also Equation 1.8.4(1) on page 612 4 of [BS02]) states that for $\gamma,\sigma>0$,

$$\Big(\int_0^\infty e^{-\sigma B(x) - \gamma x}\,dx\Big)^{-1} \sim \text{Gamma}(2\gamma\sigma^{-2}, 2\sigma^{-2})$$

(as a caution, we note that [Duf90] parameterizes the Gamma distribution by shape and scale, so the result is stated there with scale $\sigma^2/2$). Observe that

$$\int_{-\infty}^{0} e^{\sqrt{2}\beta B(x)+\lambda\beta x} = \int_0^\infty e^{\sqrt{2}\beta B(-x)-\lambda\beta x} \overset{d}{=} \int_0^\infty e^{-\sqrt{2}\beta B(x)-\lambda\beta x},$$

and so the result follows upon substituting $\sigma = \sqrt{2}\beta$ and $\gamma = \lambda\beta$. $\square$

The remainder of this appendix contains the omitted proofs from Section 2.1, along with some additional lemmas. These follow similarly as for zero temperature in [FS20] and [SS23a], with the exception of the first lemma below.

Lemma A.3. *Let $\beta > 0$ and let Y^2, Y^1, B^1 be so that the following are well-defined. Set $B^2 = R_\beta(Y^1, B^1)$. Then,*

$$D_\beta(D_\beta(B^1, Y^1), D_\beta(B^2, Y^2)) = D_\beta^{(3)}(B^1, Y^1, Y^2). \tag{A.1}$$

Proof. Written out fully, the statement reads

$$D_\beta(D_\beta(B^1, Y^1), D_\beta(R_\beta(B^1, Y^1), Y^2)) = D_\beta^{(3)}(B^1, Y^1, Y^2).$$

By applying Lemma 2.6 to each of the operations D and R, this is equivalent to

$$T_\beta D_1(D_1(T_{\beta^{-1}}B^1, T_{\beta^{-1}}Y^1), D_1(R_1(T_{\beta^{-1}}B^1, T_{\beta^{-1}}Y^1), T_{\beta^{-1}}Y^2)) = T_\beta D_1^{(3)}(T_{\beta^{-1}}B^1, T_{\beta^{-1}}Y^1, T_{\beta^{-1}}Y^2).$$

Hence, it suffices to prove the $\beta = 1$ case. For this, we drop the subscript in the mappings D, R, and $D^{(3)}$. We make repeated use of Lemma 2.1. By (2.9),

$$\exp\bigl(D^{(3)}(B^1, Y^1, Y^2)(y)\bigr) = \frac{e^{B_1(y)} \int_{-\infty<w<x<y} \exp\bigl(Y^1(x) - B^1(x) + Y^2(w) - Y^1(w)\bigr)\, dx\, dw}{\int_{-\infty<w<x<0} \exp\bigl(Y^1(x) - B^1(x) + Y^2(w) - Y^1(w)\bigr)\, dx\, dw}. \tag{A.2}$$

We turn to the left-hand side of (A.1) for $\beta = 1$. We repeatedly use the $n = 2$ case of Lemma 2.1 as follows:

$$\begin{aligned} \exp\bigl[D(D(B^1, Y^1), D(B^2, Y^2))(y)\bigr] &= \frac{e^{D(B^1,Y^1)(y)} \int_{-\infty}^{y} e^{D(B^2,Y^2)(x)-D(B^1,Y^1)(x)}\, dx}{\int_{-\infty}^{0} e^{D(B^2,Y^2)(x)-D(B^1,Y^1)(x)}\, dx} \\ &= \frac{e^{B^1(y)} \int_{-\infty}^{y} e^{Y^1(x)-B^1(x)}\, dx}{\int_{-\infty}^{0} e^{Y^1(x)-B^1(x)}\, dx} \cdot \frac{I_y}{I_0}, \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} I_y &= \int_{-\infty}^{y} e^{B^2(x)-B^1(x)} \frac{\int_{-\infty}^{x} e^{Y^2(w)-B^2(w)}\, dw \int_{-\infty}^{0} e^{Y^1(w)-B^1(w)}\, dw}{\int_{-\infty}^{0} e^{Y^2(w)-B^2(w)}\, dw \int_{-\infty}^{x} e^{Y^1(w)-B^1(w)}\, dw}\, dx \\ &= \int_{-\infty}^{y} e^{Y^1(x)-B^1(x)} \frac{\bigl(\int_{-\infty}^{0} e^{Y^1(w)-B^1(w)}\, dw\bigr)^2}{\bigl(\int_{-\infty}^{x} e^{Y^1(w)-B^1(w)}\, dw\bigr)^2} \frac{\int_{-\infty}^{x} e^{Y^2(w)-B^2(w)}\, dw}{\int_{-\infty}^{0} e^{Y^2(w)-B^2(w)}\, dw}\, dx \\ &= \int_{-\infty}^{y} e^{Y^1(x)-B^1(x)} \frac{\bigl(\int_{-\infty}^{0} e^{Y^1(w)-B^1(w)}\, dw\bigr)^2}{\bigl(\int_{-\infty}^{x} e^{Y^1(w)-B^1(w)}\, dw\bigr)^2} \frac{\int_{-\infty}^{x} e^{Y^2(w)-Y^1(w)} \int_{-\infty}^{w} e^{Y^1(z)-B^1(z)}\, dz\, dw}{\int_{-\infty}^{0} e^{Y^2(w)-Y^1(w)} \int_{-\infty}^{w} e^{Y^1(z)-B^1(z)}\, dz\, dw}\, dx. \end{aligned}$$

Therefore, $I_y/I_0 = I'_y/I'_0$, where

$$I'_y = \int_{-\infty}^{y} e^{Y^1(x)-B^1(x)} \frac{\int_{-\infty}^{x}\int_{-\infty}^{w} \exp\bigl(Y^2(w) - Y^1(w) + Y^1(z) - B^1(z)\bigr)\, dz\, dw}{\bigl(\int_{-\infty}^{x} e^{Y^1(w)-B^1(w)}\, dw\bigr)^2}\, dx. \tag{A.4}$$

Comparing (A.2), (A.3), and (A.4), to prove (A.1), it suffices to show that for each $y \in \mathbb{R}$,

$$\begin{aligned} &\int_{-\infty}^{y} e^{Y^1(x)-B^1(x)}\, dx \int_{-\infty}^{y} e^{Y^1(x)-B^1(x)} \frac{\int_{-\infty}^{x}\int_{-\infty}^{w} e^{Y^2(w)-Y^1(w)+Y^1(z)-B^1(z)}\, dz\, dw}{\bigl(\int_{-\infty}^{x} e^{Y^1(w)-B^1(w)}\, dw\bigr)^2}\, dx \\ &= \int_{-\infty<w<x<y} e^{Y^1(x)-B^1(x)+Y^2(w)-Y^1(w)}\, dx\, dw. \end{aligned} \tag{A.5}$$

On the left-hand side of (A.5), integrate by parts in the second dx integral over $(-\infty, y]$ with

$$dv = e^{Y^2(x)-B^1(x)}\Bigl(\int_{-\infty}^{x} e^{Y^1(w)-B^1(w)}\, dw\Bigr)^{-2}, \quad u = \int_{-\infty}^{x}\int_{-\infty}^{w} e^{Y^2(w)-Y^1(w)+Y^1(z)-B^1(z)}\, dz\, dw.$$

Then the left-hand side of (A.5) equals

$$\begin{aligned}
&\int_{-\infty}^{y} e^{Y^1(x)-B^1(x)}\,dx\cdot\Biggl[-\Bigl(\int_{-\infty}^{x} e^{Y^1(w)-B^1(w)}\,dw\Bigr)^{-1}\int_{-\infty}^{x}\int_{-\infty}^{w} e^{Y^2(w)-Y^1(w)+Y^1(z)-B^1(z)}\,dz\,dw\Bigg|_{x=-\infty}^{y}\\
&\qquad\qquad+\int_{-\infty}^{y}\Bigl(\int_{-\infty}^{x} e^{Y^1(x)-B^1(x)}\,dx\Bigr)^{-1}\int_{-\infty}^{x} e^{Y^2(x)-Y^1(x)+Y^1(z)-B^1(z)}\,dz\,dx\Biggr]\\
&=\int_{-\infty}^{y} e^{Y^1(x)-B^1(x)}\,dx\int_{-\infty}^{y} e^{Y^2(x)-Y^1(x)}\,dx-\int_{-\infty}^{y}\int_{-\infty}^{w} e^{Y^2(w)-Y^1(w)+Y^1(z)-B^1(z)}\,dz\,dw\\
&=\int_{(x,w)\in(-\infty,y)^2} e^{Y^1(x)-B^1(x)+Y^2(w)-Y^1(w)}\,dx\,dw\\
&\qquad-\int_{-\infty<x<w<y} e^{Y^1(x)-B^1(x)+Y^2(w)-Y^1(w)}\,dx\,dw.
\end{aligned}$$

One readily sees that the last right-hand side above equals the right-hand side of (A.5). $\square$

Lemma A.4. *Let $n\ge 2$, and let $(B^1,Y^1,\dots,Y^n)$ be such that the following operations are well-defined. Let $\beta>0$. For $2\le j\le n$, define $B^j=R_\beta(B^{j-1},Y^{j-1})$. Then, for $1\le k\le n-1$,*

$$D_\beta^{(n+1)}(B^1,Y^1,\dots,Y^n)=D_\beta^{(k+1)}\bigl(D_\beta(B^1,Y^1),\dots,D_\beta(B^k,Y^k),D_\beta^{(n-k+1)}(B^{k+1},Y^{k+1},\dots,Y^n)\bigr).\tag{A.6}$$

We note that the case $k=n-1$ of Lemma A.4 gives us

$$D_\beta^{(n+1)}(B^1,Y^1,\dots,Y^n)=D_\beta^{(n)}(D_\beta(B^1,Y^1),\dots,D_\beta(B^n,Y^n)).\tag{A.7}$$

Proof of Lemma A.4. Equation (A.1) gives us the statement for $n=2$. Assume, by induction, that the statement is true for some $n-1\ge 2$. We will show the statement is also true for n. We first prove the case $k=1$. Using (A.1) in the second equality below,

$$\begin{aligned}
&D_\beta^{(2)}(D_\beta(B^1,Y^1),D_\beta^{(n)}(B^2,Y^2,\dots,Y^n))\\
&=D_\beta(D_\beta(B^1,Y^1),D_\beta(B^2,D_\beta^{(n-1)}(Y^2,\dots,Y^n)))\\
&=D_\beta(B^1,D_\beta(Y^1,D_\beta^{(n-1)}(Y^2,\dots,Y^n)))\\
&=D_\beta(B^1,D_\beta^{(n)}(Y^1,\dots,Y^n))\\
&=D_\beta^{(n+1)}(B^1,Y^1,\dots,Y^n).
\end{aligned}$$

Now, let $2\le k\le n-1$. Then, by definition of $D^{(k+1)}$ and the induction assumption,

$$\begin{aligned}
&D_\beta^{(k+1)}(D_\beta(B^1,Y^1),\dots,D_\beta(B^k,Y^k),D_\beta^{(n-k+1)}(B^{k+1},Y^{k+1},\dots,Y^n))\\
=&D_\beta(D_\beta(B^1,Y^1),D_\beta^{(k)}(D_\beta(B^2,Y^2),\dots,D_\beta(B^k,Y^k),D_\beta^{(n-k+1)}(B^{k+1},Y^{k+1},,\dots,Y^n)))\\
=&D_\beta(D_\beta(B^1,Y^1),D_\beta^{(n)}(B^2,Y^2,\dots,Y^n))=D_\beta^{(2)}(D_\beta(B^1,Y^1),D_\beta^{(n)}(B^2,Y^2,\dots,Y^n)).
\end{aligned}$$

The lemma now follows from the $k=1$ case. $\square$

The *multiline process* is a discrete-time Markov chain on the state space $\mathcal{Y}_n^{(a,\infty)}$ of (2.5), where $a\in\mathbb{R}$. The analogous process is defined in a discrete setting for particle systems in [FM07], for lattice last-passage percolation in [FS20], and in zero temperature BLPP in [SS23b]. Starting at time $m-1$ in state $\mathbf{Y}_{m-1}=\mathbf{Y}=(Y^1,Y^2,\dots,Y^n)\in\mathcal{Y}_n^{(a,\infty)}$the time m state is given as

$$\mathbf{Y}_m=\overline{\mathbf{Y}}=(\overline{Y}^1,\overline{Y}^2,\dots,\overline{Y}^n)\in\mathcal{Y}_n$$

is defined as follows. Let $B\in C_{\text{pin}}(\mathbb{R})$ satisfy

$$\lim_{x\to-\infty}x^{-1}B(x)=a.$$

First, set $B^1=B$, and $\overline{Y}^1=D_\beta(Y^1,B^1)$. Then, iteratively for $i=2,3,\dots,n$:

$$B^i=R_\beta(B^{i-1},Y^{i-1}),\qquad\text{and}\qquad\overline{Y}^i=D_\beta(B^i,Y^i).\tag{A.8}$$

Lemma A.5. *The mapping* (A.8) *is well-defined on the state space* $\mathcal{Y}_n^{(a,\infty)}$.

Proof. This follows from Lemma 2.2: By induction, each B^i satisfies

$$\lim_{x\to-\infty}\frac{B^i(x)}{x}=a.$$

Therefore, since $\mathbf{Y}\in\mathcal{Y}_n^{(a,\infty)}$, for $1\le i\le n$,

$$\limsup_{x\to\infty}\frac{Y^i(x)-B^i(x)}{x}>0. \qquad \square$$

Theorem A.6. *At each step of the evolution of the multiline process, take the driving function B to be an independent standard, two-sided Brownian motion with drift $a\in\mathbb{R}$. For each $\bar\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n_{>0}$ with $a<\lambda_1<\cdots<\lambda_n$, the measure $\nu^{\bar\lambda}$ on $\mathcal{Y}_n^{(a,\infty)}$ is invariant for the multiline process* (A.8)

Proof. Assuming that $\mathbf{Y}=(Y^1,\dots,Y^n)\in\mathcal{Y}_n^{\mathbb{R}_{>0}}$ are i.i.d. Brownian motions with drifts $\lambda_1,\dots,\lambda_n$, we must show that the same is true for $\overline{Y}^1,\dots,\overline{Y}^n$. By Theorem A.1, $\overline{Y}^1=D_\beta(B^1,Y^1)$ is a two-sided Brownian motion with drift λ_1, independent of $B^2=R_\beta(B^1,Y^1)$, which is a two-sided Brownian motion with drift a. Hence, the random paths $\overline{Y}^1,B^2,Y^2,\dots,Y^n$ are mutually independent. We iterate this process as follows: Assume, for some $2\le k\le n-1$, that the random paths $\overline{Y}^1,\dots,\overline{Y}^{k-1},B^k,Y^k,\dots,Y^n$ are mutually independent, where for $1\le i\le k-1$, $\overline{Y}^i$ is a Brownian motion with drift λ_i. Applying Theorem A.1 again, $\overline{Y}^k=D_\beta(B^k,Y^k)$ is a two-sided Brownian motion with drift λ_k, independent of $B^{k+1}=R_\beta(B^k,Y^k)$, which is a two-sided Brownian motion with zero drift. Since $(\overline{Y}^k,B^{k+1})$ is a function of (B^k,Y^k), it follows that $\overline{Y}^1,\dots,\overline{Y}^k,B^{k+1},Y^{k+1},\dots,Z^n$ are mutually independent, completing the proof. $\square$

Proof of Lemma 2.7 (Consistency of the measures). It suffices to show that if $(\eta^1,\dots,\eta^n)$ has distribution $\mu_\beta^{\lambda_1,\dots,\lambda_n}$, then

$$(\eta^1,\dots,\eta^{i-1},\eta^{i+1},\dots,\eta^n)\sim\mu_\beta^{\lambda_1,\dots,\lambda_{i-1},\lambda_{i+1},\dots,\lambda_n}.$$

Let $\mathbf{Y}=(Y^1,\dots,Y^n)\sim\nu^{\bar\lambda}$ and $\eta=\mathbf{D}_\beta^{(n)}(\mathbf{Y})$ so $\eta=(\eta^1,\dots,\eta^n)\sim\mu_\beta^{\bar\lambda}$.

For $i=n$, the statement is immediate from the definition of the map $\mathbf{D}_\beta^{(n)}$. Next, we show the case $i=1$. For $2\le j\le n$, by (A.7), we may write

$$D_\beta^{(j)}(Y^1,\dots,Y^j)=D_\beta^{(j-1)}(D_\beta(\widetilde{Y}^1,Y^2),D_\beta(\widetilde{Y}^2,Y^3),\dots,D_\beta(\widetilde{Y}^{j-1},Y^j)),$$

where $\widetilde{Y}^1=Y^1$, and for $i>1$, $\widetilde{Y}^i=R(\widetilde{Y}^{i-1},Y^i)$. Then $(\eta^2,\dots,\eta^n)=\mathbf{D}_\beta^{(n-1)}(\hat{Y}^2,\dots,\hat{Y}^n)$, where $\hat{Y}^i=D_\beta(\widetilde{Y}^{i-1},Y^i)$ for $2\le i\le n$. By Theorem A.6, $\hat{Y}^2,\dots,\hat{Y}^n$ are independent, completing the proof in the case $i=1$. Using the definition of $D_\beta^{(j)}$ (2.3), for $i<j\le n$,

$$D_\beta^{(j)}(Y^1,\dots,Y^j)=D_\beta(D_\beta(Y^1,D_\beta(Y_2,\cdots D_\beta(Y^{i-1},D_\beta^{(j-i+1)}(Y^i,\dots,Y^j))\cdots). \tag{A.9}$$

We apply (A.7), just as in the $i=1$ case, to obtain

$$\begin{aligned}&D_\beta^{(j-i+1)}(Y^i,\dots,Y^j)\\&=D_\beta^{(j-i)}(D_\beta(\widetilde{Y}^i,Y^{i+1}),\dots,D_\beta(\widetilde{Y}^{j-1},Y^j))=D_\beta^{(j-i)}(\hat{Y}^{i+1},\dots,\hat{Y}^j),\end{aligned} \tag{A.10}$$

where, $\widetilde{Y}^i=Y^i$, and for $j>i$, $\widetilde{Y}^j=R(\widetilde{Y}^{j-1},Y^j)$. For $j>i$, we define $\hat{Y}^j=D_\beta(\widetilde{Y}^{j-1},Y^j)$. Then, by (A.9) and (A.10), when $i<j\le n$,

$$D_\beta^{(j)}(Y^1,\dots,Y^j)=D_\beta^{(j-1)}(Y^1,\dots,Y^{i-1},\hat{Y}^{i+1},\dots,\hat{Y}^j),$$

and thus,

$$(\eta^1,\dots,\eta^{i-1},\eta^{i+1},\dots,\eta^n)=\mathbf{D}_\beta^{(n-1)}(Y^1,\dots,Y^{i-1},\hat{Y}^{i+1},\dots,\hat{Y}^n). \tag{A.11}$$

By Theorem A.6, $\hat{Y}^{i+1},\dots,\hat{Y}^n$ are independent Brownian motions with drifts $\lambda_{i+1},\dots,\lambda_n$. These random paths are functions of $Y^i,\dots,Y^n$,so the paths functions $Y^1,\dots,Y^{i-1},\hat{Y}^{i+1},\dots,\hat{Y}^j$ are also independent. Thus, by (A.11),

$$(\eta^1,\dots,\eta^{i-1},\eta^{i+1},\dots,\eta^n)\sim\mu^{(\lambda_1,\dots,\lambda_{i-1},\lambda_{i+1},\dots,\lambda_n)}. \qquad \square$$

Proof of Theorem 2.8. Let $\mathbf{Y}\sim\nu^{\bar\lambda}$. Let $\boldsymbol{\eta}=\mathbf{D}_\beta^{(n)}(\mathbf{Y})$ so that $\boldsymbol{\eta}\sim\mu_\beta^{\bar\lambda}$. Then, for Brownian motion B, let $\mathcal{S}_\beta^B$ denote the mapping of a single evolution step of $\mathbf{Y}$ according to the multiline process (A.8) and $\mathcal{T}_\beta^B$ denote the mapping of a single evolution step of $\boldsymbol{\eta}$ according to the Markov chain (2.16). By definition of $D_\beta^{(k)}$ and Equation (A.7),

$$\begin{aligned}[\mathcal{T}_\beta^B(\boldsymbol{\eta})]_k =& D_\beta(\boldsymbol{\eta}^k, B) = D_\beta(B^1, D_\beta^{(k)}(Y^1,\dots,Y^k)) = D_\beta^{(k+1)}(B^1, Y^1,\dots,Y^k)\\ =& D_\beta^{(k)}(D_\beta(B^1,Y^1), D_\beta(B^2,Y^2),\dots,D_\beta(B^k,Y^k))\\ =& D_\beta^{(k)}([\mathcal{S}_\beta^B(\mathbf{Y})]_1, [\mathcal{S}_\beta^B(\mathbf{Y})]_2,\dots,[\mathcal{S}_\beta^B(\mathbf{Y})]_k) = [\mathbf{D}_\beta^{(n)}(\mathcal{S}_\beta^B(\mathbf{Y}))]_k.\end{aligned}$$

Therefore, $\mathcal{T}_\beta^B(\boldsymbol{\eta}) = \mathbf{D}_\beta^{(n)}(\mathcal{S}_\beta^B(\mathbf{Z}))$, and because $\boldsymbol{\eta} = \mathbf{D}^{(n)}(\mathbf{Z})$, we have

$$\mathcal{T}^B(\mathcal{D}^{(n)}(\mathbf{Y})) = \mathcal{D}^{(n)}(\mathcal{S}^B(\mathbf{Y})).$$

Theorem A.6 implies that $\mathcal{S}_\beta^B(\mathbf{Y}) \overset{d}{=} \mathbf{Y} \sim \nu^{\bar\lambda}$. Therefore, $\mathcal{T}_\beta^B(\boldsymbol{\eta}) \overset{d}{=} \mathbf{D}_\beta^{(n)}(\mathbf{Y}) \sim \mu^{\bar\lambda}$. □

Appendix B. Stationary horizon and the directed landscape

The stationary horizon (SH) was first introduced by Busani in [Bus21] and was later studied by Busani and the third and fourth authors in [SS23b, BSS22b, BSS22a]. We refer to those articles for a more complete description. Analogously to how the KPZH$_\beta$ describes the jointly invariant measures for the O'Connell-Yor polymer and the KPZ equation, it was proved in [SS23b, BSS22b] that the SH describes the jointly invariant measures for Brownian last-passage percolation and the KPZ fixed point. Jointly invariant measures for the KPZ fixed point are made precise through the coupling with the directed landscape. See [MQR21, DOV22, QS23, Vir20, DV21, Wu23] for more on the KPZ fixed point and directed landscape. We briefly describe the needed definition and facts about the SH, DL, and the KPZ fixed point here.

The directed landscape (DL) is a random continuous function $\mathcal{L}:\mathbb{R}^4_\uparrow \to \mathbb{R}$. By convention, we switch the ordering of space-time coordinates to $\mathcal{L}(x,s;y,t)$ (in contrast to the ordering in $Z_\beta(t,y|s,x)$). Given the DL, we can construct the KPZ fixed point started from time s as

$$h_{\mathcal{L}}(t,y|s,\mathfrak{h}) = \sup_{x\in\mathbb{R}}\{\mathfrak{h}(x) + \mathcal{L}(x,s;y,t)\}, \qquad t>s,\ y\in\mathbb{R}.$$

SH is constructed with the zero-temperature counterparts of the mappings of Section 2.1. We denote these with the same letters but without the β subscript. For functions that satisfy $Y(0)=B(0)=0$ and $\limsup_{x\to-\infty} Y(x)-B(x) = -\infty$, define

$$D(B,Y)(y) = B(y) + \sup_{-\infty<x\le y}\{Y(x)-B(x)\} - \sup_{-\infty<x\le 0}\{Y(x)-B(x)\}. \tag{B.1}$$

As in (2.3), iterate the mapping D as follows:

$$D^{(1)}(Y) = Y, \quad \text{and} \quad D^{(n)}(Y^1,Y^2,\dots,Y^n) = D(Y^1, D^{(n-1)}(Y^2,\dots,Y^n)) \quad \text{for } n\ge 2.$$

A mapping $\mathbf{D}^{(n)}:\mathcal{Y}_n^{\mathbb{R}} \to \mathcal{X}_n^{\mathbb{R}}$ is defined as follows: the image $\boldsymbol{\eta} = (\eta^1,\dots,\eta^n) = \mathbf{D}^{(n)}(\mathbf{Z}) \in \mathcal{X}_n$ is defined for $\mathbf{Y} = (Y^1,\dots,Y^n) \in \mathcal{Y}_n^{\mathbb{R}}$ by

$$\eta^i = D^{(i)}(Y^1,\dots,Y^i) \text{ for } 1\le i\le n.$$

For $\bar\lambda = (\lambda_1 < \cdots < \lambda_n)$, we define the measure $\mu^{\lambda_1,\dots,\lambda_n}$ (again without the β subscript) as

$$\mu^{\bar\lambda} = \nu^{\bar\lambda} \circ (\mathcal{D}^{(n)})^{-1}.$$

Definition B.1. The stationary horizon $\{G_\mu\}_{\mu\in\mathbb{R}}$ is a process with paths in $D(\mathbb{R}, C(\mathbb{R}))$. Its law is characterized as follows: For real numbers $\bar\lambda = (\lambda_1 < \cdots < \lambda_k)$, the k-tuple $(G^{\lambda_1},\dots,G^{\lambda_k}) \in C(\mathbb{R})^k$ has distribution $\mu^{\bar\lambda} \circ (\widetilde{T}_2^k)^{-1}$, where $\widetilde{T}_2$ is the mapping $C(\mathbb{R})^k \to C(\mathbb{R})^k$ defined by

$$\widetilde{T}_2^k(f_1,\dots,f_k)(x) = (f_1(2x),\dots,f_k(2x)).$$

In this definition, we multiply by a factor of 2 so that the marginal distributions are Brownian motions with diffusivity $\sqrt{2}$. This is the correct parameterization for invariance under the KPZ fixed point.

Lemma B.2 ([Bus21], Theorem 1.2; [SS23b], Theorems 3.6(iii), 5.4)**.** *For $c>0$ and $\nu\in\mathbb{R}$,*

$$\{cG_{c(\mu+\nu)}(c^{-2}x) - 2\nu x : x\in\mathbb{R}\}_{\mu\in\mathbb{R}} \overset{d}{=} \{G_\mu(x) : x\in\mathbb{R}\}_{\mu\in\mathbb{R}}.$$

Lemma B.3 ([SS23b], Lemma 7.2, and see Appendix D in [SS23a])**.** *For $\mathbf{Y} = (Y^1,\dots,Y^n)$, define*

$$A_n^{\mathbf{Y}}(x) = \sup_{-\infty<x_{n-1}\le\cdots\le x_1\le x}\Big\{\sum_{i=1}^n Y^i(x_i) - Y^{i-1}(x_i)\Big\}.$$

Then, if $A_n^{\mathbf{Y}}(0)$ is finite, for $n\ge 2$,

$$D^{(n)}(Y^1,Y^2,\dots,Y^n)(x) = Y^1(x) + A_n^{\mathbf{Y}}(x) - A_n^{\mathbf{Y}}(0).$$

The following states properties of the Busemann process for the DL from [BSS22b]. For a single direction λ, these properties were previously established in [RV21].

Theorem B.4. [BSS22b, Theorems 5.1–5.2] *On the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *of the directed landscape* $\mathcal{L}$*, there exists a process*

$$\{\mathcal{B}^{\lambda\square}(p;q) : \lambda \in \mathbb{R},\, \square \in \{-,+\},\, p,q \in \mathbb{R}^2\}$$

satisfying the following properties. All the properties below hold on a single event of probability one, simultaneously for all directions $\lambda \in \mathbb{R}$*, signs* $\square \in \{-,+\}$*, and points* $p,q \in \mathbb{R}^2$*, unless otherwise specified.*

(i) (Continuity) *As an* $\mathbb{R}^4 \to \mathbb{R}$ *function,* $(x,s;y,t) \mapsto \mathcal{B}^{\lambda\square}(x,s;y,t)$ *is continuous.*

(ii) (Additivity) *For all* $p,q,r \in \mathbb{R}^2$*,* $\mathcal{B}^{\lambda\square}(p;q) + \mathcal{B}^{\lambda\square}(q;r) = \mathcal{B}^{\lambda\square}(p;r)$*. In particular,* $\mathcal{B}^{\lambda\square}(p;q) = -\mathcal{B}^{\lambda\square}(q;p)$ *and* $\mathcal{B}^{\lambda\square}(p;p) = 0$.

(iii) (Backwards evolution as the KPZ fixed point) *For all* $x,y \in \mathbb{R}$ *and* $s < t$*,*

$$\mathcal{B}^{\lambda\square}(x,s;y,t) = \sup_{z\in\mathbb{R}}\{\mathcal{L}(x,s;z,t) + \mathcal{B}^{\lambda\square}(z,t;y,t)\}. \tag{B.2}$$

(iv) (Independence) *For each* $T \in \mathbb{R}$*, these processes are independent:*

$$\{\mathcal{B}^{\lambda\square}(x,s;y,t) : \lambda \in \mathbb{R},\, \square \in \{-,+\},\, x,y \in \mathbb{R},\, s,t \geq T\}$$

$$\text{and} \quad \{\mathcal{L}(x,s;y,t) : x,y \in \mathbb{R},\, s < t \leq T\}.$$

(v) (Distribution along a time level) *For each* $t \in \mathbb{R}$*, the following equality in distribution holds between random elements of the Skorokhod space* $D(\mathbb{R}, C(\mathbb{R}))$*:*

$$\{\mathcal{B}^{\lambda+}(\bullet,t;0,t)\}_{\lambda\in\mathbb{R}} \overset{d}{=} \{G_\lambda(\bullet)\}_{\lambda\in\mathbb{R}},$$

where G *is the stationary horizon.*

We also make use of the following symmetry of the directed landscape.

Lemma B.5. [DV21, Proposition 14.1] *The directed landscape satisfies the following symmetry*

$$\{\mathcal{L}(x,s;y,t) : (s,x,t,y) \in \mathbb{R}^4_\uparrow\} \overset{d}{=} \{\mathcal{L}(y,-t;x,-s) : (s,x,t,y) \in \mathbb{R}^4_\uparrow\}.$$

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