Parameter estimation for stochastic differential equations driven by fractional Brownian motion

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Outline of the Talk

- Background and motivation
- Stochastic calculus for the fractional Brownian motion
- Properties of the SDEs
- Conclusion

Background and Motivation

- We consider the SDE

$$dX_t = -f(X_t)\theta dt + \sigma dB_t\,, \quad t \geq 0\,,$$

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- $B_t = \{(B_t^1, \ldots, B_t^d), t \geq 0\}$ is a d-dimensional fBm of Hurst parameter $H \in (0,1)$, which is a zero mean Gaussian process whose components are independent and have the covariance function

$$\mathbb{E}(B_t^i B_s^i) = R_H(t,s) := \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

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for $i = 1, \ldots, d$.
- $\sigma = (\sigma_1, \ldots, \sigma_d) \in \mathbb{R}^{m \times d}$.
- The function $f = (f_{ij}) : \mathbb{R}^m \to \mathbb{R}^{m \times l} \in \mathcal{C}_p^1(\mathbb{R}^m)$. Assume that there is a positive constant L_1 independent of the initial condition $x_0 \in \mathbb{R}^m$, such that the Jacobian matrices $\nabla f_j(x) \in \mathbb{R}^{m \times m}$ satisfy $\sum_{j=1}^{l} \theta_j \nabla f_j \geq L_1 I_m$, where I_m is the $m \times m$ identity matrix.

- Under the above assumptions, f satisfies the one-sided dissipative Lipschitz condition:

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- There exists a constant $C_p>0$ such that

$$\|X_t\|_{L^p(\Omega;\mathbb{R}^m)} \leq C_p\,,$$

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- $X_t \in C^\alpha(\mathbb{R}_+;\mathbb{R}^m)$ for all $\alpha < H$.
- Assume $\theta = (\theta_1,\ldots,\theta_l) \in \mathbb{R}^l$ is an unknown parameter vector.
- Suppose we have a continuous trajectory of the SDE, we are interested in the estimation of the parameter vector θ.

Previous work

- In the linear case, X_t is known as the fOU process. There are many research results.
- Kleptsyna and Le Breton (2002) studied the maximum likelihood estimator (MLE) and prove the strong consistency.
- Brouste and Kleptsyna(2010), Bercu, Courtin and Savy (2011) obtained the central limit theorem.
- Tudor and Viens (2007) also obtain the strong consistency of MLE in linear and nonlinear cases for $H \in (0, 1)$.
- Hu and Nualart (2010) proposed the least squares estimator for $H \in (\frac{1}{2}, 1)$.
- Hu, Nualart, Zhou (2017) obtained the strong consistency and central limit theorem for all $H \in (0, 1)$.

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- We propose the LSE for θ as

$$\begin{aligned} \hat{\theta}_T &= -\left(\int_0^T (f^{tr} f)(X_t) dt\right)^{-1} \int_0^T f^{tr}(X_t) dX_t \\ &= \theta - \left(\int_0^T (f^{tr} f)(X_t) dt\right)^{-1} \int_0^T f^{tr}(X_t) \sigma dB_t \,. \end{aligned}$$

Strong consistency of LSE

Theorem (Hu, Nualart, Z ('18))

Assume that the components of f *belong to* $\mathcal{C}_p^1(\mathbb{R}^m)$ *when* $H \in [\frac{1}{2}, 1)$*, and they belong to* $\mathcal{C}_p^2(\mathbb{R}^m)$ *when* $H \in (\frac{1}{4}, \frac{1}{2})$*.* f *also satisfies the assumptions mentioned above. Then the least squares estimator* $\hat{\theta}_T$ *of the parameter* θ *is strongly consistent,*

$$\hat{\theta}_T \to \theta\,, \text{ a.s. as } T \to \infty.$$

Ingredients for the proof

- Our target is to show

$$\lim_{T\to\infty}\frac{1}{T}|\hat{\theta}_T-\theta| = \lim_{T\to\infty}\left(\int_0^T (f^{tr}f)(X_t)dt\right)^{-1}\int_0^T f^{tr}(X_t)\sigma dB_t = 0\,.$$

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$$\left(\frac{1}{T}\int_0^T (f^{tr} f)(X_t)dt\right)^{-1} \to \left(\mathbb{E}\left((f^{tr} f)(\overline{X})\right)\right)^{-1} \quad \text{a.s.}$$

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- Divergence integrals $Z_{j,T} = \frac{1}{T}\int_0^T f_j^{tr}(X_s)\sigma dB_s$ converge to 0 a.s. for $j = 1, \ldots, l$.

Ergodic property of fBm

- Consider the canonical probability space of fBm $(\Omega, \mathcal{F}, \mathbb{P})$: $\Omega = C_0(\mathbb{R}_+; \mathbb{R}^d)$, $\mathcal{F}$ is the Borel σ-algebra, and $\mathbb{P}$ is the probability measure on $(\Omega, \mathcal{F})$ s.t. the coordinate process $B_t(\omega) = \omega(t)$ is a fBm.

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- The probability measure $\mathbb{P}$ is invariant with respect to the shift operators μ_t, which are defined as

$$\mu_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \;\; t \in \mathbb{R}_+, \omega \in \Omega.$$

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- For any integrable random variable $F : \Omega \to \mathbb{R}$, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(\mu_t(\omega))dt = \mathbb{E}(F).$$

Ergodic property of SDE

Theorem (Garrido-Atienza, Kloeden, Neuenkirch ('09))

Assume the drift function f satisfies the assumptions above (polynomial growth and one-sided Lipschitz). Then, the following results hold:

(i) *There exists a random variable* $\overline{X} : \Omega \to \mathbb{R}^m$ *with* $\mathbb{E}|\overline{X}|^p < \infty$ *for all* $p \geq 1$ *such that*

$$\lim_{t\to\infty} |X_t(\omega) - \overline{X}(\mu_t \omega)| = 0$$

for $\mathbb{P}$*-almost all* $\omega \in \Omega$.

(ii) *For any function* $g \in \mathcal{C}_p^1(\mathbb{R}^m)$*, we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(X_t)dt = \mathbb{E}[g(\overline{X})] \qquad \mathrm{P}\text{-}a.s.$$

This implies that $\left(\frac{1}{T}\int_0^T (f^{tr} f)(X_t)dt\right)^{-1} \to \left(\mathbb{E}\left((f^{tr} f)(\overline{X})\right)\right)^{-1}$ a.s., given that $P(\det(f^{tr} f)(\overline{X}) > 0) > 0$.

Ingredients for the proof (cont.)

The next object is to show that divergence integrals $\frac{1}{T} Z_{j,T} = \frac{1}{T} \int_0^T f_j^{tr}(X_s) \sigma dB_s$ converge to 0 a.s. for $j = 1, \dots, l$,

- We show the sequence $\{n^{-1} Z_{j,n}\} \to 0$ a.s..

$$\sum_{n=1}^{\infty} \mathbb{P}(\left|n^{-1} Z_{j,n}\right| > \epsilon) \leq \sum_{n=1}^{\infty} \epsilon^{-p} \mathbb{E}\left(\left|n^{-1} Z_{j,n}\right|^p\right)$$

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- Next, we consider the limit of the process $\frac{1}{T}Z_{j,T}$. Let the integer k_T defined by $k_T \le T < k_T + 1$.

$$\begin{aligned} \frac{1}{T}|Z_{j,T}| &\le \frac{1}{k_T}\left|\int_0^{k_T} g_j(X_t)dB_t\right| + \frac{1}{T}\left|\int_{k_T}^{T} g_j(X_t)dB_t\right| \\ &\le \frac{1}{k_T}\left|\int_0^{k_T} g_j(X_t)dB_t\right| + \frac{1}{k_T}\sup_{t\in[k_T,k_T+1]}\left|\int_{k_T}^{t} g_j(X_s)dB_s\right| \end{aligned}$$

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- Estimation of $\|Z_{j,n}\|_{L^p(\Omega)}$ and $\| \sup_{t \in [k_T, k_T+1]} \int_{k_T}^{t} g_j(X_s) dB_s\|_{L^p(\Omega)}$.

Stochastic integrals

- The Hilbert space $\mathfrak{H}^d$ is defined as the closure of $\mathcal{E}^d$ endowed with the inner product

$$\langle (\mathbb{1}_{[0,s_1]}, \ldots, \mathbb{1}_{[0,s_d]}), (\mathbb{1}_{[0,t_1]}, \ldots, \mathbb{1}_{[0,t_d]}) \rangle_{\mathfrak{H}^d} = \sum_{i=1}^{d} R_H(s_i, t_i) .$$

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- The mapping $(\mathbb{1}_{[0,t_1]}, \ldots, \mathbb{1}_{[0,t_d]}) \mapsto \sum_{j=1}^{d} B^j_{s_j}$ can be extended to a linear isometry between $\mathfrak{H}^d$ and the Gaussian space $\mathcal{H}_1$ spanned by B.

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- For $F = f(B_{t_1}, \dots, B_{t_n})$, where $f \in C_b^\infty(\mathbb{R}^{d\times n})$, we define the Malliavin derivative as the $\mathfrak{H}^d$-valued random variable given by $DF = (D^1F, \dots, D^dF)$ whose jth component is

$$D^j_s F = \sum_{i=1}^{n} \frac{\partial f}{\partial x^j_i}(B_{t_1}, \dots, B_{t_n}) \mathbb{1}_{[0,t_j]}(s).$$

- The Sobolev space $\mathbb{D}^{p,q}$ is the closure of the space of smooth cylindrical random variables w.r.t the norm $\|\cdot\|_{p,q}$

$$\|F\|_{p,q}^q = \mathbb{E}(|F|^q) + \sum_{i=1}^{p} \mathbb{E}\left[\left(\sum_{j_1,\ldots,j_i=1}^{d} \|D^{j_1,\ldots,j_i}F\|^2_{(\mathfrak{H}^d)^{\otimes i}}\right)^{\frac{q}{2}}\right].$$

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- Let u such that $|\mathbb{E}\langle D^j F, u\rangle_{\mathfrak{H}}| \le c_u \|F\|_{L^2}$, for any $F \in \mathbb{D}^{1,2}$.
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- Define the divergence operator on $\mathfrak{H}^d$ as $\delta(u) = \sum_{j=1}^d \delta^j(u_j)$ for $u = (u_1, \dots, u_d) \in \cap_{j=1}^d \mathrm{Dom}(\delta^j)$.

p-th moment of divergence integrals

The divergence operator δ is continuous from $\mathbb{D}^{1,p}(\mathfrak{H}^d)$ into $L^p(\Omega)$, which means

$$\mathbb{E}(|\delta(u)|^p) \leq C_p \left(\mathbb{E}(\|u\|^p_{\mathfrak{H}^d}) + \mathbb{E}(\|Du\|^p_{\mathfrak{H}^d \otimes \mathfrak{H}^d}) \right) ,$$

for some constant C_p depending on p.

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Lemma

Let $H \in (\frac{1}{2}, 1)$ and let u be an element of $\mathbb{D}^{1,p}(\mathfrak{H}^d)$, $p > 1$. Then u belongs to the domain of the divergence operator δ in $L^p(\Omega)$. Moreover, we have

$$\mathbb{E}(|\delta(u)|^p) \le C_{p,H} \left(\|\mathbb{E}(u)\|^p_{L^{1/H}([0,\infty);\mathbb{R}^d)} + \mathbb{E}\left(\|Du\|^p_{L^{1/H}([0,\infty)^2;\mathbb{R}^{d\times d})} \right) \right) .$$

Proposition

Let $H \in (0, \frac{1}{2})$ and $p \geq 2$. Assume that the $\mathbb{R}^d$-valued stochastic process $\{u_t, t \geq 0\}$ satisfies the regularity Hypothesis (i)-(iv).

(i) $\|u\|_{p,0,\infty} = \sup_{t \geq 0} \|u_t\|_{L^p(\Omega;\mathbb{R}^d)} < \infty$,

(ii) $\|u_t - u_s\|_{L^p(\Omega;\mathbb{R}^d)} \leq K(t-s)^\beta$,

(iii) $\|Du_t\|_{L^p(\Omega;\mathfrak{H}^d \otimes \mathbb{R}^d)} \leq K t^\lambda$,

(iv) $\|Du_t - Du_s\|_{L^p(\Omega;\mathfrak{H}^d \otimes \mathbb{R}^d)} \leq K(t-s)^\beta s^\lambda$.

where the constants $K > 0$, $\beta > \frac{1}{2} - H$ and $\lambda \in (0, H]$. Then for any $T > 0$, the divergence integral $\delta(u\mathbb{1}_{[0,T]})$ is in $L^p(\Omega)$, and

$$\mathbb{E}(|\delta(u\mathbb{1}_{[0,T]})|^p) \leq C T^{pH}(1 + T^{p\lambda})(1 + T^{p\beta}),$$

where the constant C is independent of T.

Maximal inequality for stochastic integrals

Theorem (Hu, Nualart, Z ('18))

- *Let* $H \in (\frac{1}{2}, 1)$ *and* $\frac{1}{p} + \frac{1}{q} = H$ *with* $p > q$*. Suppose that for all* $T > 0$

  (i) $\int_0^T \mathbb{E}(|u_s|^p) ds < \infty$,

  (ii) $\int_0^T \int_0^s \mathbb{E}(|D_t u_s|^p) dt ds < \infty$.

  Then the divergence integral $\int_0^t u_s dB_s$ *is in* $L^p(\Omega)$ *for all* $t \geq 0$ *and for any interval* $[a, b]$*, we have*

$$\mathbb{E}\left(\sup_{t \in [a,b]} \left| \int_a^t u_s dB_s \right|^p \right) \leq C(b-a)^{\frac{p}{q}} \int_a^b \mathbb{E}(|u_s|^p) ds + C(b-a)^{\frac{2p}{q}} \int_a^b \int_a^s \mathbb{E}(|D_t u_s|^p) dt ds,$$

  where the constant C *does not depend on* a, b.

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Theorem (Hu, Nualart, Z ('18))

Let $\{u_t, t \geq 0\}$ be an $\mathbb{R}^d$-valued stochastic process. For the divergence integral $\int_0^t u_s dB_s$, $t \geq 0$, we have the following statements:

- *Let $H \in (\frac{1}{4}, \frac{1}{2})$ and $p > \frac{1}{H}$. Assume that the stochastic process u satisfies the regularity Hypothesis. Then the divergence integral $\int_0^t u_s dB_s$ is in $L^p(\Omega)$ for all $t \geq 0$ and for any $0 \leq a < b$ we have the estimate*

$$\mathbb{E}\left(\sup_{t\in[a,b]}\left|\int_a^t u_s dB_s\right|^p\right) \leq C(b-a)^{pH}(1+(b-a)^{p\beta})(1+b^{p\lambda}),$$

where C is a generic constant that does not depend on a, b.

- Based on factorization method,

$$\begin{aligned} &\mathbb{E}\left(\sup_{t\in[a,b]}\left|\int_a^t u_s dB_s\right|^p\right) \\ &= \left(\frac{\sin(\alpha\pi)}{\pi}\right)^p \mathbb{E}\left(\sup_{t\in[a,b]}\left|\int_a^t \left(\int_s^t (t-r)^{\alpha-1}(r-s)^{-\alpha} dr\right) u_s dB_s\right|^p\right) \\ &\leq C_{\alpha,p}(b-a)^{p\alpha-1}\int_a^b \mathbb{E}(|G_r|^p) dr\,, \end{aligned}$$

where

$$G_r := \int_a^r (r-s)^{-\alpha} u_s dB_s, \qquad r\in[a,b]\,.$$

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- Then apply the previous estimate about the p-th moment of the stochastic integrals. In the case when $H<\frac{1}{2}$, we require the regularity of $(r-s)^{-\alpha}u_s$ and this triggers the restriction of $H>\frac{1}{4}$.

Proposition

(1) *The solution X_t satisfies*

$$\|X_t\|_{L^p(\Omega;\mathbb{R}^m)} \leq C_p\,,$$

and

$$\|X_t - X_s\|_{L^p(\Omega;\mathbb{R}^m)} \leq C_p|t-s|^H$$

for all $t \geq s \geq 0$.

(2) *The Malliavin derivative of the solution X_t satisfies for all* $0 \leq s \leq t$

$$|D_s X_t| \leq |\sigma| e^{-L_1(t-s)}\,, \text{ a.s.}$$

Moreover, if $v \leq u \leq s \leq t$, we have

$$\|D_u X_t - D_v X_t\|_{L^p(\Omega;\mathbb{R}^{m\times d})} \leq Ce^{-L_1(t-u)}(1 \wedge |u-v|)\,,$$

$$\|D_u X_t - D_u X_s\|_{L^p(\Omega;\mathbb{R}^{m\times d})} \leq Ce^{-L_1(s-u)}(1 \wedge |t-s|)\,,$$

$$\|D_u X_t - D_v X_t - (D_u X_s - D_v X_s)\|_{L^p(\Omega;\mathbb{R}^{m\times d})} \leq Ce^{-L_1(s-u)}(1 \wedge |u-v|)(1 \wedge |t-s|)$$

Conclusion

- Convergence of the sequence $n^{-1}Z_{j,n} = \frac{1}{n}\int_0^n f_j^{tr}(X_s)\sigma dB_s$.

$$\sum_{n=1}^{\infty} \mathbb{P}(|n^{-1}Z_{j,n}| > \epsilon) \le \sum_{n=1}^{\infty} \epsilon^{-p}\mathbb{E}\left(|n^{-1}Z_{j,n}|^p\right)$$

$$\mathbb{E}(|Z_{j,n}|^p) \le \begin{cases} Cn^{pH} & \text{when } H \in (\frac{1}{2}, 1) \\ Cn^{p(2H+\lambda)} & \text{when } H \in (\frac{1}{4}, \frac{1}{2}) \end{cases}$$

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- Convergence of the sequence $T^{-1}Z_{j,T}$.

$$\frac{1}{T}|Z_{j,T}| \le \frac{1}{k_T}\left|\int_0^{k_T} g_j(X_t)dB_t\right| + \frac{1}{k_T}\sup_{t\in[k_T,k_T+1]}\left|\int_{k_T}^t g_j(X_s)dB_s\right|.$$

$$\mathbb{E}\left(\sup_{t\in[k_T,k_T+1]}\left|\int_{k_T}^t g_j(X_s)dB_s\right|^p\right) \le \begin{cases} C & \text{when } H \in (\frac{1}{2},1) \\ C(k_T+1)^{p\lambda} & \text{when } H \in (\frac{1}{4},\frac{1}{2}), \end{cases}$$

where $\lambda \in (0,H)$.

THANK YOU!