Parameter estimation for stochastic differential equations driven by fractional Brownian motion

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Outline of the Talk

- Background and motivation
- Stochastic calculus for the fractional Brownian motion
- Properties of the SDEs
- Conclusion
We consider the SDE
\[ dX_t = -f(X_t)\theta \, dt + \sigma \, dB_t, \quad t \geq 0, \]
where \( X_0 = x_0 \in \mathbb{R}^m \) is a given initial condition.
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where \( X_0 = x_0 \in \mathbb{R}^m \) is a given initial condition.

\( B_t = \{(B^1_t, \ldots, B^d_t), t \geq 0\} \) is a \( d \)-dimensional fBm of Hurst parameter \( H \in (0, 1) \), which is a zero mean Gaussian process whose components are independent and have the covariance function
\[ \mathbb{E}(B^i_t B^i_s) = R_H(t, s) := \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}) \]
for \( i = 1, \ldots, d \).
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\[ \sigma = (\sigma_1, \ldots, \sigma_d) \in \mathbb{R}^{m \times d}. \]
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for \( i = 1, \ldots, d \).

- \( \sigma = (\sigma_1, \ldots, \sigma_d) \in \mathbb{R}^{m \times d} \).

- The function \( f = (f_{ij}) : \mathbb{R}^m \to \mathbb{R}^{m \times l} \in C^1_p(\mathbb{R}^m) \). Assume that there is a positive constant \( L_1 \) independent of the initial condition \( x_0 \in \mathbb{R}^m \), such that the Jacobian matrices \( \nabla f_j(x) \in \mathbb{R}^{m \times m} \) satisfy

\[
\sum_{j=1}^l \theta_j \nabla f_j \geq L_1 l_m, \quad \text{where} \quad l_m \text{ is the } m \times m \text{ identity matrix.} \]
Under the above assumptions, \( f \) satisfies the one-sided dissipative Lipschitz condition:

\[
\langle x - y, (f(x) - f(y))\theta \rangle \geq L_1 |x - y|^2, \quad \forall \, x, y \in \mathbb{R}^m.
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The SDE admits a unique solution $X_t$. 

Assume $\theta = (\theta_1, \ldots, \theta_l) \in \mathbb{R}^l$ is an unknown parameter vector. Suppose we have a continuous trajectory of the SDE, we are interested in the estimation of the parameter vector $\theta$. 

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There exists a constant $C_p > 0$ such that

$$\|X_t\|_{L^p(\Omega;\mathbb{R}^m)} \leq C_p,$$

and

$$\|X_t - X_s\|_{L^p(\Omega;\mathbb{R}^m)} \leq C_p |t - s|^H$$

for all $t \geq s \geq 0$. 

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$X_t \in C^\alpha(\mathbb{R}_+; \mathbb{R}^m)$ for all $\alpha < H$.

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There exists a constant $C_\rho > 0$ such that

$$\|X_t\|_{L^p(\Omega;\mathbb{R}^m)} \leq C_\rho,$$

and

$$\|X_t - X_s\|_{L^p(\Omega;\mathbb{R}^m)} \leq C_\rho|t - s|^H$$

for all $t \geq s \geq 0$.

$X_t \in C^\alpha(\mathbb{R}^+; \mathbb{R}^m)$ for all $\alpha < H$.

Assume $\theta = (\theta_1, \ldots, \theta_l) \in \mathbb{R}^l$ is an unknown parameter vector.

Suppose we have a continuous trajectory of the SDE, we are interested in the estimation of the parameter vector $\theta$. 
In the linear case, $X_t$ is known as the fOU process. There are many research results.

Kleptsyna and Le Breton (2002) studied the maximum likelihood estimator (MLE) and prove the strong consistency.

Brouste and Kleptsyna (2010), Bercu, Courtin and Savy (2011) obtained the central limit theorem.

Tudor and Viens (2007) also obtain the strong consistency of MLE in linear and nonlinear cases for $H \in (0, 1)$.

Hu and Nualart (2010) proposed the least squares estimator for $H \in (\frac{1}{2}, 1)$.

Hu, Nualart, Zhou (2017) obtained the strong consistency and central limit theorem for all $H \in (0, 1)$. 
Least squares estimator

- We write $\sigma dB_t = dX_t + f(X_t)\theta dt$. 
Least squares estimator

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- We propose the LSE for $\theta$ as

$$\hat{\theta}_T = \theta - \left( \int_0^T (f^{tr} f)(X_t) dt \right)^{-1} \int_0^T f^{tr}(X_t) dX_t$$

$$= \theta - \left( \int_0^T (f^{tr} f)(X_t) dt \right)^{-1} \int_0^T f^{tr}(X_t) \sigma dB_t.$$
Theorem (Hu, Nualart, Z ('18))

Assume that the components of $f$ belong to $C^1_p(\mathbb{R}^m)$ when $H \in [\frac{1}{2}, 1)$, and they belong to $C^2_p(\mathbb{R}^m)$ when $H \in (\frac{1}{4}, \frac{1}{2})$. $f$ also satisfies the assumptions mentioned above. Then the least squares estimator $\hat{\theta}_T$ of the parameter $\theta$ is strongly consistent,

$$\hat{\theta}_T \rightarrow \theta , \text{ a.s. as } T \rightarrow \infty.$$
Ingredients for the proof

- **Our target is to show**

\[
\lim_{T \to \infty} \frac{1}{T} |\hat{\theta}_T - \theta| = \lim_{T \to \infty} \left( \int_0^T (f^{tr} f)(X_t) dt \right)^{-1} \int_0^T f^{tr} (X_t) \sigma dB_t = 0.
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We will show that
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Divergence integrals \( Z_{j,T} = \frac{1}{T} \int_0^T f^{tr}_j (X_s) \sigma dB_s \) converge to 0 a.s. for \( j = 1, \ldots, l \).
Consider the canonical probability space of fBm \((\Omega, \mathcal{F}, \mathbb{P})\):

\(\Omega = C_0(\mathbb{R}_+; \mathbb{R}^d)\), \(\mathcal{F}\) is the Borel \(\sigma\)-algebra, and \(\mathbb{P}\) is the probability measure on \((\Omega, \mathcal{F})\) s.t. the coordinate process \(B_t(\omega) = \omega(t)\) is a fBm.
Ergodic property of fBm

Consider the canonical probability space of fBm \((\Omega, \mathcal{F}, \mathbb{P})\):
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The probability measure \(\mathbb{P}\) is invariant with respect to the shift operators \(\mu_t\), which are defined as

\[ \mu_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}_+, \omega \in \Omega. \]
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$$\mu_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \; t \in \mathbb{R}^+, \omega \in \Omega.$$ 

For any integrable random variable $F : \Omega \to \mathbb{R}$, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T F(\mu_t(\omega)) dt = \mathbb{E}(F).$$
Theorem (Garrido-Atienza, Kloeden, Neuenkirch ('09))

Assume the drift function $f$ satisfies the assumptions above (polynomial growth and one-sided Lipschitz). Then, the following results hold:

(i) There exists a random variable $\bar{X} : \Omega \to \mathbb{R}^m$ with $\mathbb{E}|\bar{X}|^p < \infty$ for all $p \geq 1$ such that

$$\lim_{t \to \infty} |X_t(\omega) - \bar{X}(\mu_t \omega)| = 0$$

for $\mathbb{P}$-almost all $\omega \in \Omega$.

(ii) For any function $g \in C^1_p(\mathbb{R}^m)$, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T g(X_t) dt = \mathbb{E}[g(\bar{X})] \quad \mathbb{P}\text{-a.s.}$$

This implies that $\left(\frac{1}{T} \int_0^T (f^{tr} f)(X_t) dt\right)^{-1} \to \left(\mathbb{E} ((f^{tr} f)(\bar{X})))\right)^{-1}$ a.s., given that $P(\det(f^{tr} f)(\bar{X}) > 0) > 0$. 

Ergodic property of SDE
The next object is to show that divergence integrals
\[ \frac{1}{T} Z_{j,T} = \frac{1}{T} \int_0^T f_j^{tr}(X_s) \sigma dB_s \]
converge to 0 a.s. for \( j = 1, \ldots, l \),

- We show the sequence \( \{ n^{-1} Z_{j,n} \} \to 0 \) a.s.

\[
\sum_{n=1}^{\infty} \mathbb{P}(|n^{-1} Z_{j,n}| > \epsilon) \leq \sum_{n=1}^{\infty} \epsilon^{-p} \mathbb{E}
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Ingredients for the proof (cont.)

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\]

- Next, we consider the limit of the process \( \frac{1}{T} Z_{j,T} \). Let the integer \( k_T \) defined by \( k_T \leq T < k_T + 1 \).

\[
\frac{1}{T} |Z_{j,T}| \leq \frac{1}{k_T} \left| \int_0^{k_T} g_j(X_t) dB_t \right| + \frac{1}{T} \left| \int_{k_T}^T g_j(X_t) dB_t \right|
\]

\[
\leq \frac{1}{k_T} \left| \int_0^{k_T} g_j(X_t) dB_t \right| + \frac{1}{k_T} \sup_{t \in [k_T, k_T+1]} \left| \int_{k_T}^t g_j(X_s) dB_s \right|
\]
Ingredients for the proof (cont.)

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$$\frac{1}{T} |Z_{j,T}| \leq \frac{1}{k_T} \left| \int_0^{k_T} g_j(X_t) dB_t \right| + \frac{1}{T} \left| \int_{k_T}^T g_j(X_t) dB_t \right|$$

$$\leq \frac{1}{k_T} \left| \int_0^{k_T} g_j(X_t) dB_t \right| + \frac{1}{k_T} \sup_{t \in [k_T, k_T+1]} \left| \int_{k_T}^t g_j(X_s) dB_s \right|$$

- Estimation of $\|Z_{j,n}\|_{L^p(\Omega)}$ and $\| \sup_{t \in [k_T, k_T+1]} \int_{k_T}^t g_j(X_s) dB_s \|_{L^p(\Omega)}$. 
The Hilbert space $\mathcal{H}^d$ is defined as the closure of $\mathcal{E}^d$ endowed with the inner product

$$\langle (1_{[0,s_1]}, \ldots, 1_{[0,s_d]}), (1_{[0,t_1]}, \ldots, 1_{[0,t_d]}) \rangle_{\mathcal{H}^d} = \sum_{i=1}^{d} R_H(s_i, t_i).$$
The Hilbert space $\mathcal{H}^d$ is defined as the closure of $\mathcal{E}^d$ endowed with the inner product

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The mapping $(\mathbb{1}_{[0,t_1]}, \ldots, \mathbb{1}_{[0,t_d]}) \mapsto \sum_{j=1}^{d} B_{s_j}$ can be extended to a linear isometry between $\mathcal{H}^d$ and the Gaussian space $\mathcal{H}_1$ spanned by $B$. 

Stochastic integrals
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The mapping $(1_{[0,t_1]}, \ldots, 1_{[0,t_d]}) \mapsto \sum_{j=1}^{d} B_j$ can be extended to a linear isometry between $\mathcal{H}_d$ and the Gaussian space $\mathcal{H}_1$ spanned by $B$.

For $F = f(B_{t_1}, \ldots, B_{t_n})$, where $f \in C^\infty_b(\mathbb{R}^{d \times n})$, we define the Malliavin derivative as the $\mathcal{H}_d$-valued random variable given by $DF = (D^1 F, \ldots, D^d F)$ whose $j$th component is

$$D^j_s F = \sum_{i=1}^{n} \frac{\partial f}{\partial x^j_i}(B_{t_1}, \ldots, B_{t_n}) 1_{[0,t_j]}(s).$$
The Sobolev space $\mathbb{D}^{p,q}$ is the closure of the space of smooth cylindrical random variables w.r.t the norm $\| \cdot \|_{p,q}$

$$\| F \|_{p,q}^q = \mathbb{E}(|F|^q) + \sum_{i=1}^{p} \mathbb{E} \left[ \sum_{j_1, \ldots, j_i=1}^{d} \left\| D^{j_1, \ldots, j_i} F \right\|_{(\mathcal{H}^d)^\otimes i}^2 \right]^{\frac{q}{2}}.$$
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\| F \|_{p,q}^q = \mathbb{E}(|F|^q) + \sum_{i=1}^{p} \mathbb{E} \left[ \left( \sum_{j_1, \ldots, j_i=1}^{d} \| D^{j_1, \ldots, j_i} F \|_{(\mathbb{S}_{j_1}) \otimes \cdots \otimes \mathbb{S}_{j_i}}^2 \right)^{\frac{q}{2}} \right].
$$

Let $u$ such that $|\mathbb{E}\langle D^j F, u \rangle_{\mathbb{S}_{j_1}}| \leq c_u \| F \|_{L^2}$, for any $F \in \mathbb{D}^{1,2}$. 

The Sobolev space $\mathbb{D}^{p,q}$ is the closure of the space of smooth cylindrical random variables w.r.t the norm $\| \cdot \|_{p,q}$

$$\| F \|^q_{p,q} = \mathbb{E}(F^q) + \sum_{i=1}^{p} \mathbb{E} \left[ \left( \sum_{j_1, \ldots, j_i = 1}^{d} \| D^{j_1, \ldots, j_i} F \|^2_{(S^d) \otimes i} \right)^{\frac{q}{2}} \right].$$

Let $u$ such that $|\mathbb{E} \langle D^j F, u \rangle_{S^d}| \leq c_u \| F \|_{L^2}$, for any $F \in \mathbb{D}^{1,2}$.

The divergence operator $\delta^j$ is defined as the adjoint of the Malliavin derivative $D^j$.

$$\mathbb{E}(F \delta^j(u)) = \mathbb{E} \langle D^j F, u \rangle_{S^d}.$$
The Sobolev space \( \mathbb{D}^{p,q} \) is the closure of the space of smooth cylindrical random variables w.r.t the norm \( \| \cdot \|_{p,q} \)

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\| F \|_{p,q}^q = \mathbb{E}(|F|^q) + \sum_{i=1}^{p} \mathbb{E} \left[ \left( \sum_{j_1, \ldots, j_i = 1}^{d} \| D^{j_1, \ldots, j_i} F \|_{(\mathbb{S}^d)^{\otimes i}}^2 \right)^{\frac{q}{2}} \right].
\]

Let \( u \) such that \( |\mathbb{E}\langle D^j F, u \rangle_{\mathbb{S}^d}| \leq c_u \| F \|_{L^2} \), for any \( F \in \mathbb{D}^{1,2} \).

The divergence operator \( \delta^i \) is defined as the adjoint of the Malliavin derivative \( D^i \).

\[
\mathbb{E}(F \delta^i(u)) = \mathbb{E}\langle D^i F, u \rangle_{\mathbb{S}^d}.
\]

Define the divergence operator on \( \mathbb{S}^d \) as \( \delta(u) = \sum_{j=1}^{d} \delta^j(u_j) \) for \( u = (u_1, \ldots, u_d) \in \cap_{j=1}^{d} \text{Dom}(\delta^j) \).
The divergence operator $\delta$ is continuous from $\mathbb{D}^{1,p}(\mathcal{S}^d)$ into $L^p(\Omega)$, which means

$$
\mathbb{E}(|\delta(u)|^p) \leq C_p \left( \mathbb{E}(\|u\|_{\mathcal{S}^d}^p) + \mathbb{E}(\|Du\|_{\mathcal{S}^d \otimes \mathcal{S}^d}^p) \right),
$$

for some constant $C_p$ depending on $p$. 
p-th moment of divergence integrals

The divergence operator $\delta$ is continuous from $\mathbb{D}^{1,p}(\mathcal{F}_d)$ into $L^p(\Omega)$, which means

$$\mathbb{E}(|\delta(u)|^p) \leq C_p \left( \mathbb{E}(\|u\|^p_{\mathcal{F}_d}) + \mathbb{E}(\|Du\|^p_{\mathcal{F}_d \otimes \mathcal{F}_d}) \right),$$

for some constant $C_p$ depending on $p$.

**Lemma**

Let $H \in (\frac{1}{2}, 1)$ and let $u$ be an element of $\mathbb{D}^{1,p}(\mathcal{F}_d)$, $p > 1$. Then $u$ belongs to the domain of the divergence operator $\delta$ in $L^p(\Omega)$. Moreover, we have

$$\mathbb{E}(|\delta(u)|^p) \leq C_{p,H} \left( \|\mathbb{E}(u)\|^p_{L^1/H([0,\infty);\mathbb{R}^d)} + \mathbb{E} \left( \|Du\|^p_{L^1/H([0,\infty)^2;\mathbb{R}^d \times \mathbb{R}^d)} \right) \right).$$
p-th moment of divergence integrals when $H \in (0, \frac{1}{2})$

**Proposition**

Let $H \in (0, \frac{1}{2})$ and $p \geq 2$. Assume that the $\mathbb{R}^d$-valued stochastic process \{u_t, t \geq 0\} satisfies the regularity Hypothesis (i)-(iv).

(i) \[\|u\|_{p,0,\infty} = \sup_{t \geq 0} \|u_t\|_{L^p(\Omega;\mathbb{R}^d)} < \infty,\]

(ii) \[\|u_t - u_s\|_{L^p(\Omega;\mathbb{R}^d)} \leq K(t - s)^\beta,\]

(iii) \[\|Du_t\|_{L^p(\Omega;\mathcal{S}^d \otimes \mathbb{R}^d)} \leq K t^\lambda,\]

(iv) \[\|Du_t - Du_s\|_{L^p(\Omega;\mathcal{S}^d \otimes \mathbb{R}^d)} \leq K(t - s)^\beta s^\lambda.\]

where the constants $K > 0$, $\beta > \frac{1}{2} - H$ and $\lambda \in (0, H]$. Then for any $T > 0$, the divergence integral $\delta(u 1_{[0,T]})$ is in $L^p(\Omega)$, and

\[\mathbb{E}(\|\delta(u 1_{[0,T]})\|^p) \leq CT^{pH}(1 + T^{p\lambda})(1 + T^{p\beta}),\]

where the constant $C$ is independent of $T$. 
Maximal inequality for stochastic integrals

Theorem (Hu, Nualart, Z ('18))

- Let \( H \in (\frac{1}{2}, 1) \) and \( \frac{1}{p} + \frac{1}{q} = H \) with \( p > q \). Suppose that for all \( T > 0 \)
  - (i) \( \int_0^T \mathbb{E}(|u_s|^p)ds < \infty \),
  - (ii) \( \int_0^T \int_0^s \mathbb{E}(|D_t u_s|^p)dt ds < \infty \).

Then the divergence integral \( \int_0^t u_s dB_s \) is in \( L^p(\Omega) \) for all \( t \geq 0 \) and for any interval \([a, b] \), we have

\[
\mathbb{E} \left( \sup_{t \in [a, b]} \left| \int_a^t u_s dB_s \right|^p \right) \leq C(b - a)^{p \frac{q}{q}} \int_a^b \mathbb{E}(|u_s|^p)ds \\
+ C(b - a)^{2p \frac{q}{q}} \int_a^b \int_a^s \mathbb{E}(|D_t u_s|^p)dt ds ,
\]

where the constant \( C \) does not depend on \( a, b \).
Maximal inequality for stochastic integrals

Theorem (Hu, Nualart, Z (’18))

Let \( \{u_t, t \geq 0\} \) be an \( \mathbb{R}^d \)-valued stochastic process. For the divergence integral \( \int_0^t u_s dB_s, t \geq 0 \), we have the following statements:

- Let \( H \in (\frac{1}{4}, \frac{1}{2}) \) and \( p > \frac{1}{H} \). Assume that the stochastic process \( u \) satisfies the regularity Hypothesis. Then the divergence integral \( \int_0^t u_s dB_s \) is in \( L^p(\Omega) \) for all \( t \geq 0 \) and for any \( 0 \leq a < b \) we have the estimate

\[
\mathbb{E} \left( \sup_{t \in [a,b]} \left| \int_a^t u_s dB_s \right|^p \right) \leq C(b - a)^{pH} \left( 1 + (b - a)^{p\beta} \right) \left( 1 + b^{p\lambda} \right),
\]

where \( C \) is a generic constant that does not depend on \( a, b \).
Based on factorization method,

$$\mathbb{E} \left( \sup_{t \in [a, b]} \left| \int_a^t u_s dB_s \right|^p \right)$$

$$= \left( \frac{\sin(\alpha \pi)}{\pi} \right)^p \mathbb{E} \left( \sup_{t \in [a, b]} \left| \int_a^t \left( \int_s^t (t - r)^{\alpha - 1} (r - s)^{-\alpha} \, dr \right) u_s dB_s \right|^p \right)$$

$$\leq C_{\alpha, p} (b - a)^{p\alpha - 1} \int_a^b \mathbb{E}(|G_r|^p) \, dr,$$

where

$$G_r := \int_a^r (r - s)^{-\alpha} u_s dB_s, \quad r \in [a, b].$$
Based on factorization method,

\[
\mathbb{E} \left( \sup_{t \in [a,b]} \left| \int_{a}^{t} u_s dB_s \right|^p \right)
= \left( \frac{\sin(\alpha \pi)}{\pi} \right)^p \mathbb{E} \left( \sup_{t \in [a,b]} \left| \int_{a}^{t} \left( \int_{s}^{t} (t - r)^{\alpha - 1} (r - s)^{-\alpha} \ dr \right) u_s dB_s \right|^p \right)
\leq C_{\alpha,p} (b - a)^{p\alpha - 1} \int_{a}^{b} \mathbb{E}(|G_r|^p) dr ,
\]

where

\[ G_r := \int_{a}^{r} (r - s)^{-\alpha} u_s dB_s, \quad r \in [a, b]. \]

Then apply the previous estimate about the \( p \)-th moment of the stochastic integrals. In the case when \( H < \frac{1}{2} \), we require the regularity of \( (r - s)^{-\alpha} u_s \) and this triggers the restriction of \( H > \frac{1}{4} \).
Proposition

(1) The solution $X_t$ satisfies

$$\|X_t\|_{L^p(\Omega;\mathbb{R}^m)} \leq C_p,$$

and

$$\|X_t - X_s\|_{L^p(\Omega;\mathbb{R}^m)} \leq C_p|t - s|^H$$

for all $t \geq s \geq 0$.

(2) The Malliavin derivative of the solution $X_t$ satisfies for all $0 \leq s \leq t$

$$|D_s X_t| \leq |\sigma|e^{-L_1(t-s)}, \text{ a.s.}$$

Moreover, if $v \leq u \leq s \leq t$, we have

$$\|D_u X_t - D_v X_t\|_{L^p(\Omega;\mathbb{R}^{m \times d})} \leq Ce^{-L_1(t-u)}(1 \wedge |u - v|),$$

$$\|D_u X_t - D_u X_s\|_{L^p(\Omega;\mathbb{R}^{m \times d})} \leq Ce^{-L_1(s-u)}(1 \wedge |t - s|),$$

$$\|D_u X_t - D_v X_t - (D_u X_s - D_v X_s)\|_{L^p(\Omega;\mathbb{R}^{m \times d})} \leq Ce^{-L_1(s-u)}(1 \wedge |u - v|)(1 \wedge |t - s|)$$
Convergence of the sequence \( n^{-1}Z_{j,n} = \frac{1}{n} \int_0^n f_j^r(X_s) \sigma dB_s \).

\[
\sum_{n=1}^\infty \mathbb{P}(\left| n^{-1}Z_{j,n} \right| > \epsilon) \leq \sum_{n=1}^\infty \epsilon^{-p} \mathbb{E} \left( \left| n^{-1}Z_{j,n} \right|^p \right)
\]

\[
\mathbb{E}(\left| Z_{j,n} \right|^p) \leq \begin{cases} 
Cn^{pH} & \text{when } H \in \left(\frac{1}{2}, 1\right) \\
Cn^{p(2H+\lambda)} & \text{when } H \in \left(\frac{1}{4}, \frac{1}{2}\right)
\end{cases}
\]
Conclusion

- Convergence of the sequence $n^{-1} Z_{j,n} = \frac{1}{n} \int_0^n f_j^{tr}(X_s)\sigma dB_s$.

\[
\sum_{n=1}^{\infty} \mathbb{P}(|n^{-1} Z_{j,n}| > \epsilon) \leq \sum_{n=1}^{\infty} \epsilon^{-p} \mathbb{E}\left(|n^{-1} Z_{j,n}|^p\right)
\]

\[
\mathbb{E}(|Z_{j,n}|^p) \leq \begin{cases} C n^{pH} & \text{when } H \in \left(\frac{1}{2}, 1\right) \\ C n^{p(2H+\lambda)} & \text{when } H \in \left(\frac{1}{4}, \frac{1}{2}\right) \end{cases}
\]

- Convergence of the sequence $T^{-1} Z_{j,T}$.

\[
\frac{1}{T} |Z_{j,T}| \leq \frac{1}{k_T} \left| \int_0^{k_T} g_j(X_t)dB_t \right| + \frac{1}{k_T} \sup_{t \in [k_T, k_T+1]} \left| \int_{k_T}^t g_j(X_s)dB_s \right|.
\]

\[
\mathbb{E}\left(\sup_{t \in [k_T, k_T+1]} \left| \int_{k_T}^t g_j(X_s)dB_s \right|^p\right) \leq \begin{cases} C & \text{when } H \in \left(\frac{1}{2}, 1\right) \\ C(k_T + 1)^p\lambda & \text{when } H \in \left(\frac{1}{4}, \frac{1}{2}\right), \end{cases}
\]

where $\lambda \in (0, H)$. 

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THANK YOU!