Random self-similar trees and their applications

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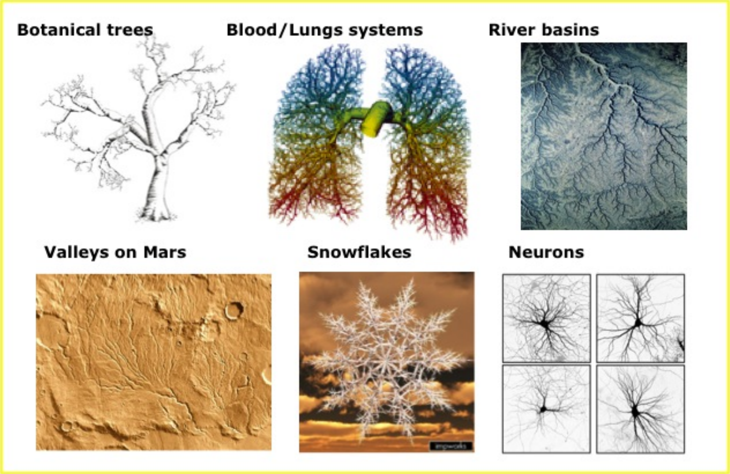
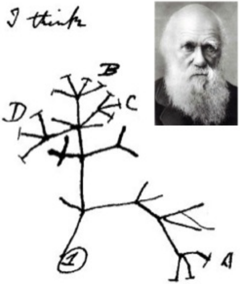
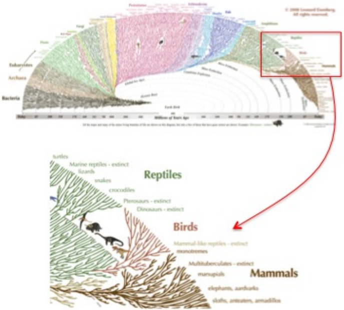
Trees in Nature



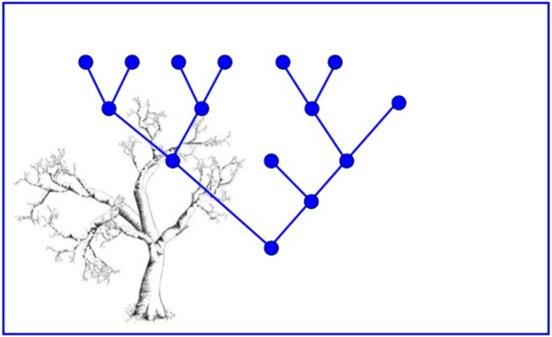
Trees in Nature

Phylogenetic tree of life





Rooted binary tree



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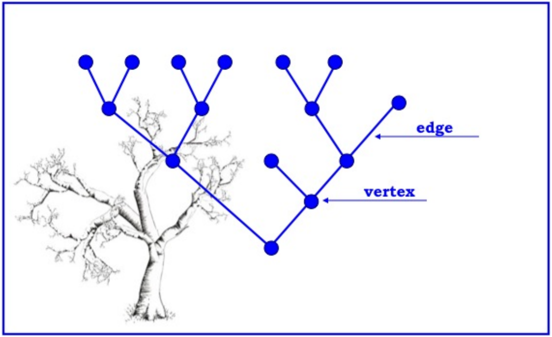

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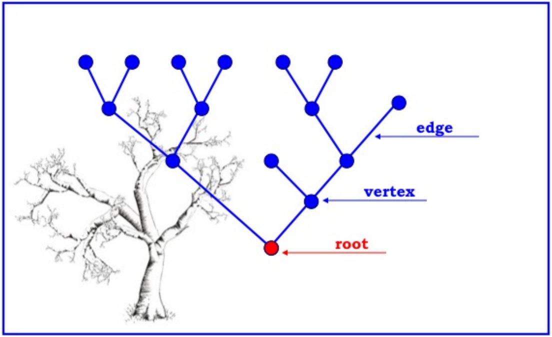

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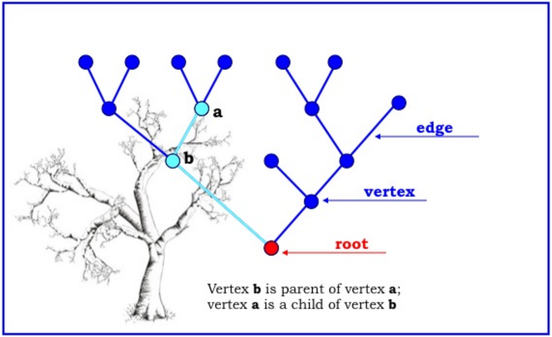

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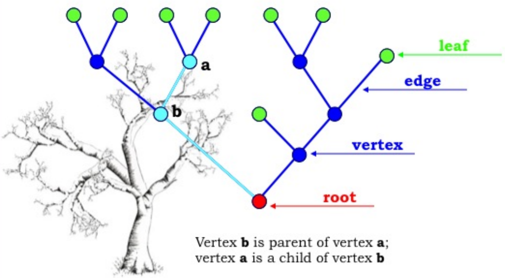

Vertex **b** is parent of vertex **a**;
vertex **a** is a child of vertex **b**

Rooted binary tree

- $\mathcal{L}_{\text{plane}}$ - space of finite unlabeled rooted reduced binary trees with edge lengths, including an empty tree $\phi = \{\rho\}$ comprised of a root vertex ρ and no edges.
- $d(x, y)$: the length of the minimal path within T between x and y.
- The length of a tree T is the sum of the lengths of its edges:

$$\textsc{length}(T) = \sum_{i=1}^{\#T} l_i.$$

- The height of a tree T is the maximal distance between the root and a vertex:

$$\textsc{height}(T) = \max_{1 \le i \le \#T} d(v_i, \rho).$$

Galton-Watson trees

A Galton-Watson process is a simple (Markov) model of population growth.

- The process starts with a single progenitor at time $t = 0$.
- At each integer instant $t > 0$ each member terminates and leaves a random number k of offspring according to a distribution $\{p_k\}$, $k = 0, 1, \ldots$.
- If $p_0 + p_2 = 1$ (only zero or two offspring are possible), the process is called binary.
- If $\mathsf{E}(k) = 1$ (constant expected progeny), the process is called critical.
- A Galton-Watson tree describes a trajectory of the process.
- A Galton-Watson tree with i.i.d. exponential edge lengths with parameter λ is called exponential GW tree $\mathsf{GW}(\lambda)$.

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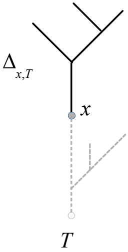
Prune invariance results

- Neveu (1986): established invariance of an exponential critical and sub-critical binary Galton-Watson tree GW(λ) with respect to the tree erasure (a.k.a. leaf-length pruning, trimming).
- Burd, Waymire, and Winn (2000): established invariance of the critical binary Galton-Watson tree (with no edge lengths) with respect to Horton pruning (cutting the tree leaves).
- Burd, Waymire, and Winn (2000): established that Horton prune invariance is a characteristic property of the critical binary tree, in the space of (not necessarily binary) Galton-Watson trees with no edge lengths.
- Duquesne and Winkel (2012): established invariance of the Galton-Watson (non-binary) trees with respect to hereditary reduction.

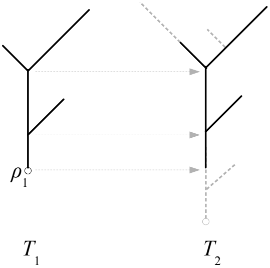
Next: give a unified description of various pruning operations and generalize the invariance results.

Partial order on trees

- $\Delta_{x,T}$ is the subtree descendant to point $x \in T$.
- Partial order: $T_1 \preceq T_2$ if and only if $\exists$ an isometry $f : (T_1, d) \to (T_2, d)$.



(a) Descendant tree



(b) Isometry

Generalized dynamical pruning

- Consider a monotone non-decreasing $\varphi : \mathcal{L}_{\rm plane} \to \mathbb{R}^+$, i.e. $\varphi(T_1) \le \varphi(T_2)$ whenever $T_1 \preceq T_2$.
- Generalized dynamical pruning operator

$$\mathcal{S}_t(\varphi, T) : \mathcal{L}_{\rm plane} \to \mathcal{L}_{\rm plane}$$

  induced by φ at any $t \ge 0$ cuts all subtrees $\Delta_{x,T}$ for which the value of φ is below threshold t.
- Formally,

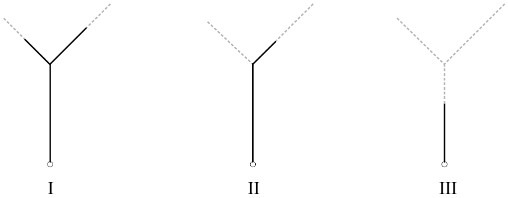
$$\mathcal{S}_t(\varphi, T) := \rho \cup \Big\{ x \in T \setminus \rho \ : \ \varphi\big(\Delta_{x,T}\big) \ge t \Big\}.$$

- Here, $S_s(\varphi, T) \preceq S_t(\varphi, T)$ whenever $s \ge t$.

Generalized dynamical pruning

Informally,

- Consider a function $\varphi : T \to \mathbb{R}^+$, that is non-decreasing along every path from a leaf to the root.
- Generalized dynamical pruning operator $\mathcal{S}_t(\varphi, T)$ induced by φ at any $t \geq 0$ cuts all points $x \in T$ for which the value of φ is below threshold t.



Generic stages in dynamical pruning of a tree: an example.

Example 1: Pruning by tree height (tree erasure)

- Let the function $\varphi(T)$ equal the height of T:

$$\varphi(T) = \textsc{height}(T).$$

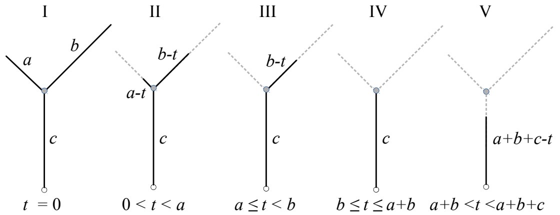
- Semigroup property: $\mathcal{S}_t \circ \mathcal{S}_s = \mathcal{S}_{t+s}$ for any $t, s \geq 0$.
- It coincides with the tree erasure introduced by Neveu (1986), and further examined by Le Jan (1991), Duquesne and Winkel (2007, 2012), Evans, Pitman and Winter (2006), and others.

Example 2: Pruning by tree length

- Let the function $\varphi(T)$ equal the total lengths of T:

$$\varphi(T) = \textsc{length}(T).$$

- No semigroup property.
- Closely related to the dynamics of a particular Hamilton-Jacobi equation.



Example 3: Horton pruning

- Let
$$\varphi(T) = \mathsf{k}(T) - 1,$$
where the Horton-Strahler order $\mathsf{k}(T)$ is the minimal number of Horton prunings $\mathcal{R}$ (cutting the tree leaves and applying series reduction) necessary to eliminate all points in tree T except ρ.
- Semigroup property with $\mathcal{S}_t = \mathcal{R}^{\lfloor t \rfloor}$.
- The Horton-Strahler order $\mathsf{k}(T)$ is known as the register number as it equals the minimum number of memory registers necessary to evaluate an arithmetic expression described by a tree T.
- Studied by Burd, Waymire, and Winn (2000).

Prune invariance for Galton-Watson trees

Theorem (Arnold, Kovchegov, IZ [2017])

Let $T \stackrel{d}{=} \mathsf{GW}(\lambda)$ *be an exponential critical binary Galton-Watson tree with parameter* $\lambda > 0$*. Then, for any monotone non-decreasing function* $\varphi : \mathcal{L}_{\rm plane} \to \mathbb{R}^+$*, the pruned tree* T^t *conditioned on surviving is an exponential critical binary Galton-Watson tree with parameter*

$$\mathcal{E}_t(\lambda, \varphi) = \lambda p_t(\lambda, \varphi).$$

Formally,

$$T^t := \{\mathcal{S}_t(\varphi, T) | \mathcal{S}_t(\varphi, T) \neq \phi\} \stackrel{d}{=} \mathsf{GW}(\lambda p_t(\lambda, \varphi)),$$

where $p_t(\lambda, \varphi) = \mathsf{P}(\mathcal{S}_t(\varphi, T) \neq \phi)$.

Prune invariance results

Moreover,

Theorem (Arnold, Kovchegov, Z [2017])

(a) *If $\varphi(T)$ equals the total length of T ($\varphi = \text{LENGTH}(T)$), then*

$$\mathcal{E}_t(\lambda, \varphi) = \lambda e^{-\lambda t}\Big[I_0(\lambda t) + I_1(\lambda t)\Big].$$

(b) *If $\varphi(T)$ equals the height of T ($\varphi = \text{HEIGHT}(T)$), then*

$$\mathcal{E}_t(\lambda, \varphi) = \frac{2\lambda}{\lambda t + 2}.$$

(c) *If $\varphi(T) + 1$ equals the Horton-Strahler order of the tree T, then*

$$\mathcal{E}_t(\lambda, \varphi) = \lambda 2^{-\lfloor t \rfloor}.$$

Generalized prune-invariance

Definition (Prune-invariance)

Consider a probability measure μ *on* $\mathcal{L}_{\mathrm{plane}}$ *such that* $\mu(\phi) = 0$. *Let*

$$\nu(T) = \mu \circ \mathcal{S}_t^{-1}(T) = \mu\big(\mathcal{S}_t^{-1}(T)\big).$$

Measure μ *is called invariant with respect to the pruning operator* $\mathcal{S}_t(\varphi, T)$ *if for any tree* $T \in \mathcal{L}_{\mathrm{plane}}$ *we have*

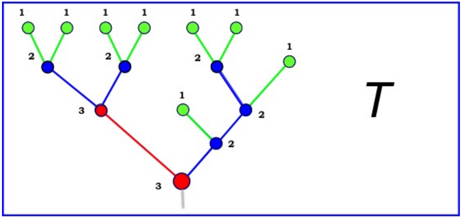
$$\mu(T) = \nu(T | T \neq \phi).$$

- *Also need the invariance of the distribution of edge lengths in the pruned tree* $T_t := \mathcal{S}_t(\varphi, T)$. *Kovchegov & Z [2017] - arXiv:1608.05032*
- *A weaker mean invariance only preserves the means of selected branch statistics.*

Open question: finding and classifying all the invariant probability measures μ on $\mathcal{L}_{\mathrm{plane}}$.

Horton pruning

- *Pruning* $\mathcal{R}(T)$ of a finite tree T cuts the leaves and degree-2 chains connected to leaves.
- Nodes cut at k-th pruning, $\mathcal{R}^{k-1}(T) \setminus \mathcal{R}^k(T)$, have order k, $k \geq 1$.
- A chain of the same order vertices is called *branch*.
- Let N_k is the number of branches of order k; and N_{ij} is the number of instances when an order-i branch merges an order-j branch.



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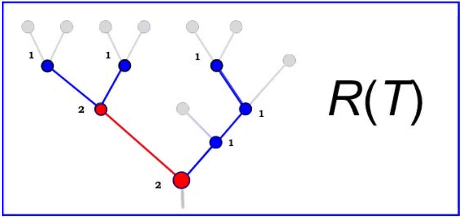

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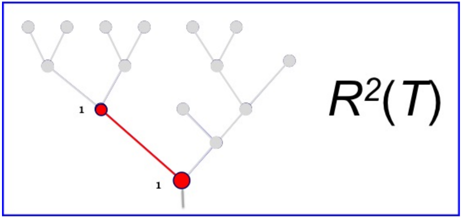

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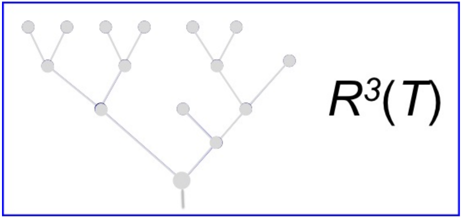

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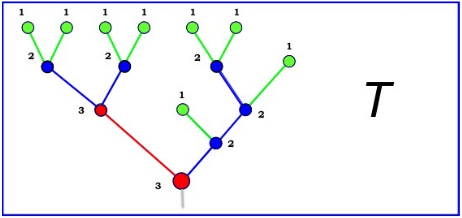

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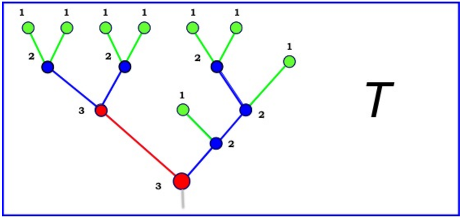

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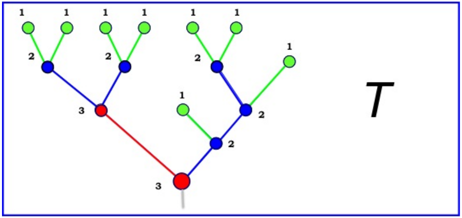
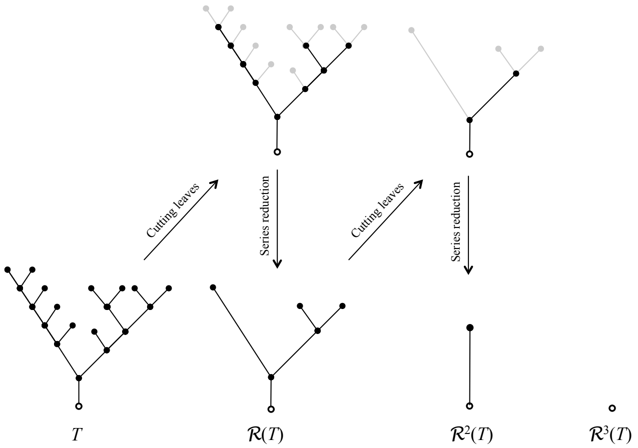


Horton pruning

Tree T has order $\mathsf{k}(T) = 3$ since it is eliminated in three prunings.



Statistical approach to prune invariance

[Kovchegov and Z, *Fractals*, 2016]

- The Tokunaga coefficient T_{ij} is the average number of branches of order i that merge with a branch of order j:

$$T_{ij} = \frac{\mathsf{E}[N_{ij}]}{\mathsf{E}[N_j]}, \quad 1 \le i < j \le K.$$

- The coefficients T_{ij} form the Tokunaga matrix

$$\mathbb{T}_K = \begin{bmatrix} 0 & T_{1,2} & T_{1,3} & \dots & T_{1,K} \\ 0 & 0 & T_{2,3} & \dots & T_{2,K} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & T_{K-1,K} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Statistical approach to prune invariance

Theorem (Kovchegov and Z [2016])

(Subject to some conditions) a probability measure μ on $\mathcal{L}_{\mathrm{plane}}$ is mean Horton invariant if and only if

$$T_{i,i+k} = T_k \quad \textit{for all } i, k > 0$$

for some sequence $T_k \geq 0$.

- The Tokunaga matrix becomes Toeplitz

$$\mathbb{T}_K = \begin{bmatrix} 0 & T_1 & T_2 & \dots & T_{K-1} \\ 0 & 0 & T_1 & \dots & T_{K-2} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & T_1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Burd, Waymire and Winn [2000] established this for Galton-Watson trees.

Geometric branching process

Kovchegov and Z [2017, 2018]

- A Geometric Branching Process (GBP) produces prune-invariant trees for arbitrary Tokunaga sequences $\{T_k\}$.
- A special class of critical Tokunaga processes appears with

$$T_k = (c-1)c^{k-1}, \quad c \geq 1.$$

(The case $c = 2$ corresponds to the critical binary GW tree.)

- The critical Tokunaga property is equivalent to the time shift invariance of the GBP.
- The critical Tokunaga trees are characterized by the property that each of their sub-trees (properly defined) has the same distribution as a random tree.
- A more general property $T_k = a\,c^{k-1}$ is equivalent to the asymptotic (in branch order) time shift invariance of the GBP.
- Interestingly, the Tokunaga trees with $T_k = a\,c^{k-1}$ are well known in applied literature...

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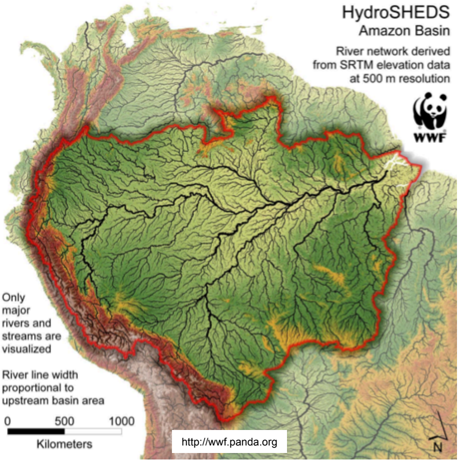
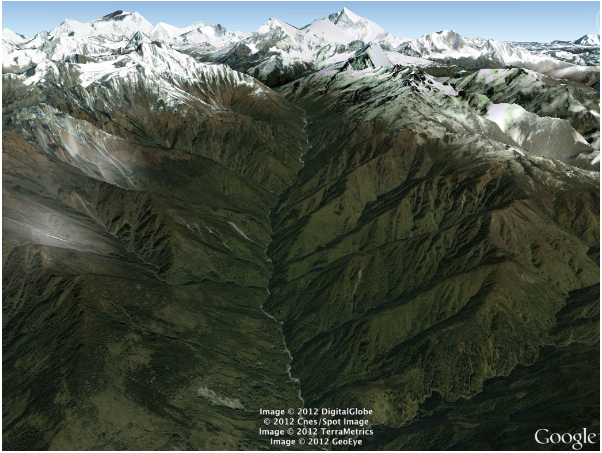
River networks

- [Shreve 1966, 1969; Tokunaga, 1978; Peckham, 1995; Burd et al., 2000; Z et al., 2009; Zanardo et al., 2013]



Hillslope drainage networks

- [Z et al., 2009]

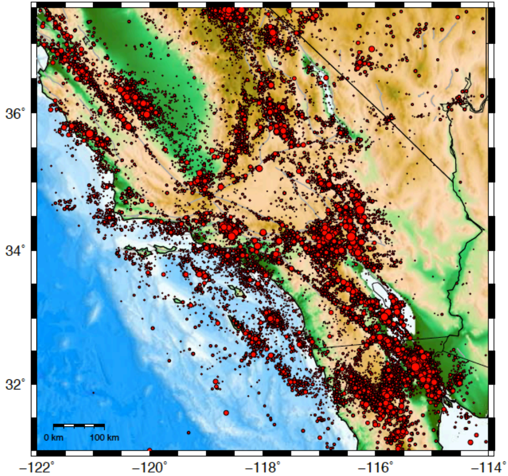


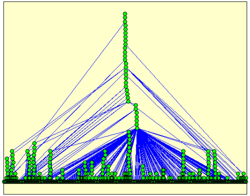
Vein structure of botanical leaves

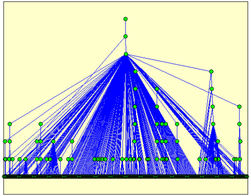
- [Newman et al., 1997; Turcotte et al., 1998]

Earthquake aftershock clusters

- [Yoder et al., 2011]

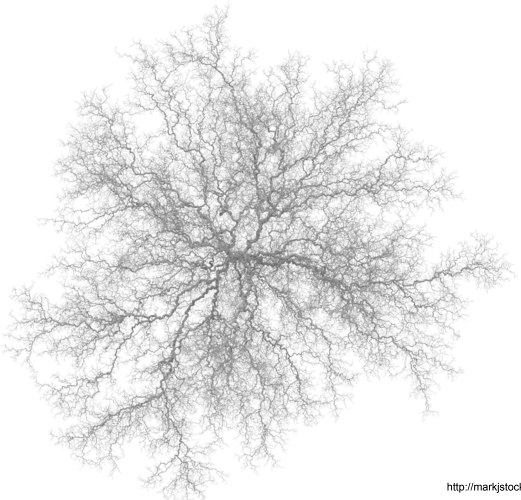






Diffusion limited aggregation

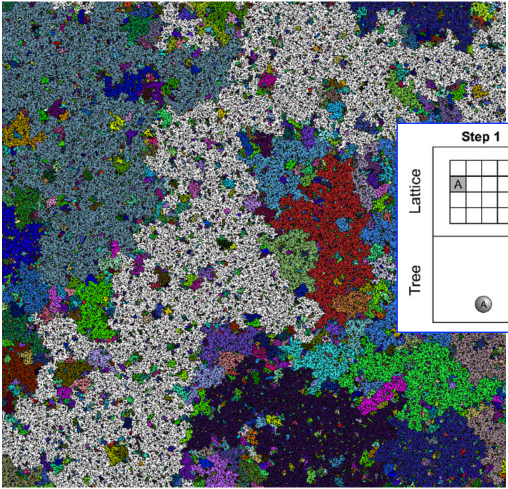
- [Ossadnik, 1992; Masek and Turcotte, 1993]

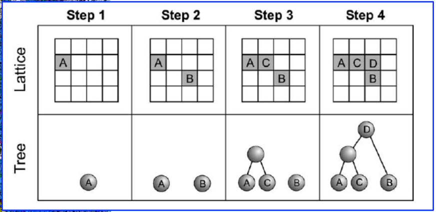


http://markjstock.org/dla3d/images/save47c_e.png

Two dimensional site percolation

- [Turcotte et al., 1999; Yakovlev et al., 2005; Z et al., 2006]

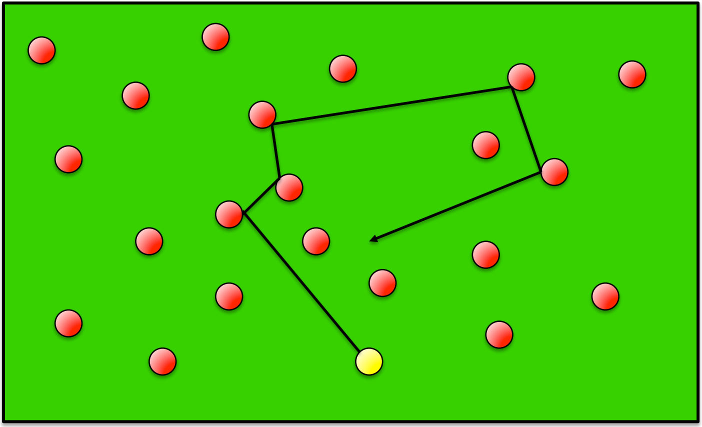




http://www.opencourse.info/anderson/l1024pc.png

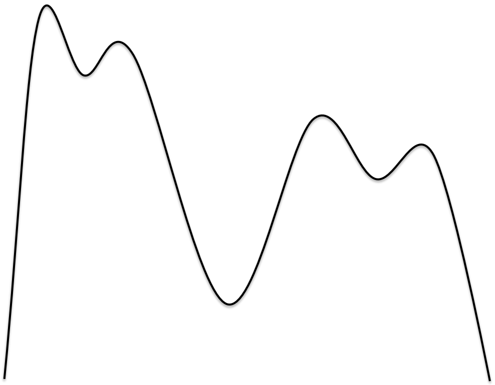
Dynamics of billiards

- [Gabrielov et al., 2008; Patterson et al., 2016]



Tree representation of a function, $\mathrm{LEVEL}(X_t)$

Z and Kovchegov, 2012



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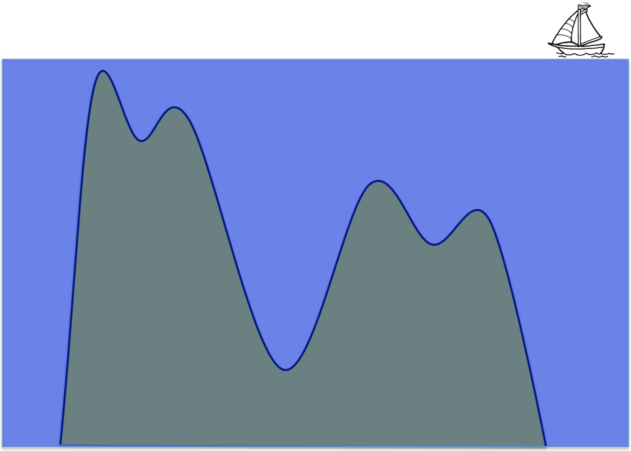

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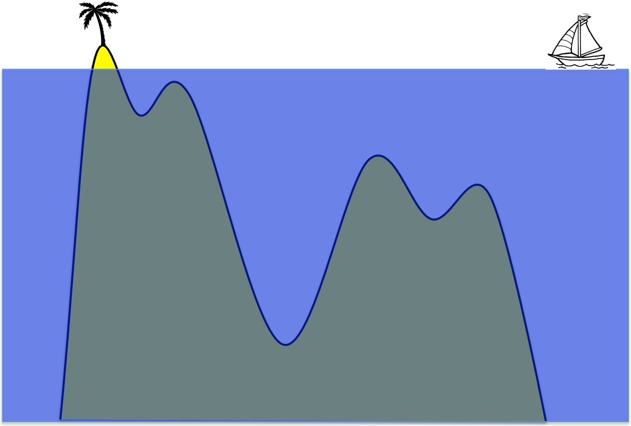

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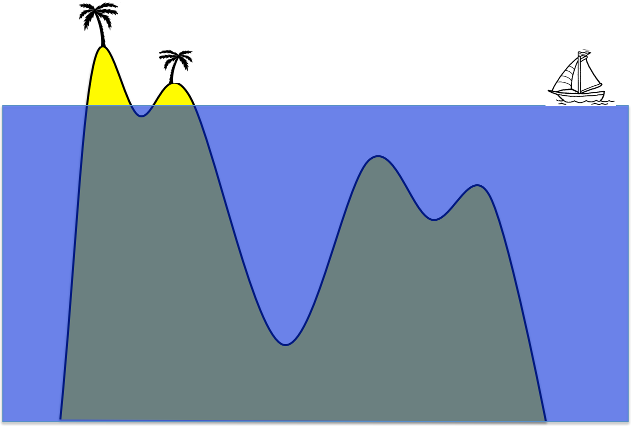

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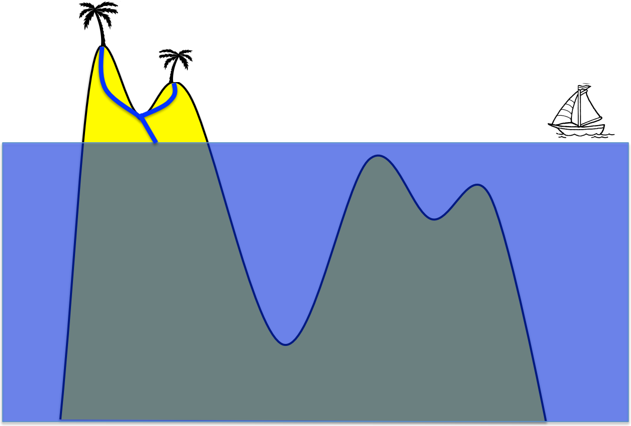

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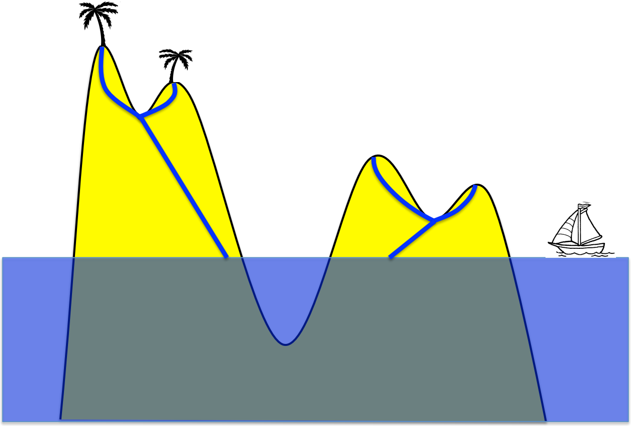

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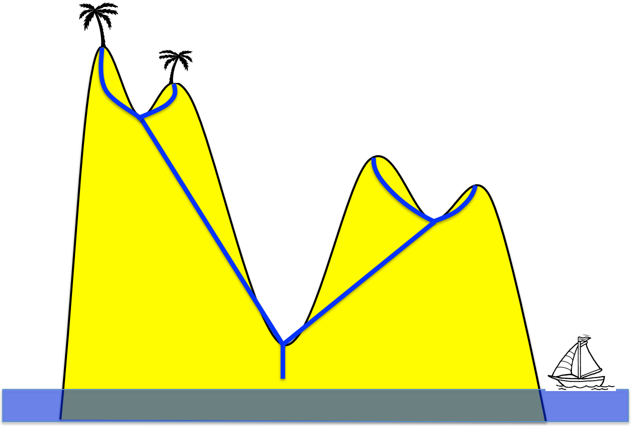

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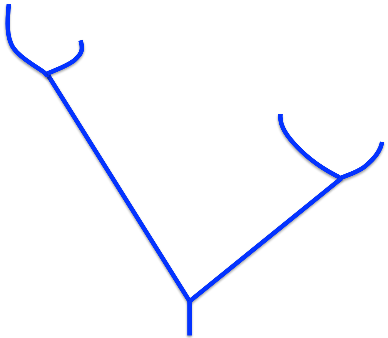

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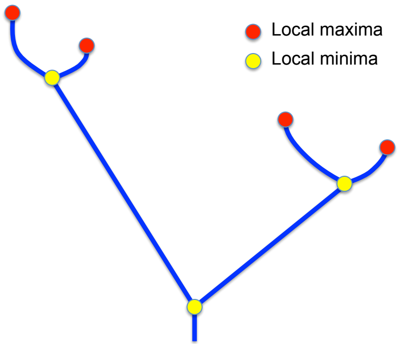
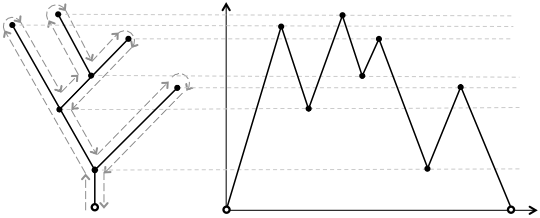


Tree representation of a function



(a) Tree T (b) Harris path H_T

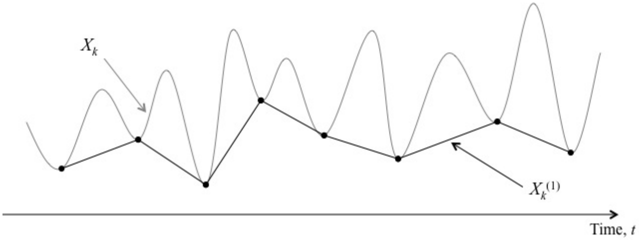
Theorem (J. Neveu and J. Pitman [1989], J. F. Le Gall [1993])

The level-set tree $\text{LEVEL}(X_t)$ *is an exponential critical binary Galton-Watson tree* $\mathrm{GW}(\lambda)$ *if and only if the rises and falls of* X_t*, excluding the last fall, are i.i.d. exponential random variables with parameter* $\lambda/2$.

Pruning time series

Proposition (Z and Kovchegov [2012])

1. *The transition from a time series X_k to the time series X_k^{min} of its local minima corresponds to the Horton pruning of the level-set tree* $\text{LEVEL}(X)$.
2. *A symmetric random walk in discrete time corresponds to a critical Tokunaga tree with $c = 2$ (i.e., $T_k = 2^{k-1}$). In particular, the critical Galton-Watson tree corresponds a symmetric exponential random walk.*



Open problem: Finding all the time series models invariant with respect to Horton pruning.

Summary

- Generalized dynamical pruning encompasses a number of continuous and discrete pruning operations.
- Horton prune invariance is equivalent to the existence of Tokunaga coefficients $T_k := T_{i,i+k}$ (under some natural conditions).
- Geometric Branching Model generates prune-invariant trees for an arbitrary sequence T_k.
- Tokunaga trees, $T_k = ac^{k-1}$, is a subclass widely seen in observations. It is equivalent to the time shift invariance.
- Future: A possibility to study non-linear wave dynamics as a dynamical pruning (proof-of-concept results for Hamilton-Jacobi systems).

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