On Hydrodynamic Limits of Young Diagrams

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Frontier Probability Days, March 2018

Joint work with I. Fatkullin, S. Sethuraman.

- Static Models of Young Diagrams
- Evolutional Models of Young Diagrams
- Main Results
- Sketch of Proof

Young diagrams are related with:
combinatorics; representation theory; . . .
polymer physics; genetics; zero-temperature Ising model; . . ..

2D/3D Young diagrams: static theory (statistical mechanics),
dynamical theory

We will be focusing on models of 2D Young diagrams.

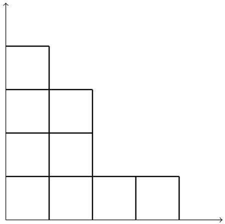
Let $p = (p_1, p_2, p_3, \ldots, p_n)$, $p_k \geq p_{k+1}$, be a partition of the integer

$$M(p) = \sum_{k=1}^{n} p_k.$$

For example, $p = (4, 2, 2, 1)$ is a partition of

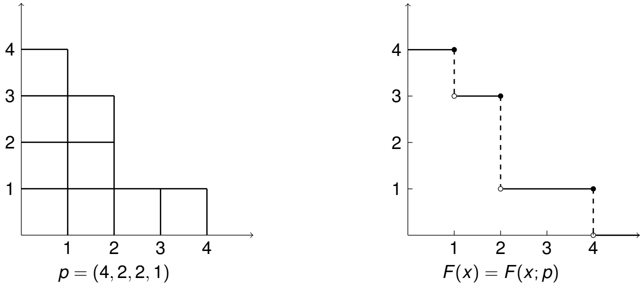
$$9 = 4 + 2 + 2 + 1$$

The corresponding Young diagram:



$p = (4, 2, 2, 1)$

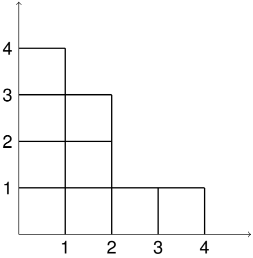
Shape function $F(x)$:



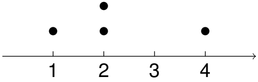
Clearly:

$$M(p) := \sum_{k=1}^{n} p_k = \int_0^\infty F(x)dx.$$

Size density (or configuration of particles) $\eta = (\eta(k))_{k \in \mathbb{N}}$:



$p = (4, 2, 2, 1)$



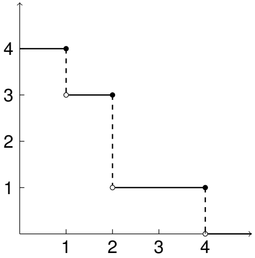
$\eta = (1, 2, 0, 1, 0, \ldots)$

$$M(p) = \sum_{k=1} k\eta(k).$$

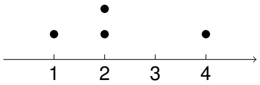
Relations between F and η:

$$F(x) = \sum_{k \geq x} \eta(k), \qquad \eta(k) = F(k) - F(k+1)$$

$\eta(k)$ can be viewed as negative gradient of F at k.

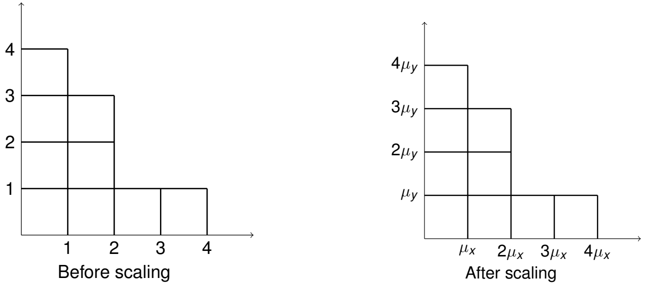


$F(x)$



$\eta = (1, 2, 0, 1, 0, \ldots)$

The size of a diagram grows, of course, as the $M(p)$ grows. For the limit $M \to \infty$, rescale the diagram by setting the width and height of one square by μ_x and μ_y respectively. After rescaling, the area of the Young diagram is $\mu_x \, \mu_y \, M$.



If we set $\mu_x = \mu_y = \dfrac{1}{\sqrt{M}}$ rescaled shape function

$$F_M(x;p) = \frac{1}{\sqrt{M}} F(x\sqrt{M};p)$$

A classical result of A. Vershik [V]: Let $\mathcal{P}_M$ be the uniform probability on all partitions of M, e.g. $(4,2,2,1)$ and $(5,4)$ are equally likely. As $M \to \infty$, F_M concentrate near

$$F(x) = -\frac{\sqrt{6}}{\pi} \ln\left(1 - e^{-\pi x/\sqrt{6}}\right)$$

Precisely, for all $M > M_0(a,b,\varepsilon)$

$$\mathcal{P}_M \left\{ \sup_{x\in[a,b]} |F_M(x;p) - F(x)| > \varepsilon \right\} < \varepsilon.$$

The uniform measure $\mathcal{P}_\mu$ can be thought as a canonical ensemble. Similar result as above holds for other choices of measures. For example

$$\mathcal{P}_\mu(p) = \frac{1}{Z_\mu} e^{-\mu M(p)}.$$

or the general Grand-canonical ensemble (cf. e.g. [EG], [FS], [V], [VY])

$$\mathcal{P}_{\beta,\mu}(p) = \frac{1}{Z_{\beta,\mu}} e^{-\beta \sum_{k\in p} E_k - \mu M(p)}$$

Rescale $F_\mu(x;p) := \dfrac{1}{\mu \mathbb{E}_\mu(M)} F(x/\mu;p)$.

- $\beta = 0$: $F_\mu \to \dfrac{6}{\pi^2} \ln(1-e^{-x})$
- $E_k \sim \ln k$, $0 < \beta < 1$: $F_\mu \to \dfrac{1}{\Gamma(2-\beta)} \displaystyle\int_x^\infty u^{-\beta} e^{-u} du$
- $E_k \ll \ln k$, $\beta > 0$: $F_\mu \to e^{-x}$

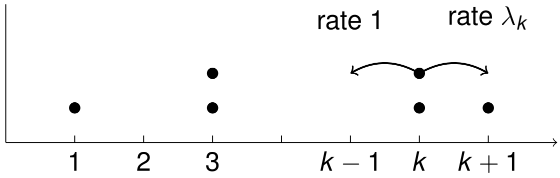
For our evolutional models, start with the particle systems directly. Introduce generator

$$Lf(\eta) = \sum_{k=1}^{\infty} \Big\{ \lambda_k \left[f\left(\eta^{k,k+1}\right) - f(\eta) \right] \chi_{\{\eta(k)>0\}} + \left[f\left(\eta^{k,k-1}\right) - f(\eta) \right] \chi_{\{\eta(k)>0,k>1\}} \Big\}$$

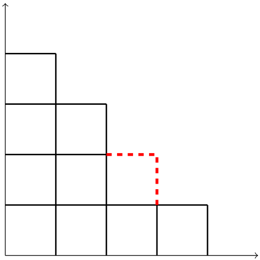
where

$$\lambda_k = e^{-\beta(E_{k+1}-E_k)-\mu}, \quad \eta^{x,y}(k) = \begin{cases} \eta(k) - 1 & k = x \\ \eta(k) + 1 & k = y \\ \eta(k) & \text{otherwise} \end{cases} .$$

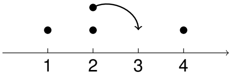
Weakly asymmetric zero range process on $\mathbb{Z}^+$.



Remember $\lambda_k = e^{-\beta(E_{k+1}-E_k)-\mu}$.

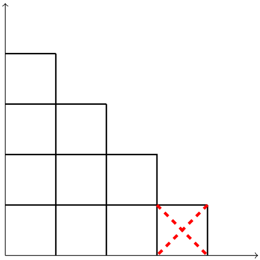


Growth at $(2,1)$

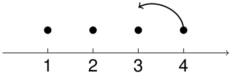


a particle jumps from site 2 to 3

In this example, a particle at site 2 jumps (with rate λ_2) to site 3 corresponds to creation of a square at the corner $(2,1)$.



Loss at $(3, 0)$



a particle jumps from site 4 to 3

Here, a particle at site 4 jumps (with rate λ_4) to site 3 corresponds to annihilation of a square at the corner $(3, 0)$.

Remember $\eta(k) = F(k) - F(k+1)$. Since

$$F_\mu(x;p) := \frac{1}{\mu \mathbb{E}_\mu(M)} F(x/\mu;p)$$

we consider rescaled empirical measures

$$\pi_t^\mu(dx) = \pi^\mu(\eta_t, dx) = \mu\gamma_\mu \sum_{k=1}^{\infty} \eta_t(k)\delta_{k\mu}(dx).$$

where $\gamma_\mu = \dfrac{1}{\mu^2 \mathbb{E}_\mu(M)}$. Since $\mathbb{E}_\mu(M) \sim \mu^{-2} e^{-\beta E_{1/\mu}}$ (c.f. [FS])

$$\gamma_\mu = \begin{cases} 1 & \beta = 0 \\ \\ \mu^{-\beta} & 0 < \beta < 1, E_k \sim \ln k \\ \\ (\ln \frac{1}{\mu})^\beta & 0 < \beta, E_k \sim \ln(\ln k) \end{cases}$$

Theorem (Case $\beta = 0$)

With appropriate initial measures, for any test function $G \in C_c^\infty(0,\infty)$, for all $0 < t \le T$, as $\mu \to 0$

$$\langle G, \pi^\mu_{t/\mu^2} \rangle \to \int_0^\infty G(x)\rho(t,x)dx, \quad \textit{in probability}$$

where $\rho(t,x)$ is the unique weak solution of the equation

$$\begin{cases} \partial_t \rho = \partial_x^2 \dfrac{\rho}{\rho+1} + \partial_x \dfrac{\rho}{\rho+1} \\ \rho(0,\cdot) = \rho_0(\cdot), \quad \displaystyle\int_0^\infty \rho(t,x)dx = \int_0^\infty \rho_0(x)dx \\ \rho(t,\cdot) \le \phi(\cdot) \text{ for all } t \le T \end{cases} .$$

Theorem (Case $E_k \sim \ln k$)

With appropriate initial measures, for any test function $G \in C_c^\infty(0,\infty)$, for all $0 < t \leq T$, as $\mu \to 0$

$$\langle G, \pi^{\mu}_{t/\mu^2} \rangle \to \int_0^\infty G(x)\rho(t,x)dx, \quad \textit{in probability}$$

where $\rho(t,x)$ is the unique weak solution of the equation

$$\begin{cases} \partial_t \rho = \partial_x^2 \rho + \partial_x \left(\dfrac{\beta + x}{x} \rho \right) \\ \rho(0,\cdot) = \rho_0(\cdot), \quad \displaystyle\int_0^\infty \rho(t,x)dx = \int_0^\infty \rho_0(x)dx \\ \rho(t,\cdot) \leq \phi_c(\cdot) \text{ for all } t \leq T \end{cases} . \tag{1}$$

Theorem (Case $E_k \ll \ln k$)

With appropriate initial measures, for any test function $G \in C_c^\infty(0,\infty)$, for all $0 < t \leq T$, as $\mu \to 0$

$$\langle G, \pi^{\mu}_{t/\mu^2} \rangle \to \int_0^\infty G(x)\rho(t,x)dx, \quad \textit{in probability}$$

where $\rho(t,x)$ is the unique weak solution of the equation

$$\begin{cases} \partial_t \rho = \partial_x^2 \rho + \partial_x \rho \\ \rho(0,\cdot) = \rho_0(\cdot), \quad \int_0^\infty \rho(t,x)dx = \int_0^\infty \rho_0(x)dx \\ \rho(t,\cdot) \leq \phi_c(\cdot) \text{ for all } t \leq T \end{cases} . \qquad (2)$$

The macroscopic equations:

- $\beta = 0$:

$$\partial_t \rho = \partial_x^2 \frac{\rho}{\rho + 1} + \partial_x \frac{\rho}{\rho + 1}$$

- $E_k \sim \ln k$:

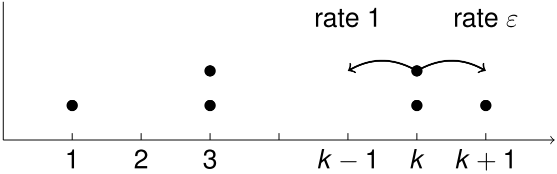
$$\partial_t \rho = \partial_x^2 \rho + \partial_x \left(\frac{\beta + x}{x} \rho \right)$$

- $E_k \ll \ln k$:

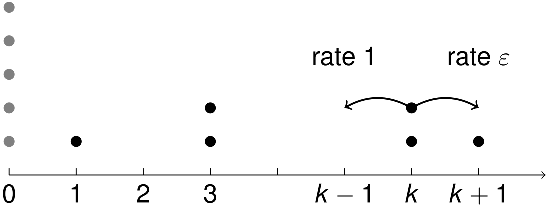
$$\partial_t \rho = \partial_x^2 \rho + \partial_x \rho$$

Funaki and Sasada [FuSa] obtained the the same equation, as in the case $\beta = 0$, for a different model.

- Case $\beta = 0$: $\lambda_k = e^{-\mu} =: \varepsilon$



- [FuSa] model: a weakly asymmetric reservoir at site 0



Invariant measures:

- model in [FuSa]:

$$\mathcal{P}_\mu(\eta) = \frac{1}{Z_\mu} e^{-\mu \sum_k k\eta(k)} = \frac{1}{Z_\mu} \prod_k \left(e^{-k\mu}\right)^{\eta(k)}$$

- Case $\beta = 0$: for all $0 < c \leq 1$

$$\mathcal{P}_{\mu,c}(\eta) = \frac{1}{Z_{\mu,c}} \prod_k \left(c\, e^{-k\mu}\right)^{\eta(k)}$$

Initial conditions:

- model in [FuSa] : $\int_0^\infty \rho_0(x)dx = \infty$.
- Case $\beta = 0$: $\rho_0 < \phi$, $\int_0^\infty \rho_0(x)dx < \infty$

Formal derivation of macroscopic equations:

$$\langle G, \pi(\eta_t)\rangle = \langle G, \pi(\eta_0)\rangle + \int_0^t \mu^{-2} L \langle G, \pi(\eta_s)\rangle \, ds + M_t^G$$

with

$$\mu^{-2} L \langle G, \pi_t(\eta)\rangle = \mu \sum_{k=2}^{\infty} \Delta_\mu G(k\mu)\, \gamma_\mu \chi_{\{\eta_t(k)>0\}} + \mu \sum_{k=2}^{\infty} \frac{\lambda_k - 1}{\mu} \nabla_\mu G(k\mu)\, \gamma_\mu \chi_{\{\eta_t(k)>0\}}$$

As $k\mu \to x$

$$\frac{\lambda_k - 1}{\mu} \to \begin{cases} 1 & \beta = 0 \\ \dfrac{\beta + x}{x} & E_k \sim \ln k \\ 1 & E_k \ll \ln k \end{cases}$$

Equilibrium measures are products of geometrics with parameters very close locally.

$$\gamma_\mu \chi_{\eta(k)} \sim \gamma_\mu \mathbb{E}_{\eta^{\varepsilon/\mu}(k)}(\chi_{\eta>0}) = \frac{\gamma_\mu \eta^{\varepsilon/\mu}}{1+\eta^{\varepsilon/\mu}}.$$

Notice that typically $\gamma_\mu \eta_t^{\varepsilon/\mu}(k) \to \rho^\varepsilon(t,x)$ then

$$\frac{\gamma_\mu \eta^{\varepsilon/\mu}}{1+\eta^{\varepsilon/\mu}} \sim \frac{\rho(x)}{1+\gamma_\mu^{-1}\rho(x)}$$

- $\beta = 0$: $\gamma_\mu = 1$, $\gamma_\mu \chi_{\eta(k)} \sim \dfrac{\rho(x)}{1+\rho(x)}$
- $E_k \sim \ln k$ or $E_k \ll \ln k$: $\gamma_\mu \to \infty$, $\gamma_\mu \chi_{\eta(k)} \sim \rho(x)$

Brief sketch of proof for the case $\beta = 0$:

- 1-block estimate:

$$\limsup_{l\to\infty}\limsup_{N\to\infty} \mathbb{E}^N \left| \frac{1}{N} \sum_{aN\le k\le bN} \int_0^T D_{N,k}^{G,t} \left(\chi_{\eta_{N^2t}(k)>0} - \frac{\eta_{N^2t}^l(k)}{1+\eta_{N^2t}^l(k)} \right) dt \right| = 0.$$

- 2-block estimate:

$$\limsup_{l\to\infty}\limsup_{\tau\to 0}\limsup_{N\to\infty} \mathbb{E}^N \left| \frac{1}{N} \sum_{aN\le k\le bN} \int_0^T D_{N,k}^{G,t} \left(\frac{\eta_{N^2t}^l(k)}{1+\eta_{N^2t}^l(k)} - \frac{\eta_{N^2t}^{\tau N}(k)}{1+\eta_{N^2t}^{\tau N}(k)} \right) dt \right| = 0.$$

A 1-block estimate will be sufficient for the cases when $\beta \neq 0$.

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Thank you!