Stability of Hilbert Lyapunov exponents

Anthony Quas

March 31st 2018

Bob

A remarkable paper

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Commun. Math. Phys. 121, 501–505 (1989)

Communications in Mathematical Physics

Density and Uniqueness in Percolation

R. M. Burton[1] and M. Keane[2]

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[2] Department of Mathematics and Informatics, Delft University of Technology, NL-2628 BL Delft, The Netherlands

Abstract. Two results on site percolation on the d-dimensional lattice, $d \geqq 1$ arbitrary, are presented. In the first theorem, we show that for stationary underlying probability measures, each infinite cluster has a well-defined density with probability one. The second theorem states that if in addition, the probability measure satisfies the finite energy condition of Newman and Schulman, then there can be at most one infinite cluster with probability one. The simple arguments extend to a broad class of finite-dimensional models, including bond percolation and regular lattices.

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The paper in mathematics I most wish I had written

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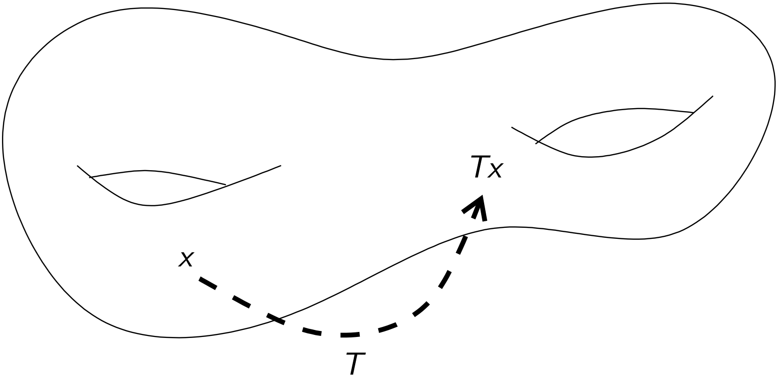

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Look for *equivariant* vector fields:

$$DT(x)v_1(x) /\!/ v_1(Tx)$$
$$DT(x)v_2(x) /\!/ v_2(Tx).$$

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The subspaces spanned by the $v_i(x)$ are the *Oseledets spaces*.

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A nice proof of this version was subsequently given by Raghunathan using the Kingman sub-additive ergodic theorem (or Furstenberg-Kesten) and singular values.

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Oseledets theorem: semi-invertible version [Froyland, Lloyd, Q]

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Let σ be a invertible ergodic measure-preserving transformation of a probability space $(\Omega, \mathbb{P})$. Let $\mathcal{L}_\omega$ be a *quasi-compact* family of operators on a Banach space X satisfy $\int \log \|\mathcal{L}_\omega\| \, d\mathbb{P} < \infty$.

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Then there exist $1 \leq k \leq \infty$ $\lambda_1 > \lambda_2 > \ldots > \lambda_k > -\infty$ and closed subspaces $V_1(\omega), \ldots, V_k(\omega), R(\omega)$ such that:

- $(V_i), R$ is a decomposition of X: $V_1(\omega) \oplus \ldots \oplus V_k(\omega) \oplus R(\omega) = X$;
- $\mathcal{L}_\omega V_i(\omega) = V_i(\sigma\omega)$ for μ-a.e. x;
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Previous infinite-dimensional versions due to Ruelle, Mañé, Thieullen, Lian and Lu,... (essentially all 'invertible')

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The n step evolution of the ocean is given by

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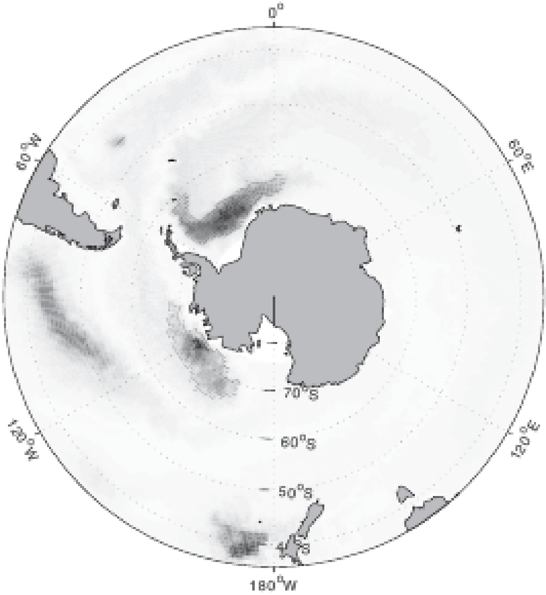
Based on an analogy with autonomous dynamical systems, we expect (sub)-level sets of Oseledets vectors to be *almost equivariant regions*.

A picture



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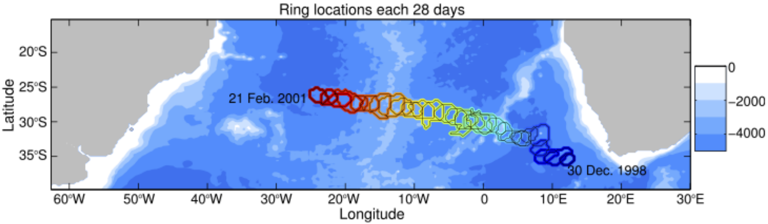
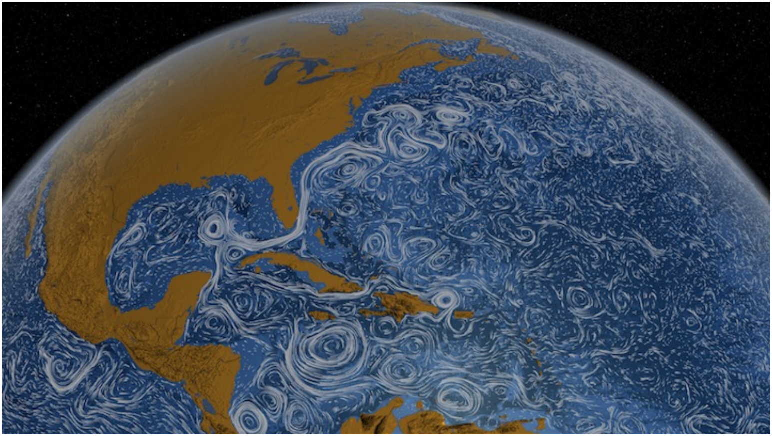


And another one: - See NASA YouTube Movie Perpetual Ocean



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The density operators $\mathcal{L}$ that we're looking at are very far from invertible (but the assumption that σ is invertible is fine), so we're naturally in the semi-invertible situation.

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Question: Can the sub-level sets obtained from the approximations $\mathcal{L}_\omega^\epsilon$ be shown to be close to those obtained from $\mathcal{L}_\omega$?

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He considers an ergodic system σ and a matrix cocycle A_ω taking values in $SL_2(\mathbb{R})$ (so $\lambda_1 + \lambda_2 = 0$). If $V_1(\omega)$ and $V_2(\omega)$ are not uniformly separated, then there exist arbitrarily small perturbations of the cocycle so that $\lambda_1^\epsilon = \lambda_2^\epsilon = 0$.

Bochi and Viana also proved higher-dimensional versions.

Answer 2: Yes

Theorem[Froyland, González-Tokman, Q] Suppose σ is an invertible measure-preserving transformation and (A_ω) is a $M_d(\mathbb{R})$-valued cocycle.

Then if the cocycle is perturbed by adding i.i.d. absolutely continuous noise to the matrices, $A_\omega^\epsilon = A_\omega + \epsilon \cdot \mathit{Noise}$, then

- $\lambda_i^\epsilon \to \lambda_i$
- $V_i^\epsilon(\omega) \to V_i(\omega)$ in probability

The proof uses ideas from an earlier proof due to Ledrappier and Young in the case where the A_ω and A_ω^{-1} are uniformly bounded.

Answer 3: Yes

Theorem[Froyland, González-Tokman, Q] Suppose σ is an invertible measure-preserving transformation and $\mathcal{L}_\omega$ is an (exponentially) Hilbert-Schmidt cocycle.

Then if $\mathcal{L}_\omega$ is perturbed by adding i.i.d. faster decaying Gaussian perturbations, $\mathcal{L}_\omega = \mathcal{L}_\omega + \epsilon\Delta$, then

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One slide about the proof

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This is a sub-additive quantity: $\Xi_k(\mathcal{L}_1\mathcal{L}_2) \leq \Xi_k(\mathcal{L}_1) + \Xi_k(\mathcal{L}_2)$. So

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We introduce an approximately super-additive quantity $\tilde{\Xi}_k(\mathcal{L}) = \mathbb{E}\Xi_k(\Delta\mathcal{L}\Delta)$ and work (hard!) to estimate $\Xi_k - \tilde{\Xi}_k$.

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Now: split blocks into good blocks and bad blocks. Bad blocks = rare; almost no control. Good blocks: Very strong attraction towards maximally expanding subspace.

Use the ϵ perturbation to steer you back towards the good directions in case you're deeply in the weeds (cost = $O(\log\epsilon \times L^1)$, but do this once every $o(1/|\log\epsilon|)$ steps); estimate how deeply in the weeds you can be using a corollary of Kingman. ($\mathbb{E}cost = O(\log\epsilon)$, but do this once every $o(1/|\log\epsilon|)$ steps.)