Sofic and percolative entropies of Gibbs measures on regular infinite trees

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Ising model on finite graphs

- Finite graph $G = (V, E)$.
- $\mathcal{F} = \big\{\{v_1, v_2\} : \{v_1, v_2\} \text{ is an edge}\big\}$.
- Set of spins $\{-1, 1\}^V$.
- Interaction Φ defined on the edges. For spin configuration $\sigma \in \{-1, 1\}^V$,

$$\Phi\left(\sigma_{\{v_1, v_2\}}\right) = \beta \sigma_{v_1} \sigma_{v_2}.$$

- β called *inverse temperature*.
- Ising measure μ defined as

$$\mu(\sigma) \propto \exp\left\{\sum_{\{v_1, v_2\} \in E} \beta \sigma_{v_1} \sigma_{v_2}\right\}, \quad \sigma \in \{-1, 1\}^V.$$

- The exponent, called *Hamiltonian*, denotes total energy of system.

Ising model on infinite graphs

- Cannot be defined directly as Hamiltonian, being an infinite sum, may diverge.
- Defined via consistency conditions – *DLR equations*.
- $G = (V, E)$ infinite graph. Call μ on $\{-1, 1\}^V$ a *Gibbs state* for Ising model if for every finite $U \subseteq V$ and spin configuration η on $V \setminus U$, marginal of μ on U coincides with finite Gibbs measure on U with boundary condition η.
- Ising measure μ need not be unique on infinite graphs.

The infinite graph we are interested in

- The infinite d-regular tree, T_d.
- Root of T_d denoted ϕ.
- For $r \in \mathbb{N}$, let T_d^r denote closed ball of radius r around ϕ.
- Let δT_d^r denote the boundary of T_d^r, i.e. all vertices at distance r from ϕ.

Ising measures on T_d

- Let μ_+^r Gibbs measure on T_d^r conditioned on all positive spins on δT_d^r. Let μ^+ be weak* limit of μ_+^r as $r \to \infty$.
- Let μ_-^r Gibbs measure on T_d^r conditioned on all negative spins on δT_d^r. Let μ^- be weak* limit of μ_-^r as $r \to \infty$.
- Let μ^r Gibbs measure on T_d^r with no restrictions on boundary conditions (called *free boundary*). Let μ be weak* limit of μ^r as $r \to \infty$.
- When Ising measure non-unique, the marginals at root ϕ of μ_+^r, μ_-^r and μ^r, even for r large, are distinct. The impact of boundary conditions does not decay with $r \to \infty$.
- Exists critical β_c such that for lower β, there is a unique Ising Gibbs measure.

More on the set-up for our result

- Finite d-regular graphs $G_n = (V_n, E_n)$.
- $\{G_n\}$ locally converges to T_d, i.e. for each $r \in \mathbb{N}$ and uniformly random I in V_n,

$$P[\text{neighbourhood } B(I,r) \text{ isomorphic to } T_d(r)] \to 1 \text{ as } n \to \infty.$$

  $B(I,r)$ is the closed ball of radius r around I.
- In ergodic theory, such a sequence called *sofic approximation* to T_d.

Aim of the result

- No direct definition of entropy for Ising measure μ on T_d, since infinite sample space $\{-1, 1\}^{T_d}$.
- Two notions of entropy available:
  - Use Shannon entropies $H(\mu_n)$ of Gibbs measures μ_n on G_n, where μ_n obtained from μ via a suitable *pull-back*.
  - Use *percolative entropy* $H_{\text{perc}}(\mu)$ of μ – function of T_d and Φ alone.
- We show, under strong mixing conditions, that $|V_n|^{-1}H(\mu_n)$ converges to $H_{\text{perc}}(\mu)$ as $n \to \infty$.
- As strong mixing conditions used, our result is true in high temperature regimes.

Shannon entropy

- $G = (V, E)$ finite graph.
- μ Ising measure on G.
- Shannon entropy $H(\mu)$ defined as

$$H(\mu) = -\sum_{\sigma \in \{-1,1\}^V} \mu(\sigma) \log \mu(\sigma).$$

- *Specific entropy* defined as $|V|^{-1} H(\mu)$.

Percolative entropy – definition

- μ Ising measure on T_d. Random configuration $\sigma \sim \mu$.
- For $S \subseteq T_d$, set $H_\mu\left(\sigma_\phi | \sigma_{S\setminus\phi}\right)$ Shannon entropy at root ϕ conditioned on configuration $\sigma_{S\setminus\phi} \in \{-1,1\}^{S\setminus\phi}$.
- S random subset of T_d, where each vertex included, independently, with probability p. Let θ_p denote law of S.
- *Percolative entropy* defined as

$$H_{\text{perc}}(\mu) = \int_0^1 \int_{S\subseteq T_d} H_\mu\left(\sigma_\phi | \sigma_{S\setminus\phi}\right) \theta_p(dS)dp.$$

Story behind percolative entropy

- Introduced by ergodic theorist John C. Kieffer in "A generalized Shannon-McMillan theorem for the action of an amenable group on a probability space".
- Γ a countable group.
- $\{U_g : g \in \Gamma\}$ collection of i.i.d. $U[0,1]$ random variables.
- Almost surely all distinct – hence induce random total ordering on Γ as follows: $g \prec h$ if $U_g < U_h$.
- Total entropy per element in Γ can be defined, via chain rule, by the formal average

$$\frac{1}{|\Gamma|} \sum_{g \in \Gamma} H\big(U_g | \{U_h : h \prec g\}\big).$$

- Percolative entropy is a rigorous version of the above, and makes sense even when $|\Gamma| = \infty$.

Our result

Theorem (Austin, P.)

- *μ Ising measure on T_d.*
- *μ exhibits strong spatial mixing (strong spatial mixing $\Longrightarrow$ weak spatial mixing $\Longrightarrow$ uniqueness of Ising measure).*
- *μ_n Gibbs measure on $\{-1,1\}^{V_n}$ derived from μ via suitable pull-back map.*

Then the limit of specific Shannon entropies of μ_n's equals percolative entropy of μ, i.e.

$$\lim_{n\to\infty} \frac{1}{|V_n|} H(\mu_n) = H_{\text{perc}}(\mu).$$

Pull-back maps

- Fix $v \in V_n$ and any surjection $\varphi_v : T_d \to V_n$ with $\varphi_v(\phi) = v$.
- For configuration $\sigma \in A^{V_n}$, set *pull-back configuration centered at* v as

$$\Pi_{v,\varphi_v}(\sigma) = \left(\sigma_{\varphi_v(g)} : g \in T_d\right).$$

- Note that $\Pi_{v,\varphi_v}(\sigma) \in A^{T_d}$.
- Gibbs measure μ_n defined on A^{V_n} as

$$\mu_n(\sigma) \propto \exp\left\{\sum_{v \in V_n} \sum_{u \sim \phi} \beta \sigma_{\varphi_v(u)} \sigma_v\right\}.$$

Strong spatial mixing

- For $r \in \mathbb{N}$, and η spin configuration on $T_d \setminus T_d^r$, let $\mu_{\phi,T_d^r,\eta}$ denote the marginal of the Ising measure μ at root ϕ conditioned on η.
- μ exhibits *strong spatial mixing* if

$$\max_{\eta,\tau} \left|\left| \mu_{\phi,T_d^r,\eta} - \mu_{\phi,T_d^r,\tau} \right|\right|_{\mathrm{TV}} \to 0,$$

  where maximum taken over all spin configurations η, τ on $T_d \setminus T_d^r$.
- Our definition is more general, where we add arbitrary self-interactions, and convergence is uniform in this self-interaction.

Strong spatial mixing for well-known models

- In the uniqueness regime, Ising model on T_d exhibits strong spatial mixing, which can be deduced from results given in Noam Berger, Claire Kenyon, Elchanan Mossel, and Yuval Peres. "Glauber dynamics on trees and hyperbolic graphs".
- Dror Weitz in "Counting independent sets up to the tree threshold" showed strong spatial mixing for independent set model on T_d with activity parameter λ when $\lambda \leq \lambda_c(d) = \frac{(d-1)^{d-1}}{(d-2)^d}$.

Generalization and further questions of interest

Our result holds for very general statistical models. The interaction should be translation invariant, of bounded range, and exhibit strong spatial mixing.

Further questions:

- What about removing assumption of strong spatial mixing? We no longer necessarily have uniqueness of Gibbs measure on T_d.
- Related to above – what happens at low temperatures?
- What about random trees (may be with bounded degree)?

Suggestions of other questions are most welcome.

Thank you!