Anomalous Diffusion and the Generalized Langevin Equation

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Frontier Probability Days, Oregon State University
March 30, 2018

Overview

The Generalized Langevin Equation (GLE)

Formal definition:

$$m\ddot{x}(t) = -\gamma\dot{x}(t) - \Phi'(x(t)) - \int_{-\infty}^{t} K(t-s)\dot{x}(s)\mathrm{d}s + F(t) + \sqrt{2\gamma}\dot{W}(t)$$

The GLE models the motion of microparticles moving in viscoelastic fluids.

1. Introduction: Anomalous diffusion and the linear GLE
2. GLE in a non-linear potential well with a special class of power-law memory kernels

Goal: Investigating unique ergodicity in non-linear potential wells.

I. Anomalous diffusion and the linear GLE
(joint work with S. McKinley)

Linear GLE

Classical Langevin Equation (LE): Describes the diffusion of a particle with mass m in a viscous medium.

$$m\dot{v}(t) = -\gamma v(t) + \sqrt{2\gamma}\dot{W}(t),$$

$$\begin{aligned} v(t) &: \text{velocity of the particle at time } t \\ \gamma &: \text{drag constant} \\ W(t) &: \text{standard Brownian Motion} \end{aligned}$$

Note: take $k_B T = 1$ throughout the talk.

$$x(t) := \int_0^t v(s)ds, \text{ position at time } t$$

$\Rightarrow$ Mean-Squared Displacement (MSD) $\mathbb{E}\left[x^2(t)\right] \sim t, \quad t \to \infty$,
(Asymptotically diffusion)
Here, $f(t) \sim g(t)$, $t \to \infty$ means $\lim_{t\to\infty} f(t)/g(t) = c \in (0, \infty)$.

Anomalous Diffusion is observed in nature, particularly in viscoelastic fluids, e.g. mucus and cytoplasm.

$$\mathbb{E}\left[x(t)^2\right] \sim t^{\alpha}, \quad t \to \infty, \quad \alpha \neq 1.$$

- $\alpha \in (0, 1)$: Sub-diffusion
- $\alpha > 1$: Super-diffusion

References

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- etc.

Assumption (Kubo 1966, Mason & Weitz 1995)

Fluctuation-Dissipation relationship

The memory kernel of the drag term is the same as the covariance of the noise term.

	Langevin	Generalized Langevin
Force due to drag	$-\gamma v(t)$	$-\int_{-\infty}^{t} K(t-s)v(s)\mathrm{d}s$
Thermal fluctuation	$\sqrt{2\gamma}\dot{W}(t)$	$F(t)$ with $\mathbb{E}\left[F(t)F(s)\right] = K(t-s)$

Generalized Langevin Equation (GLE)

$$\begin{cases} m\dot{v}(t) = -\int_{-\infty}^{t} K(t-s)v(s)\mathrm{d}s + F(t), \\ \mathbb{E}\left[F(t)F(s)\right] = K(t-s). \end{cases}$$

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GLE is able to produce subdiffusive diffusion.

Question: What type of memory $K(t) \Rightarrow$ subdiffusive diffusion?

Physicists' Guess:(Morgado, 2002) $x(t) = \int_0^t v(s)\mathrm{d}s$,

$$\alpha \in (0,1), \quad K(t) \sim t^{-\alpha}, \quad t \to \infty,$$

$$\Rightarrow \mathbb{E}\left[x(t)^2\right] \sim t^{\alpha}, \quad t \to \infty.$$

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$$\Rightarrow \mathbb{E}\left[x(t)^2\right] \sim t^{\alpha}, \quad t \to \infty.$$

Theorem (Kou, 2008)

$$\alpha \in (0,1),\ K(t) = |t|^{-\alpha} \Rightarrow \mathbb{E}\left[x(t)^2\right] \sim t^{\alpha}, \quad t \to \infty.$$

Stationary statistics of MSD

$$\begin{cases} m\dot{v}(t) = -\int_{-\infty}^{t} K(t-s)v(s)\mathrm{d}s + F(t), \\ \mathbb{E}\left[F(t)F(s)\right] = K(t-s). \end{cases}$$

Well-posedness:

- Theory of stationary random distributions, (Ito 1954) + Fourier Analysis
- (Weak) Solutions are understood as an operator
$$V : \mathrm{Dom}(V) \subset \mathcal{S}' \to L^2(\Omega)$$
- $v(t) := \langle V, \delta_t \rangle$ and $x(t) := \langle V, 1_{[0,t]} \rangle$ when they are well-defined.

Theorem 1 (McKinley, N., 2017, arXiv:1711.00560 (in review))

Under extra assumptions on the memory kernel $K(t)$,

	$K \in L^1(\mathbb{R})$	$K \notin L^1(\mathbb{R})$ $\exists \alpha \in (0,1)$, $K(t) \sim t^{-\alpha}$, $t \to \infty$
Asymptotics of MSD	$\mathbb{E}\left[x(t)^2\right] \sim t$, $t \to \infty$	$\mathbb{E}\left[x(t)^2\right] \sim t^{\alpha}$, $t \to \infty$

II. GLE in a potential well with a class of power-law memory kernels (joint work with N. Glatt-Holtz, D. Herzorg and S. McKinley)

GLE in potential well

Formal definition of GLE in a potential well with viscous drag: $m, \gamma > 0$,

$$\begin{cases} m\ddot{x}(t) = -\gamma\dot{x}(t) - \Phi'(x(t)) - \int_{-\infty}^{t} K(t-s)\dot{x}(s)\mathrm{d}s + F(t) + \sqrt{2\gamma}\dot{W}(t), \\ \mathbb{E}\left[F(t)F(s)\right] = K(t-s). \end{cases}$$

where

- $\Phi(x)$: potential well.
- $W(t)$: standard Brownian motion.
- $F(t)$: stationary Gaussian process with $\mathbb{E}\left[F(t)F(s)\right] = K(|t-s|)$.
- $K(t) \sim t^{-\alpha}$, $t \to \infty$, $\alpha > 0$.

$$\begin{cases} m\ddot{x}(t) = -\gamma\dot{x}(t) - \Phi'(x(t)) - \int_{-\infty}^{t} K(t-s)\dot{x}(s)\mathrm{d}s + F(t) + \sqrt{2\gamma}\dot{W}(t), \\ \mathbb{E}\left[F(t)F(s)\right] = K(t-s). \end{cases}$$

- Theory of linear stationary Gaussian processes does not apply.

Question: Does there exist a measure π on $\mathbb{R}^2$ s.t. $\forall f$ bounded

$$\text{“}\lim_{t\to\infty} \frac{1}{T}\int_0^T f(x(t), v(t))dt = \int_{\mathbb{R}^2} f(u, v)\pi(du, dv).\text{”}$$

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$\Rightarrow$ We will use a Markov representation of the GLE
Mori, 1965; Zwanzig, 1970 & 2001; Kupferman 2004; Goychuk, 2009; Pavliotis, 2014.

Well-posedness, stationary structure, unique ergodicity??

A toy model with a double-well potential

$$dx(t) = -\Phi'(x(t))dt + dW(t),$$

$$\Phi(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{4},$$

The density of the unique invariant probability measure:

$$p(x) \propto e^{-\Phi(x)}$$

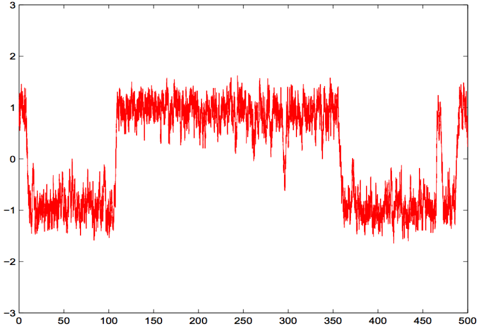
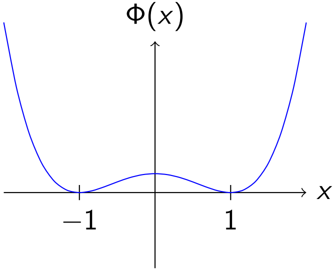


Figure: Trajectory of $x(t)$



Markov representation for exponentials

Theorem (Doob's Theorem)

$F(t)$ stationary Gaussian & $\mathbb{E}[F(t)F(s)] = K(|t-s|) = ce^{-\lambda|t-s|}$
$\Rightarrow F(t)$ is an Ornstein-Uhlenbeck process, i.e.

$$F(t) = \sqrt{c}\Big(e^{-\lambda t}F_0 + \sqrt{2\lambda}\int_0^t e^{-\lambda(t-s)}dW_1(s)\Big),$$

where $F_0 \sim \mathcal{N}(0,1)$.

$$m\ddot{x}(t) = -\gamma\dot{x}(t) - \Phi'(x(t)) + \sqrt{2\gamma}\dot{W}_0(t) - \int_{-\infty}^{t} K(t-s)\dot{x}(s)\mathrm{d}s + F(t)$$

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$$m\ddot{x}(t) = -\gamma\dot{x}(t) - \Phi'(x(t)) + \sqrt{2\gamma}\dot{W}_0(t)$$
$$- c\int_{-\infty}^t e^{-\lambda(t-s)}\dot{x}(s)\mathrm{d}s - \sqrt{c}e^{-\lambda t}F_0 - \sqrt{2c\lambda}\int_0^t e^{-\lambda(t-s)}\mathrm{d}W_1(s)$$

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where $F_0 \sim \mathcal{N}(0,1)$.

$$m\ddot{x}(t) = -\gamma\dot{x}(t) - \Phi'(x(t)) + \sqrt{2\gamma}\dot{W}_0(t)$$
$$-\sqrt{c}\Big[\underbrace{\sqrt{c}\int_{-\infty}^t e^{-\lambda(t-s)}\dot{x}(s)\mathrm{d}s + e^{-\lambda t}F_0 + \sqrt{2\lambda}\int_0^t e^{-\lambda(t-s)}\mathrm{d}W_1(s)}_{z}\Big]$$

$$\Rightarrow \begin{cases} m\ddot{x}(t) = -\gamma\dot{x}(t) - \Phi'(x(t)) + \sqrt{2\gamma}\dot{W}_0(t) - \sqrt{c}z(t) \\ \dot{z}(t) = -\lambda z(t) + \sqrt{c}\dot{x}(t) + \sqrt{2\lambda}\dot{W}_1(t) \end{cases}$$

Markov representation for finite sum of exponentials

$K(t) = \sum_{k=1}^{N} c_k e^{-\lambda_k |t|}, \ c_k, \lambda_k > 0, \ k = 1, \ldots, N$

$$\begin{cases} \mathrm{d}x(t) = v(t)\mathrm{d}t \\ m\,\mathrm{d}v(t) = \left(-\gamma v(t) - \Phi'(x(t)) - \sum_{k=1}^{N} \sqrt{c_k} z_k(t)\right)\mathrm{d}t + \sqrt{2\gamma}\mathrm{d}W_0(t), \\ \mathrm{d}z_k(t) = -\lambda_k z_k(t)\mathrm{d}t + \sqrt{c_k} v(t)\mathrm{d}t + \sqrt{2\lambda_k}\mathrm{d}W_k(t), k = 1, \ldots, N. \end{cases}$$

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- etc.

Markov approximation for power-tail memory kernel

Proposition (Abate, 1999)

Given $\alpha, \beta > 0$, *define* $K(t)$

$$K(t) = \sum_{k \geq 1} c_k e^{-\lambda_k |t|}, \text{ where } c_k = \frac{1}{k^{1+\alpha\beta}}, \ \lambda_k = \frac{1}{k^{\beta}}, \ k \geq 1.$$

Then, $K(t) \sim t^{-\alpha}, \quad t \to \infty.$

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Then, $K(t) \sim t^{-\alpha}, \quad t \to \infty.$

We arrive at

$$\begin{cases} \mathrm{d}x(t) = v(t)\mathrm{d}t \\ m\,\mathrm{d}v(t) = \Big(-\gamma v(t) - \Phi'(x(t)) - \sum_{k=1}^{\infty} \sqrt{c_k} z_k(t) \Big)\mathrm{d}t + \sqrt{2\gamma}\mathrm{d}W_0(t), \\ \mathrm{d}z_k(t) = -\lambda_k z_k(t)\mathrm{d}t + \sqrt{c_k} v(t)\mathrm{d}t + \sqrt{2\lambda_k}\mathrm{d}W_k(t), \quad k = 1, 2, \ldots. \end{cases}$$

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Recall Theorem 1 for linear GLE,

$$\begin{cases} m\dot{v}(t) = -\int_{-\infty}^{t} K(t-s)v(s)\mathrm{d}s + F(t) \\ \mathbb{E}\left[F(t)F(s)\right] = K(t-s) \end{cases}$$

$K \in L^1$, ($\alpha > 1$)	$\mathbb{E}\left[x(t)^2\right] \sim t, \quad t \to \infty$, (diffusion)
$\alpha \in (0,1)$, $K(t) \sim t^{-\alpha}$	$\mathbb{E}\left[x(t)^2\right] \sim t^{\alpha}, \quad t \to \infty$, (subdiffusion)
$\alpha = 1$, $K(t) \sim t^{-1}$	N/A

$$\begin{cases} \mathrm{d}x(t) = v(t)\mathrm{d}t, \qquad c_k = \dfrac{1}{k^{1+\alpha\beta}}, \qquad \lambda_k = \dfrac{1}{k^{\beta}}, \\ m\mathrm{d}v(t) = \Big(-\gamma v(t) - \Phi'(x(t)) - \sum_{k=1}^{\infty} \sqrt{c_k} z_k(t) \Big)\mathrm{d}t + \sqrt{2\gamma}\,\mathrm{d}W_0(t), \\ \mathrm{d}z(t) = -\lambda_k z_k(t)\mathrm{d}t + \sqrt{c_k}\, v(t)\mathrm{d}t + \sqrt{2\lambda_k}\,\mathrm{d}W_k(t), \quad k = 1, 2, \dots . \end{cases}$$

	Diffusion, $\alpha > 1$	Critical case $\alpha = 1$	Subdiffusion, $0 < \alpha < 1$
Well-posedness			
Existence of invariant measure			
Uniqueness of invariant measure			

Well-posedness

Potential $\Phi \in C^\infty(\mathbb{R})$ satisfies

$$c(\Phi(x) + 1) \geq x^2.$$

Examples:

1. Polynomial of even order, e.g., $\Phi(x) = x^{2n}$, $n \in \mathbb{N}^+$.
2. $\Phi(x) = e^{x^2}$

Definition

For $s \in \mathbb{R}$, define

$$\mathcal{H}_{-s} = \left\{ X = (x, v, z_1, \dots, z_k, \dots) : x^2 + v^2 + \sum_{k \geq 1} k^{-2s} z_k^2 < \infty \right\},$$

equipped with the norm $\|X\|^2_{\mathcal{H}_{-s}} = x^2 + v^2 + \sum_{k\geq 1} k^{-2s} z_k^2$.

Note: $\ell^2 = \mathcal{H}_0$.

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Proposition (Glatt-Holtz, Herzog, McKinley, N., 2018, in prep)

Under appropriate assumptions, for all initial conditions $X_0 \in \mathcal{H}_{-s}$, *there exists a unique strong solution* $X(\cdot, X_0) : \Omega \times [0, \infty) \to \mathcal{H}_{-s}$.

$$\begin{cases} \mathrm{d}x(t) = v(t)\mathrm{d}t, \qquad c_k = \dfrac{1}{k^{1+\alpha\beta}}, \qquad \lambda_k = \dfrac{1}{k^{\beta}}, \\ m\mathrm{d}v(t) = \Big(-\gamma v(t) - \Phi'(x(t)) - \sum_{k=1}^{\infty} \sqrt{c_k} z_k(t) \Big)\mathrm{d}t + \sqrt{2\gamma}\mathrm{d}W_0(t), \\ \mathrm{d}z_k(t) = -\lambda_k z_k(t)\mathrm{d}t + \sqrt{c_k} v(t)\mathrm{d}t + \sqrt{2\lambda_k}\mathrm{d}W_k(t), \quad k = 1, 2, \dots . \end{cases}$$

	Diffusion, $\alpha > 1$	Critical case $\alpha = 1$	Subdiffusion, $0 < \alpha < 1$
Well-posedness	✓		
Existence of invariant measure			
Uniqueness of invariant measure			

Invariant measure for finite-dimensional system

$$\begin{cases} \mathrm{d}x(t) = v(t)\mathrm{d}t \\ m\mathrm{d}v(t) = \Big(-\gamma v(t) - \Phi'(x(t)) - \sum_{k=1}^{N} \sqrt{c_k} z_k(t) \Big)\mathrm{d}t + \sqrt{2\gamma}\mathrm{d}W_0(t), \\ \mathrm{d}z_k(t) = -\lambda_k z_k(t)\mathrm{d}t + \sqrt{c_k} v(t)\mathrm{d}t + \sqrt{2\lambda_k}\mathrm{d}W_k(t), \quad k = 1, \dots, N. \end{cases}$$

Theorem (Pavliotis, 2014)

Let p_N be the density probability measure on $\mathbb{R}^{N+2}$ given by

$$p(x, v, z_1, \dots, z_N) \propto \exp\left\{ -\Phi(x) - \frac{mv^2}{2} - \sum_{k=1}^{N} \frac{z_k^2}{2} \right\}.$$

Then p_N is the density of the unique invariant probability measure for the finite-dimensional system.

Note: p_N does **not** depend on c_k, λ_k!

Existence of invariant measure for infinite-dimensional system

Definition

Denote by μ *the probability measure on* $\mathbb{R}^\infty$ *given by*

$$\mu = \left(c\, e^{-\Phi(x)} dx\right) \times \mathcal{N}(0, 1/m) \times \prod_{k \geq 1} \mathcal{N}(0, 1).$$

Note: $\mu(\mathcal{H}_{-s}) = \begin{cases} 1, & s > 1/2 \\ 0, & s \leq 1/2 \end{cases}$, where

$\mathcal{H}_{-s} = \left\{ X = (x, v, z_1, \dots, z_k, \dots) : x^2 + v^2 + \sum_{k\geq 1} k^{-2s} z_k^2 < \infty \right\}$

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Theorem (Glatt-Holtz, Herzog, McKinley, N., 2018, in prep)

Under appropriate assumptions and $s > 1/2$, μ is an invariant measure.

Finite-dimensional space: (Pavliotis 2014) it suffices to check that

$$\mathcal{L}^* p = 0,$$

where p is the density of the candidate measure and $\mathcal{L}^*$ is the dual of $\mathcal{L}$, the infinitesimal generator.

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Infinite-dimensional space: $\forall \psi \in C_b^2(\mathcal{H}_{-s})$, we show that

$$\langle \psi, \mathcal{L}^* \mu \rangle := \int_{\mathcal{H}_{-s}} \mathcal{L}\psi(X)\mu(\mathrm{d}X) = 0.$$

Then, by an approximating argument,

$$\int_{\mathcal{H}_{-s}} \mathcal{P}_t \psi(X)\mu(\mathrm{d}X) = \int_{\mathcal{H}_{-s}} \psi(X)\mu(\mathrm{d}X),$$

where $\mathcal{P}_t \psi(X) = \mathbb{E}_X[\psi(X(t))]$ is the Markov process associated with $\mathcal{L}$.

$$\begin{cases} \mathrm{d}x(t) = v(t)\mathrm{d}t, \qquad c_k = \dfrac{1}{k^{1+\alpha\beta}}, \qquad \lambda_k = \dfrac{1}{k^{\beta}}, \\ m\mathrm{d}v(t) = \Big(-\gamma v(t) - \Phi'(x(t)) - \sum_{k=1}^{\infty} \sqrt{c_k} z_k(t) \Big)\mathrm{d}t + \sqrt{2\gamma}\mathrm{d}W_0(t), \\ \mathrm{d}z_k(t) = -\lambda_k z_k(t)\mathrm{d}t + \sqrt{c_k} v(t)\mathrm{d}t + \sqrt{2\lambda_k}\mathrm{d}W_k(t), \quad k = 1, 2, \dots . \end{cases}$$

	Diffusion, $\alpha > 1$	Critical case $\alpha = 1$	Subdiffusion, $0 < \alpha < 1$
Well-posedness	✓		
Existence of invariant measure	$\mu = \left(ce^{-\Phi(x)}\mathrm{d}x\right) \times \mathcal{N}(0, 1/m) \times \prod_{k\geq 1} \mathcal{N}(0,1)$ is invariant		
Uniqueness of invariant measure			

Uniqueness of invariant measure in diffusion

$$\begin{cases} dx(t) = v(t)dt, & c_k = \frac{1}{k^{1+\alpha\beta}}, \quad \lambda_k = \frac{1}{k^\beta}, \\ m dv(t) = \Big(-\gamma v(t) - \Phi'(x(t)) - \sum_{k=1}^{\infty} \sqrt{c_k} z_k(t) \Big) dt + \sqrt{2\gamma} dW_0(t), \\ dz_k(t) = -\lambda_k z_k(t)dt + \sqrt{c_k} v(t)dt + \sqrt{2\lambda_k} dW_k(t), \quad k = 1, 2, \dots . \end{cases}$$

Theorem (Glatt-Holtz, Herzog, McKinley, N., 2018, in prep)

Under appropriate assumptions and assume that $\alpha > 1$, μ is the unique invariant probability measure.

Strategy: Asymptotic coupling

Goal: (Hairer, 2002) uniqueness is implied if we can show that $\forall X_0, \widetilde{X}_0 \in \mathcal{H}_{-s}$,

$$\mathbb{P}\Big\{ \lim_{t\to\infty} \|X(X_0,t) - \widetilde{X}(\widetilde{X}_0,t)\|_{-s} = 0 \Big\} = 1$$

This holds if Φ is a 4th-degree polynomial.

Strategy: Asymptotic coupling

Goal: (Hairer, 2002) uniqueness is implied if we can show that $\forall X_0, \widetilde{X}_0 \in \mathcal{H}_{-s}$,

$$\mathbb{P}\left\{ \lim_{t\to\infty} \|X(X_0,t) - \widetilde{X}(\widetilde{X}_0,t)\|_{-s} = 0 \right\} = 1$$

This holds if Φ is a 4th-degree polynomial.
More recent results show that

$$\mathbb{P}\left\{ \lim_{t\to\infty} \|X(X_0,t) - \widetilde{X}(\widetilde{X}_0,t)\|_{-s} = 0 \right\} > 0 \text{ works!}$$

- Hairer, Mattingly, Scheutzow, 2011; Glatt-Holtz, Richards, Mattingly, 2015; Kulik, Scheutzow, 2016.

Constructing stochastic control.

$$\mathrm{d}X(t) = LX(t)\mathrm{d}t + \Psi(X(t))\mathrm{d}t + B\,\mathrm{d}W(t)$$
$$\mathrm{d}\widetilde{X}(t) = L\widetilde{X}(t)\mathrm{d}t + \Psi(\widetilde{X}(t))\mathrm{d}t + B\,\underbrace{\left(\mathrm{d}W(t) + U(X(t),\widetilde{X}(t))1_{(t\le\tau)}\mathrm{d}t\right)}_{d\widetilde{W}(t)}$$

Setting $\overline{X}(t) = X(t) - \widetilde{X}(t)$. Pick $U(X(t),\widetilde{X}(t))$ and τ s.t.

- $U(X(t),\widetilde{X}(t))$ forces $\overline{X}(t) \to 0,\ t \to \infty$.
- τ shuts down $U(X(t),\widetilde{X}(t))$ if $\overline{X}(t) \not\to 0$

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a. Girsanov shift $\widetilde{W}(t) := W(t) + \int_0^t U(X(r),\widetilde{X}(r))1_{(r\le\tau)}\mathrm{d}r$ satisfies

$$\widetilde{W}(\cdot) << W(\cdot) \text{ on } [\mathbf{0},\infty).$$

b. $\mathbb{P}\left\{\|\overline{X}(t)\|_{\mathcal{H}_{-s}} \to 0,\ t \to \infty | \tau = \infty\right\} = 1.$
"τ is never activated $\Leftrightarrow \overline{X}(t) \to 0\ t \to \infty$"

c. $\mathbb{P}\{\tau = \infty\} > 0$. "There is a chance that τ is never activated"

Choice of U: $U(X(t), \widetilde{X}(t)) = (0, u(X(t), \widetilde{X}(t)), 0, \dots)$

$$\begin{cases} d\overline{x}(t) = \overline{v}(t)dt, \\ m d\overline{v}(t) = \Big(-\gamma \overline{v}(t) - [\Phi'(x(t)) - \Phi'(\widetilde{x}(t))] - \sum_{k=1}^{\infty} \sqrt{c_k}\,\overline{z}_k(t) \Big) dt \\ \qquad + u(X(t), \widetilde{X}(t))dt, \\ d\overline{z}_k(t) = -\lambda_k \overline{z}_k(t)dt + \sqrt{c_k}\,\overline{v}(t)dt, \quad k = 1, 2, \dots. \end{cases}$$

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$$

- $u(X(t), \widetilde{X}(t)) = -c\overline{x}(t) + [\Phi'(x(t)) - \Phi'(\widetilde{x}(t))] + \sum_{k=1}^{\infty} \sqrt{c_k}\,\overline{z}_k(t)$

  "u cancels the non-linear term and the memory"

  $$\Rightarrow \begin{cases} d\overline{x}(t) = \overline{v}(t)dt, \\ d\overline{v}(t) = (-\gamma\overline{v}(t) - c\overline{x}(t))dt, \end{cases}$$ deterministic, **dissipative**

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- The structure of Φ + Lyapunov function +choice of τ

  $\Rightarrow \overline{x}(t), \overline{v}(t)$ force $\overline{z}_k(t) \to 0$, $t \to 0$

  $\Rightarrow$ (a) + (b).

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  $\Rightarrow \overline{x}(t), \overline{v}(t)$ force $\overline{z}_k(t) \to 0$, $t \to 0$

  $\Rightarrow$ (a) + (b).
- (c) requires $\int_0^\infty K(t)dt < \infty \Leftrightarrow \alpha > 1$, (recall $K(t) \sim t^{-\alpha}$, $t \to \infty$).

$$\begin{cases} \mathrm{d}x(t) = v(t)\mathrm{d}t, & c_k = \frac{1}{k^{1+\alpha\beta}}, \quad \lambda_k = \frac{1}{k^{\beta}}, \\ m\mathrm{d}v(t) = \Big(-\gamma v(t) - \Phi'(x(t)) - \sum_{k=1}^{\infty} \sqrt{c_k} z_k(t) \Big)\mathrm{d}t + \sqrt{2\gamma}\mathrm{d}W_0(t), \\ \mathrm{d}z_k(t) = -\lambda_k z_k(t)\mathrm{d}t + \sqrt{c_k} v(t)\mathrm{d}t + \sqrt{2\lambda_k}\mathrm{d}W_k(t), \quad k = 1, 2, \dots . \end{cases}$$

<table>
<tr><th></th><th>Diffusion,
$\alpha > 1$</th><th>Critical case
$\alpha = 1$</th><th>Subdiffusion,
$0 < \alpha < 1$</th></tr>
<tr><td>Well-posedness</td><td colspan="3">✓</td></tr>
<tr><td>Existence of invariant measure</td><td colspan="3">$\mu = \Big(ce^{-\Phi(x)}\mathrm{d}x\Big) \times \mathcal{N}(0, 1/m) \times \prod_{k\geq 1} \mathcal{N}(0,1)$
is invariant</td></tr>
<tr><td>Uniqueness of invariant measure</td><td>✓</td><td colspan="2">Open question</td></tr>
</table>

Summary

- Use a Markovian system to represent GLE when memory kernel $K(t) \sim t^{-\alpha}$, $t \to \infty$ admits a form of infinite sum of exponentials.
- There exists an invariant structure for the Markovian system.
- Unique ergodicity is obtained in diffusive regime ($\alpha > 1$). The marginal distribution in (x, v) of the invariant probability measure is given by
$$\pi(x, v) \propto \exp\left\{-\Phi(x) - \frac{mv^2}{2}\right\},$$
which is independent of c_k, λ_k.
- Unique ergodicity when $\alpha \in (0, 1]$ remains open question.

Thank You!