Particle representations for SPDEs with boundary conditions:
An example.

Chris Janjigian

University of Utah

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Example: A SAC equation on the unit ball

Let D denote the open unit ball in $\mathbb{R}^d$ and consider the SPDE

$$\begin{aligned} \partial_t v &= \Delta v + v - v^3 + \dot{W}_\epsilon \\ v(0,x) &= h(x), \qquad x \in D \\ v(t,x) &= g(x), \qquad x \in \partial D, t > 0. \end{aligned}$$

where g is continuous, h is bounded and

$$W_\epsilon(t,x) = \int_{(0,t]\times\mathbb{R}^d} \psi_\epsilon(x-u) W(ds \otimes du)$$

is a spatially mollified space-time white noise.

SAC on the disk - weak form

Denote by π the normalized Lebesgue measure on D and β is the surface measure with the same prefactor, and η the unit inward normal. v should solve

$$
\begin{aligned}
\int_D \varphi(t,x)v(t,x)\pi(dx) = & \int_D \varphi(0,x)h(x)\pi(dx) \\
& + \int_0^t \int_D (\partial_t \varphi(s,x) + \Delta\varphi(s,x))v(s,x)\pi(dx)ds \\
& + \int_0^t \int_D \varphi(s,x)(v(s,x) - v(s,x)^3)\pi(dx)ds \\
& + \int_{(0,t]\times\mathbb{R}^d} \int_D \varphi(s,x)\psi_\epsilon(x-u)\pi(dx)W(ds\otimes du) \\
& + \int_0^t \int_{\partial D} g(x)\nabla\varphi(s,x)\cdot\eta(x)\beta(dx)ds
\end{aligned}
$$

for all $\varphi \in C_b^2(\mathbb{R}_+ \times \overline{D})$ with $\varphi|_{\mathbb{R}_+ \times \partial D} = 0$.

Weighted particle representations

Suppose that X_i are i.i.d. (and independent of W) stationary reflected diffusions on D with stationary distribution π and we introduce integrable weights A_i so that $\{A_i, X_i\}_{i=1}^{\infty}$ forms an exchangeable sequence. By de Finetti's theorem, a signed measure valued process $V(t)$ exists satisfying

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} A_i(t)\delta_{X_i(t)} = V(t)$$

and one can check that for φ compactly supported in D, a process of densities exists:

$$\int_D \varphi(x)V(t)(dx) = \int_D \varphi(x)v(t,x)\pi(dx).$$

A **particle representation** is a family $\{X_i, A_i\}_{i=1}^{\infty}$ with the property that the process of densities $v(t,x)$ above is a weak solution to the SPDE.

Particle representations - the particles

Suppose that $\{X_i\}_{i=1}^{\infty}$ are stationary normally reflected (rate 2) Brownian motions in D and let $\varphi \in C_b^2(\mathbb{R}_+ \times \overline{D})$. Then

$$
\begin{aligned}
\varphi(t, X_i(t)) = \varphi(0, X_i(0)) &+ \int_0^t (\partial_t + \Delta)\varphi(s, X_i(s))ds \\
&+ \int_0^t \nabla\varphi(s, X_i(s)) \cdot \eta(X_i(s))dL_i^X(s) \\
&+ \int_0^t \nabla\varphi(s, X_i(s)) \cdot dB_i(s)
\end{aligned}
$$

One can check that π is the stationary distribution for this process and β is the boundary measure.

Weights - first idea: ignore the boundary condition

Suppose that

$$A_i(t) = h(X_i(0)) + \int_0^t (1 - v(s, X_i(s))^2) A_i(s) ds + \int_{(0,t]\times\mathbb{R}^d} \psi_\epsilon(X_i(s) - u) W(ds \otimes du).$$

Then for $\varphi \in C_c^2(D)$,

$$\begin{aligned}\varphi(t, X_i(t)) A_i(t) = {} & \varphi(0, X_i(0)) h(X_i(0)) \\ & + \int_0^t (1 - v(s, X_i(s))^2) \varphi(s, X_i(s)) A_i(s) ds \\ & + \int_{(0,t]\times\mathbb{R}^d} \varphi(s, X_i(s)) \psi_\epsilon(X_i(s) - u) W(ds \otimes du) \\ & + \int_0^t (\partial_t + \Delta) \varphi(s, X_i(s)) A_i(s) ds \\ & + \int_0^t A_i(s) \nabla \varphi(s, X_i(s)) \cdot dB_i(s)\end{aligned}$$

Weights - first idea: ignore the boundary condition

Averaging yields

$$\begin{aligned}\int_D \varphi(x)v(t,x)\pi(dx) = &\int_D \varphi(x)h(x)\pi(dx)\\ &+ \int_0^t \int_D (1 - v(s,x)^2)\varphi(x)v(s,x)\pi(dx)ds\\ &+ \int_{(0,t]\times\mathbb{R}^d} \int \varphi(x)\psi_\epsilon(x-u)\pi(dx)W(ds\otimes du)\\ &+ \int_0^t \int_D (\partial_t + \Delta)\varphi(x)v(s,x)\pi(dx)ds.\end{aligned}$$

Weights - second idea: how to include the boundary term

Let $\tau_i(t) = \sup\{s \leqslant t : X_i(s) \in \partial D\} \vee 0$ and suppose that

$$
\begin{aligned}
A_i(t) &= g(X_i(\tau_i(t)))1_{\{\tau_i(t)>0\}} + h(X_i(0))1_{\{\tau_i(t)=0\}} \\
&\quad + \int_{\tau_i(t)}^t (1 - v(s, X_i(s))^2) A_i(s)\,ds \\
&\quad + \int_{(\tau_i(t),t]\times\mathbb{R}^d} \psi_\epsilon(X_i(s) - u) W(ds \otimes du)
\end{aligned}
$$

1. Intuitively: Whenever $X_i(t) \in \partial D$, the value of $A_i(t)$ resets to $g(X_i(t))$ and then the process starts evolving again.
2. $A_i(t)$ is a difference of the previous expression for A_i and a process that only changes when $\tau_i(t)$ changes; i.e. when $X_i(t) \in \partial D$.
3. $A_i(t)$ is not a semi-martingale.

"Ito's formula"

We cannot directly apply Ito's formula, but for $\varphi \in C_b^2(\mathbb{R}_+ \times \overline{D})$ with $\varphi|_{\mathbb{R}_+ \times \partial D} = 0$, we can show that for A_i as above

$$\begin{aligned}\varphi(t, X_i(t))A_i(t) &= \varphi(0, X_i(0))h(X_i(0)) \\ &+ \int_0^t (1 - v(s, X_i(s))^2)\varphi(s, X_i(s))A_i(s)ds \\ &+ \int_{(0,t]\times\mathbb{R}^d} \varphi(s, X_i(s))\psi_\epsilon(X_i(s) - u)W(ds \otimes du) \\ &+ \int_0^t (\partial_t + \Delta)\varphi(s, X_i(s))A_i(s)ds \\ &+ \int_0^t g(X_i(s))\nabla\varphi(s, X_i(s)) \cdot \eta(X_i(s))dL_i^X(s) \\ &+ \int_0^t A_i(s)\nabla\varphi(s, X_i(s)) \cdot dB_i(s)\end{aligned}$$

Intuitively, this works because we only run into issues is when $\tau_i(t)$ is changing. This occurs when $X_i(t) \in \partial D$, in which case $\varphi(X_i(t)) = 0$.

Existence/uniqueness of the representation/SPDE

Theorem (Crisan, J., Kurtz)

There exists a unique solution $\{A_i\}_{i=1}^{\infty}$ to the system of equations

$$\begin{aligned} A_i(t) &= g(X_i(\tau_i(t)))1_{\{\tau_i(t)>0\}} + h(X_i(0))1_{\{\tau_i(t)=0\}} \\ &\quad + \int_{\tau_i(t)}^{t} (1 - v(s, X_i(s))^2) A_i(s) ds \\ &\quad + \int_{(\tau_i(t),t]\times\mathbb{R}^d} \psi_\epsilon(X_i(s) - u) W(ds \otimes du) \end{aligned}$$

where $v(t,x)$ is given by the process of densities for the measures

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} A_i(t)\delta_{X_i(t)} := v(t,x)\pi(dx).$$

$v(t,x)$ is the unique weak solution to the SPDE which satisfies $\sup_{t\leqslant T} \mathbb{E}[\int_D e^{\epsilon|v(t,x)|^2\pi(dx)}] < \infty$ and is compatible with the driving noise W.

Some key ideas: the basic problem

Suppose that U is an $\mathcal{F}_t^W$ adapted process and

$$
\begin{aligned}
A_i^U(t) = {} & g(X_i(\tau_i(t)))1_{\{\tau_i(t)>0\}} + h(X_i(0))1_{\{\tau_i(t)=0\}} \\
& + \int_{\tau_i(t)}^t (1 - U(s, X_i(s))^2) A_i(s) ds \\
& + \int_{(\tau_i(t),t]\times\mathbb{R}^d} \psi_\epsilon(X_i(s) - u) W(ds \otimes du).
\end{aligned}
$$

Let ΦU to be the process of densities satisfying

$$
\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n A_i^U(t)\varphi(X_i(t)) = \int_D \Phi U(t,x)\varphi(x)\pi(dx)
$$

We are looking for a fixed point of this map.

Some key ideas: density representation

If U is $\mathcal{F}_t^W$ adapted, then by the ergodic theorem we also have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} A_i^U(t)\varphi(X_i(t)) = \mathbb{E}[A_i^U(t)\varphi(X_i(t))|\sigma(W)].$$

Take test functions φ and F. Then, since ΦU will be a $\sigma(W)$ measurable process,

$$\begin{aligned}\mathbb{E}[\varphi(X_i(t))A_i^U(t)F(W)] &= \mathbb{E}\left[\int_D \varphi(x)\Phi U(t,x)\pi(dx)F(W)\right] \\ &= \mathbb{E}\left[\varphi(X_i(t))\Phi U(t,X_i(t))F(W)\right].\end{aligned}$$

This gives a representation:

$$\Phi U(t,X_i(t)) = \mathbb{E}[A_i^U(t)|\sigma(X_i(t)) \vee \sigma(W)],$$

Moment estimates and iteration

A Gronwall argument leads to the *a priori* bound for $t \leqslant T$,

$$|A_i^U(t)| \leqslant \left(\|g\|_\infty \vee \|h\|_\infty + 2 \sup_{0 \leqslant t \leqslant T} \left| \int_{(0,t] \times \mathbb{R}^d} \psi_\epsilon(X_i(s) - u) W(ds \otimes du) \right| \right) e^T.$$

This combined with Jensen's inequality implies that there is $\epsilon_T > 0$ such that for all $t \leqslant T$,

$$\begin{aligned} \mathbb{E}\left[\int_D e^{\epsilon_T \Phi U(t,x)^2} \pi(dx) \right] &= \mathbb{E}\left[e^{\epsilon_T \Phi U(t, X_i(t))^2} \right] \\ &= \mathbb{E}\left[e^{\epsilon_T \, \mathbb{E}[A_i^U(t) | \sigma(X_i(t)) \vee \sigma(W)]^2} \right] < \infty. \end{aligned}$$

This bound is a key ingredient to showing that an iterative scheme $\Phi^{(n)} U$ converges to a unique fixed point of the particle map.

Uniqueness of the non-linear SPDE

First, suppose that we consider the weak SPDE that is represented by $\{A_i^U, X_i\}$ when U is fixed.

$$
\begin{aligned}
\int_D \varphi(t,x)\Phi U(t,x)\pi(dx) = &\int_D \varphi(0,x)h(x)\pi(dx) \\
&+ \int_0^t \int_D (\partial_t \varphi(s,x) + \Delta\varphi(s,x))\Phi U(s,x)\pi(dx)ds \\
&+ \int_0^t \int_D \varphi(s,x)(1 - U(s,x)^2)\Phi U(s,x)\pi(dx)ds \\
&+ \int_{(0,t]\times\mathbb{R}^d} \int_D \varphi(s,x)\psi_\epsilon(x-u)\pi(dx)W(ds\otimes du) \\
&+ \int_0^t \int_{\partial D} g(x)\nabla\varphi(s,x)\cdot\eta(x)\beta(dx)ds
\end{aligned}
$$

The difference between two solutions $\Phi U^{(1)}$ and $\Phi U^{(2)}$ solves a linear PDE for which we can show uniqueness so long as U has subgaussian tails.

Suppose that V is the fixed point of the particle system we constructed. This solves the SPDE in the weak sense. Let U be any other weak solution with a density which satisfies

$$\mathbb{E}\left[\int_D e^{\epsilon_T U(t,x)^2}\pi(dx)\right] < \infty$$

for some $\epsilon_T > 0$ and $t \leqslant T$ and which is compatible with W. Take U as the input in A_i^U and define ΦU via

$$\Phi U(t,x)\pi(dx) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} A_i^U(t)\delta_{X_i(t)}.$$

Then ΦU solves the same weak linear SPDE as U and so $U = \Phi U$. But there is only one process which satisfies $V = \Phi V$ and therefore we have $U = V$.

A more general picture

The same construction works to give representations to (unique) weak-form solutions to

$$\begin{aligned} \partial_t v &= \mathcal{L}^* v + vG(v,x) + b(x,t) + \dot{W}_\epsilon \\ v(0,x) &= h(x), \qquad x \in D \\ v(t,x) &= g(x), \qquad x \in \partial D, t > 0. \end{aligned}$$

Conditions:

1. D should be to be open, bounded, connected, and sufficiently smooth (C^2 is sufficient).
2. $\mathcal{L}$ is a uniformly elliptic differential operator with bounded continuous coefficients associated to a reflecting diffusion with stationary distribution π and associated boundary measure β, $\mathcal{L}^*$ is the adjoint with respect to π.
3. $\|h\|_\infty, \|g\|_\infty, \|b\|_\infty < \infty$, $G(v,x) \leqslant C$

$$\sup_{v,x} \frac{|G(v,x)|}{1+|v|^2} < \infty, \qquad \sup_{v,x} \frac{|G(v_1,x) - G(v_2,x)|}{|v_1 - v_2|(1+|v_1|+|v_2|)} < \infty$$

Thanks!