Analysis of a Kraichnan-type Fluid Model

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The basic problem

We start the following initial-value problem on $\mathbb{R}_+ \times \mathbb{R}^2$

$$\frac{\partial \theta(t,x,y)}{\partial t} = \frac{\nu}{2}\Delta\theta(t,x,y) + \frac{\partial \theta(t,x,y)}{\partial y} V(t,x)\,, \quad x,y \in \mathbb{R}\,,$$

subject to a nice initial data.

- $V(t,x)$ is the perturbation(Reynolds 1883?)

$$V(t,x) = \underbrace{\mu(t,x)}_{\text{mean part}} + \underbrace{\widetilde{V}(t,x)}_{\text{fluctuation part}}$$

- We assume $\mu(t,x) = 0$, but it's not hard to extend to the general setting.
- We assume that $V(t,x)$ is a mean-zero generalized Gaussian random field with

$$\mathrm{Cov}[V(t,x)V(t',x')] = \delta(t-t')\rho(x-x')\,,$$

(Kraichnan 87' and earlier, ... 60's)
- $V(t,x)$ is in the distributional sense, not defined pointwise.

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- Two schools of existence and uniqueness theory:
  - Stratonovich type theory (Kunita, 93')
  - Itô-type theory (Chow '14, Rozovsky, mid '90s; Krylov, '90s-2000s)
- Technical setbacks:
  - The Stratonovich-style theory requires $\rho \in \cup_{\epsilon>0} C^{6+\epsilon}(\mathbb{R})$.
  - Qualitative analysis not readily available.
- For our analysis next, consider the equation($\nu_1, \nu_2 > 0$)

$$\partial_t \theta = \nu_1 \partial_x^2 \theta + \nu_2 \partial_y^2 \theta + \partial_y \theta V(t, x)$$

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We say that $(t,x,y) \to \theta(t,x,y)$ is a mild solution (Itô sense) to above equation if

- for every $t > 0$ and $x \in \mathbb{R}$, $y \to \theta(t,x,y)$ is a.s. C^1;
- and for all $t > 0$ and $x, y \in \mathbb{R}$,

$$\theta(t,x,y) = \int_{\mathbb{R}^2} p_t^{\nu_1}(x-a)p_t^{\nu_2}(y-b)\theta_0(a,b)dadb + \int_{\mathbb{R}_+\times\mathbb{R}^2} p_{t-s}^{\nu_1}(x-a)p_{t-s}^{\nu_2}(y-b)\partial_b\theta(s,a,b)V(s,a)dsdadb\,.$$

p_t^ν is heat kernel, the stochastic integral is in the sense of Walsh.

Classical theory: existence and uniqueness

- For equations of the form $\partial_t u(t,x) = \Delta u(t,x) + u(t,x)W(t,x)$, Picard iteration.
- Let $u_0(t,x) = p_t * u_0(x)$

$$\text{and} \quad u_{n+1}(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) u_n(s,y) W(s,y) dyds$$

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$$\begin{aligned}
&\mathbb{E}|u_{n+1}(t,x) - u_n(t,x)|^2 \\
=&\mathbb{E} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} p_{t-s}(x-y) p_{t-s}(x-y') \\
&\times \left(u_n(s,y) - u_{n-1}(s,y)\right) \left(u_n(s,y') - u_{n-1}(s,y')\right) \rho(y-y') dydy'ds \\
\le&\rho(0) \int_0^t \sup_{z\in\mathbb{R}} \mathbb{E}|u_n(s,z) - u_{n-1}(s,z)|^2 ds
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Existence and uniqueness

$\partial_t\theta = \nu_1\partial_x^2\theta + \nu_2\partial_y^2\theta + \partial_y\theta V(t,x)$ | $\textbf{Cov}[V(t,x), V(t',x')] = \delta(t-t')\rho(x-x')$

- For our SPDE, Fourier transform $U(t,x,\xi) = \int_{-\infty}^{\infty} e^{i\xi y}\theta(t,x,y)dy$

$$\Rightarrow \partial_t U(t,x,\xi) = \nu_1\partial_x^2 U(t,x,\xi) - \nu_2\xi^2 U(t,x,\xi) + i\xi U(t,x,\xi)V(t,x)\,.$$

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- For every $\xi \in \mathbb{R}$, there is a unique solution $u(t,x,\xi)$ such that for any $\epsilon \in (0,1), t > 0, x, \xi \in \mathbb{R}$,

$$\sup_{x\in\mathbb{R}} \|u(t,x,\xi)\|_{L^2(\Omega)} \leq \epsilon^{-1}\exp\left(\frac{\rho(0)\xi^2}{2(1-\epsilon)^2}t\right)\sup_{x\in\mathbb{R}}|u_0(x,\xi)|\,.$$

- $\theta(t,x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi y}e^{-\nu_2\xi^2 t}u(t,x,\xi)d\xi$, we need $\nu_2 > \frac{\rho(0)}{2}$.

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Probabilistic representation of the solution

$\partial_t u = \nu_1 \partial_x^2 u + i\xi u V(t,x) \mid \theta(t,x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi y} e^{-\nu_2 \xi^2 t} u(t,x,\xi) d\xi$

Cov$[V(t,x), V(t',x')] = \delta(t-t')\rho(x-x')$

- Let $\varphi_\epsilon = \varphi_\epsilon(t,x)$ define a smooth approximation to the identity, i.e.

$$\varphi_\epsilon(t,x) = \frac{1}{\epsilon^2}\varphi(\frac{t}{\epsilon}, \frac{x}{\epsilon})$$

for some nonnegative smooth $\varphi(t,x)$ such that $\int_{\mathbb{R}^2} \varphi(t,x) dtdx = 1$.

- Define the Wiener integral $V_\epsilon(t,x) = V * \varphi_\epsilon(t,x)$.i.e.,

$$V_\epsilon(t,x) = \int_{\mathbb{R}^2} \varphi_\epsilon(t-s, x-y) V(s,y) dyds$$

$V_\epsilon(t,x)$ is a smooth, classical Gaussian random field.

- Let B_t be a Brownian motion with $\mathrm{Var}(B_1) = 2\nu_1$, then $\int_0^t V(s, x+B_{t-s})ds := \lim_{\epsilon \downarrow 0} \int_0^t V_\epsilon(t, x+B_{t-s})ds$ exists in $L^2(\Omega)$.

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- Feynman-Kac formula for $u(t,x,\xi)$:

$$u(t,x,\xi) = e^{t\xi^2\rho(0)/2}\mathbb{E}_B\left[\hat{\theta}_0(x+B_t,\xi)\exp\left(i\xi\int_0^t V(s,x+B_{t-s})ds\right)\right],$$

- Probabilistic representation for $\theta(t,x,y)$:

$$\begin{aligned}\theta(t,x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi y}e^{-\nu_2\xi^2 t}u(t,x,\xi)d\xi \\ &= \mathbb{E}_{B,\bar{B}}\theta_0\left(x+B_t, y+\bar{B}_t - \int_0^t V(s,x+B_{t-s})ds\right),\end{aligned}$$

$\bar{B}_t$ is another Brownian motion independent with B, with $\mathrm{Var}(\bar{B}_1) = 2(\nu_2 - \frac{\rho(0)}{2})$.

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$$\begin{aligned}\theta(t,x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi y}e^{-\nu_2\xi^2 t}u(t,x,\xi)d\xi \\ &= \mathbb{E}_{B,\bar{B}}\theta_0\left(x+B_t, y+\bar{B}_t-\int_0^t V(s,x+B_{t-s})ds\right),\end{aligned}$$

$\bar{B}_t$ is another Brownian motion independent with B, with $\mathrm{Var}(\bar{B}_1) = 2(\nu_2 - \frac{\rho(0)}{2})$.

The basic result

$\partial_t \theta = \nu_1 \partial_x^2 \theta + \nu_2 \partial_y^2 \theta + \partial_y \theta V(t,x)$ | **Cov**$[V(t,x), V(t',x')] = \delta(t-t')\rho(x-x')$

Theorem (H+Khoshnevisan '17+)

If $\sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} |\theta_0(x,y)| dy < \infty$ *and* $\nu_2 > \frac{\rho(0)}{2}$*, then our SPDE has a unique solution* θ *such that*

1. $y \to \theta(t,x,y)$ *is* C^∞ *a.s.* $\forall t > 0$ *and* $x \in \mathbb{R}$.
2. $\sup_{x,y\in\mathbb{R}} \mathbb{E}|\theta(t,x,y)|^2 = O(1/t)$ *as* $t \to \infty$.
3. *Decay of the solution*

$$\sup_{x,y\in\mathbb{R}} |\theta(t,x,y)| = O(1/\sqrt{t}) \ \ a.s. \ \ as\ t \to \infty\,.$$

Idea of the proofs

- For result 1:

$$\frac{\partial^m}{\partial y^m}\theta(t,x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\xi)^m e^{-i\xi y} e^{-\nu_2\xi^2 t} u(t,x,\xi)d\xi$$

- For result 2:

$$\begin{aligned}\sup_{x,y\in\mathbb{R}}\|\theta(t,x,y)\|_{L^2(\Omega)} &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\nu_2\xi^2 t}\sup_{x\in\mathbb{R}}\|u(t,x,\xi)\|_{L^2(\Omega)}d\xi \\ &\le C\int_{-\infty}^{\infty}\exp(-\nu_2\xi^2 t)\exp\left(\frac{\rho(0)\xi^2}{2(1-\epsilon)^2}t\right)d\xi \\ &\le \frac{C}{t}\end{aligned}$$

- Handicap: Engineers want to study the case as $\nu_1, \nu_2 \to 0$.
- $\nu_2 > \rho(0)/2$, only possible that $\rho(0) = 0$, i.e., no noise!

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A Stratonovich-type solution

$\partial_t\theta = \nu_1\partial_x^2\theta + \nu_2\partial_y^2\theta + \partial_y\theta V(t,x)$ | **Cov**$[V(t,x), V(t',x')] = \delta(t-t')\rho(x-x')$

- Recall $V_\epsilon(t,x) := V * \varphi_\epsilon(t,x)$, i.e., the regularized V.
- Now consider the classical PDE

$$\partial_t\theta_\epsilon = \nu_1\partial_x^2\theta_\epsilon + \nu_2\partial_y^2\theta_\epsilon + \partial_y\theta_\epsilon V_\epsilon(t,x)\,, \quad \theta_\epsilon(0) = \theta_0$$

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θ_ϵ *with* (ν_1, ν_2) *converges to*

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as $\epsilon \to 0^+$.

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$$\partial_t\theta = \nu_1\partial_x^2\theta + (\nu_2 + \frac{\rho(0)}{2})\partial_y^2\theta + \partial_y\theta V(t,x)$$

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Diffusive decay

- If the initial condition θ_0 is a nice function,

$$\sup_{x\in\mathbb{R}} \sup_{y\in\mathbb{R}} |\tilde{\theta}(t,x,y)| \le \frac{1}{\sqrt{2\pi\nu_2 t}} \sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} |\theta_0(x,w)| dw = O(1/\sqrt{t})$$

- Consider $\theta_0(x,y) = p_1^{\nu_2}(y) := \frac{1}{\sqrt{4\pi\nu_2}} e^{-\frac{y^2}{4\nu_2}}$ and $\rho(x) \equiv 1$, then

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W_t is a standard Brownian motion.

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Delta initial condition

- Proposition (diffusive decay). If $\theta_0 = \delta_0 \times \delta_0$, then the exact a.s. decay rate $\approx 1/t$.
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Multifractal decay

- Assume $\theta_0 = \delta_0 \times \delta_0$, $\rho(x) \equiv 1$, $\nu_1 = \nu_2 = \nu$.
- $\theta(t, x, y) = p_t^\nu(x) p_t^\nu(y - W_t)$.
- The set of times where $\theta(t, 0, 0)$ behaves largely different from $1/t$, has a macroscopic multi fractal structure.
- Define $\mathcal{C}_m(A) = \#\{j \in \{0, \ldots, m\} : [j, j+1] \cap A \neq \emptyset\}$ for $A \subset \mathbb{R}_+$ and

$$\overline{\mathrm{Dim}}_M(A) := \limsup_{m \to \infty} \frac{\log \mathcal{C}_m(A)}{\log m} \qquad \underline{\mathrm{Dim}}_M(A) := \liminf_{m \to \infty} \frac{\log \mathcal{C}_m(A)}{\log m}$$

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- Define $\mathcal{C}_m(A) = \#\{j \in \{0, \ldots, m\} : [j, j+1] \cap A \neq \emptyset\}$ for $A \subset \mathbb{R}_+$ and

$$\overline{\mathrm{Dim}}_M(A) := \limsup_{m \to \infty} \frac{\log \mathcal{C}_m(A)}{\log m} \qquad \underline{\mathrm{Dim}}_M(A) := \liminf_{m \to \infty} \frac{\log \mathcal{C}_m(A)}{\log m}$$

- If $\overline{\mathrm{Dim}}_M(A) = \underline{\mathrm{Dim}}_M(A)$, write $\mathrm{Dim}_M(A)$ for their common value: Minkowski dimension of A.

Multifractal decay

- Assume $\theta_0 = \delta_0 \times \delta_0$, $\rho(x) \equiv 1$, $\nu_1 = \nu_2 = \nu$.
- $\theta(t,x,y) = p_t^\nu(x) p_t^\nu(y - W_t)$.
- The set of times where $\theta(t,0,0)$ behaves largely different from $1/t$, has a macroscopic multi fractal structure.
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Multifractal decay

$\partial_t \theta = \nu \Delta \theta + \partial_y \theta V(t,x)$ | *Stratonovich solution,* $\theta(0) = \delta_0 \times \delta_0$, $\rho \equiv 1$

Theorem (H+Khoshnevisan '17+)

$\forall K > 0$,

$$Dim_M \left\{ t : \tilde{\theta}(t,0,0) > \frac{K}{t} \right\} = \begin{cases} 1 & \text{if } K < (4\pi\nu)^{-1} \\ 0 & \text{if } K \geq (4\pi\nu)^{-1} \end{cases} \quad a.s.$$

Theorem (H+Khoshnevisan '17+)

$\forall \delta > 0$,

$$Dim_M \left(\log \left\{ t > e : \tilde{\theta}(t,0,0) < \frac{1}{t(\log t)^{\delta}} \right) \right\} = (1 - 2\delta\nu)_+ \quad a.s.$$

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Thank you.