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RÉGION NORMANDIE

Fluctuations of Lévy processes

from Wiener-Hopf to the Scattering

Sonia Fourati

INSA de Rouen +LPSM

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$$\phi(0) = 0 \qquad \mathbf{E}(e^{-iuX_1}) = e^{-\phi(iu)}$$

Examples

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Other important example : The **Gamma process** (γ_t), γ_1 is a standard exponential r.v. and $\phi(\lambda) = \text{Ln}(1 + \lambda)$.

Simple facts

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• The **killing formalism**:

$\phi(0) + c$ ($c > 0$) is the exponent of a (still called) Lévy process killed at an exponential independent time with rate c.

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Followers (non-exhaustive list) : .L.Alili, J. Bertoin, L. Chaumont, R. Doney, T.Duquesne, A.Kyprianou, A. Kuznetsov, J.C Prado, V. Rivero, V.Vigon, and S.Asmussen, M.Pistorius, E. Eberlein (with financial applications).

Undoubtly a central result for fluctuations of Lévy processes.

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Concisely,

1) Every exponent of a (possibly killed) Lévy process is the **product** of the exponent of a subordinator and of the opposite of a subordinator: there exists two exponents κ and $\hat{\kappa}$ of subordinators such that

$$\phi(iu) = \kappa(iu)\hat{\kappa}(iu)$$

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3) If X is a killed process with death time ζ, or if X_t goes to $-\infty$, for $t \to +\infty$, then the r.v. $H_\zeta = \sup\{X_s; s \leq \zeta\}$ has Laplace transform $\frac{\kappa(0)}{\kappa(\lambda)}$.

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$$(-iu + h_{-}(a,c))(iu + h_{+}(a,c)) = \frac{u^2}{2} + aiu + c$$

$h_{-}(a,c) \in \mathbb{R}^{+}$ and $-h_{+}(a,c) \in \mathbb{R}^{+}$ are the 2 solutions of the equation : $-\frac{z^2}{2} + az + c = 0$

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• If ϕ has only positive jumps (but is not a subordinator)

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• If ϕ has a meromorphic continuation on the right half plan (a fortiori in the whole complex plane), we obtain closed formulas.

Ideas of the proofs

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2) The pair $(\kappa, \hat{\kappa})$ can be identified as a solution of a **Riemann-Hilbert Problem** :
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$$f(\lambda) := \kappa(\lambda) 1_{\Re(\lambda)>0} + \frac{1}{\check{\kappa}(\lambda)} 1_{\Re(\lambda)<0}$$

a) f is holomorphic on the two half-planes $\{\Re(\lambda) > 0\} \cup \{\Re(\lambda) < 0\}$,

$$\lim_{\varepsilon \to 0^+} \frac{f(iu+\varepsilon)}{f(iu-\varepsilon)} = \kappa(iu)\check{\kappa}(iu) = \phi(iu)$$

b) $(|f(\lambda)| + |\frac{1}{f(\lambda)}|). \inf(|\lambda|, \frac{1}{|\lambda|})$ is bounded on $\mathbb{C}$.
$\Longrightarrow$ A complete characterization of f (thus of κ and $\check{\kappa}$), up to a multiplicative constant.

Connection with the additive decomposition

Put $\phi(0) = 1$, then Ln ϕ is the exponent of the subordinate process X_{γ_t}, where:

- (γ_t) a gamma-Process and X is the non killed Lévy process with exponent $\phi(iu) - \phi(0)$
- (X_{γ_t}) is a Lévy process which does not die, has no drift and is with bounded variations,

$$\text{Ln}\,\left(\frac{\kappa(iu)}{\kappa(0)}\right) + \text{Ln}\,\left(\frac{\hat{\kappa}(iu)}{\hat{\kappa}(0)}\right) = \text{Ln}\,\phi(iu)$$

is the additive decomposition of ϕ.

The second step : the bilateral problem

The problem : Assuming that the Lévy process dies, what is the joint distribution of the maximum value (M) and minimum value (m) ? What is the distribution of the amplitude ($M - m$) ? What is the distribution of the first exit from a bounded interval ?

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Define for $x > 0$, $\lambda \in \mathbb{C}$

$$M^{-}_{1,2}(x,\lambda) = M^{+}_{1,1}(x,\lambda) = \mathbf{P}(e^{-\lambda M}; M - m \leq x)$$

$$M^{-}_{2,2}(x,\lambda) = M^{+}_{2,1}(x,\lambda) = \mathbf{P}(e^{-\lambda m}; M - m \leq x)$$

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then

$$\frac{d}{dx}\begin{pmatrix} M^{+}_{1,1} \\ M^{+}_{2,1} \end{pmatrix} = \begin{pmatrix} e^{-\lambda x} & v(x) \\ \hat{v}(x) & e^{\lambda x} \end{pmatrix} \begin{pmatrix} M^{+}_{1,1} \\ M^{+}_{2,1} \end{pmatrix} \qquad (1)$$

$(v(x), \hat{v}(x))$ is the **potential**.

Auxilliary solutions

Let , for $\Re(\lambda) > 0$,

$$\begin{pmatrix} M^{+}_{12}(x,\lambda) \\ M^{+}_{22}(x,\lambda) \end{pmatrix}$$

and for $\Re(\lambda) < 0$,

$$\begin{pmatrix} M^{-}_{11}(x,\lambda) \\ M^{-}_{21}(x,\lambda) \end{pmatrix}$$

be solutions of the equation (1), holomorphic in λ, on their own half-planes.

Finally, the 2x2 matrices :

$$\begin{aligned} M^{+}(x,\lambda) &\quad \text{for} \quad \Re(\lambda) > 0 \\ M^{-}(x,\lambda) &\quad \text{for} \quad \Re(\lambda) < 0 \end{aligned}$$

satisfy the differential equation :

$$\frac{d}{dx} M^{+/-}(x,\lambda) = \begin{pmatrix} e^{-\lambda x} & v(x) \\ \hat{v}(x) & e^{\lambda x} \end{pmatrix} M^{+/-}(x,\lambda)$$

To switch from the exponent ϕ to the solution of the bilateral problem is again a Riemann-Hilbert problem.

Theorem

For all $x > 0$, the matrices $M^{+}(x,.)$ and $M^{-}(x,.)$ satisfy for all $iu \in i\mathbb{R}$,

$$[M^{-}(x,iu)]^{-1}.M^{+}(x,iu) = \begin{pmatrix} 0 & -e^{-iux} \\ e^{iux} & \phi(iu) \end{pmatrix}$$

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and the additionnal boundary properties :
a) $\lambda \to M^+(x,\lambda)$ (resp. $\lambda \to M^-(x,\lambda)$) is holomorphic on $\{\Re(\lambda) > 0\}$ (resp. on $\{\Re(\lambda) < 0\}$), continuous on the closed half-plane $\{\Re(\lambda) \geq 0\}$ (resp. on $\{\Re(\lambda) \leq 0\}$).

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b) $M^+(x,\lambda) \sim M^+(y,\lambda)$ for $\lambda \to +\infty$, $M^-(x,\lambda) \sim M^-(y,\lambda)$ for $\lambda \to -\infty$ and $\det M^-(x,\lambda) = \det M^+(x,\lambda) = 1$.

Starting from the **Scattering matrix**

$$\begin{pmatrix} 0 & -1 \\ 1 & \phi(iu) \end{pmatrix},$$

the preceding properties entirely characterize the two holomorphic functions $\lambda \to M^{+}(x,\lambda)$ and $\lambda \to M^{-}(x,\lambda)$ for all $x \in]0,+\infty[$ (up to a constant of x and λ) and the potential $(v(x), \hat{v}(x))$.

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Explicit Solutions

• Levitan et Marchenko in the 50's or 60's have solved the case that corresponds to the Brownian motion.

• Explicit solutions can be obtained when ϕ is a meromorphic function, on one half-plane and a fortiori on the two half-planes, and this corresponds to the so called "Bargmann equations" in Scattering Theory.

Some words on Scattering Theory

Scattering Theory : To a **potential** [the two functions $(v(x), \hat{v}(x))$], we associate two parametrized differential equations :

$$M'(x,\lambda) = \begin{pmatrix} e^{-\lambda x} & v(x) \\ \hat{v}(x) & e^{\lambda x} \end{pmatrix} M(x,\lambda)$$

The wronskian identity gives

$$[M^-(x,iu)]^{-1} M^+(x,iu) = \begin{pmatrix} 0 & -e^{-iux} \\ e^{iux} & \phi(iu) \end{pmatrix}$$

The matrix $\Phi = \begin{pmatrix} 0 & -1 \\ 1 & \phi(iu) \end{pmatrix}$ is called the Scattering Matrix.

The mapping $(v(x), \hat{v}(x)) \to \Phi$ is injective. To start from ϕ in order to determine the potential is called the **inverse scattering problem**.

Thus computing the distributions related to the bilateral problem when starting from the Lévy exponent is part of that physics problem.

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