Liouville first-passage percolation: subsequential scaling limit at high temperature

Alexander Dunlap

Stanford University

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Joint work with Jian Ding

First-passage percolation in $\mathbf{Z}^2$

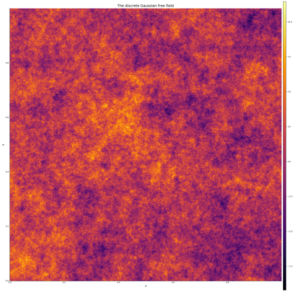
- Nearest-neighbor graph on $\mathbf{Z}^2$ (edges go one unit north, south, east, west).
- Randomly assign a weight (“cost”) to each node.
- Distance between two nodes is the total weight of the lowest-weight (“cheapest”) path between them.
- This gives a *random metric space*.
- Introduced by Hammersley and Welsh (1965).

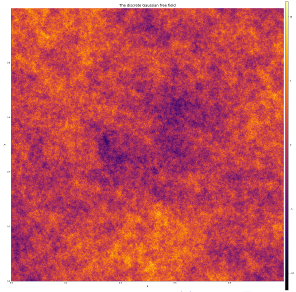
Discrete Gaussian free field

- Let $\mathfrak{R} \subsetneq \mathbf{Z}^2$ connected. Consider DGFF on $\mathfrak{R}$ with Dirichlet boundary conditions on $\partial\mathfrak{R}$: Gaussian field Y on $\mathfrak{R}$ with covariance function

$$\mathbf{E}\,Y(x)Y(y) = G_{\mathfrak{R}}(x,y),$$

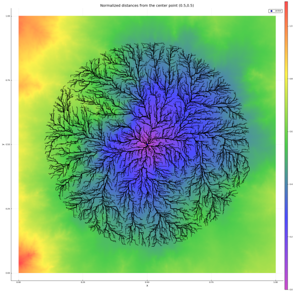
  where $G_{\mathfrak{R}}$ is the Green's function of simple random walk on $\mathfrak{R}$ killed on $\partial\mathfrak{R}$.
- Log-correlated Gaussian field: $G_{\mathfrak{R}}(x,y) \sim \log \frac{|\mathfrak{R}|}{|x-y|}$.
- Similar correlation structure to branching random walk.

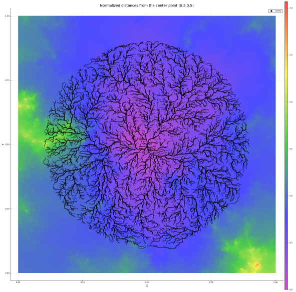




Liouville first-passage percolation

- Liouville first-passage percolation (LFPP) is first-passage percolation on $e^{\gamma Y}$, where Y is a Gaussian free field and γ is an inverse-temperature parameter:





- Duplantier–Sheffield (2008) and Rhodes–Vargas (2008), along with many others subsequently, have studied the *measure* arising from the exponential of log-correlated Gaussian fields. We are concerned instead with the *metric*.

Questions

- How does the metric (left–right crossing distance, point-to-point distance, diameter. . .) scale with the box size S and the temperature γ?
    - Ding–Goswami (2016) prove that the exponent of the expectation is strictly less than 1 in the small-γ regime via "switchings."
- As $S \to \infty$ (with γ held constant), does the (normalized) metric converge to some limiting metric?
    - We make progress in the small-γ regime.

Main result

Let $d_s(x, y)$ be the linearly-interpolated first-passage percolation distance between x and y in $[0, 1]^2$, using lattice of size $S = 2^s$, normalized so that the expected left-right distance of the square is 1.

Theorem (Ding–D.)

If γ is sufficiently small, then the sequence $\{d_s\}_{s \in \mathbf{N}}$ is tight in the Gromov–Hausdorff topology. In fact, it is tight in the uniform topology on functions $([0, 1]^2)^2 \to \mathbf{R}_{\geq 0}$.

By Prokhorov's theorem, this implies subsequential convergence.

Small-γ regime

- Any constant number of scales (at the top or at the bottom) are negligible—can use any fixed amount of independence needed.
  - Crossing weights are highly concentrated.
- At a constant scale, geodesics are almost straight (Hausdorff dimension close to 1).

Tightness

- Obtain *sub*sequential convergence by proving tightness of the normalized metric.
- Follows by Arzelà–Ascoli theorem from equicontinuity of the metric as the scale increases.
- Need to get good tail bounds on the diameter of a box so we can max over many boxes.

Coefficient of variation bound

Theorem (Ding–D.)

The coefficient of variation ($\mathrm{CV} = \sigma/\mu$) of the crossing weights can be made arbitrarily small by making γ sufficiently small.

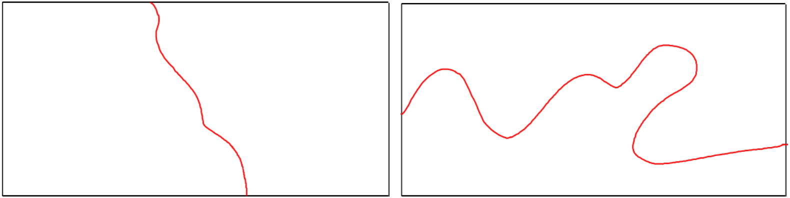
Corollary

Arbitrarily high and low crossing quantiles are multiplicatively related as long as γ is sufficiently small.

Bounding CV^2

- Use induction from scale s to scale $s+k$, where k is constant but large.
- Bound the variance from above and the expectation from below.
- Without contributions from boxes between scales s and $s+k$, variance and expectation "should" both go like $K = 2^k$.
  - Need to relate the coefficients (as constant multiples of the expected crossing weight at scale s).
- So by making γ small and k large, can make $\mathrm{Var}/\mathbf{E}^2$ as small as we like.

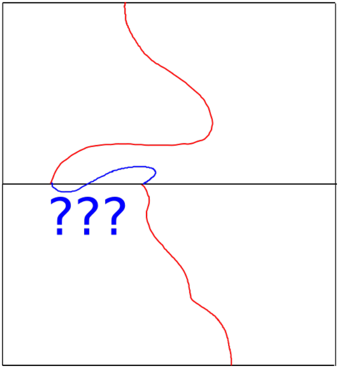
Easy and hard crossings



- Easy crossings (left) at a smaller scale are *necessary* to cross at a larger scale.
  - $\Longrightarrow$ inductive lower bounds for crossing weights
- Hard crossings (right) at a smaller scale are *sufficient* to cross at a larger scale.
  - $\Longrightarrow$ inductive upper bounds for crossing weights
- RSW result: *easy and hard crossings can be related*.
  - not obvious, and the crux of our results

Russo–Seymour–Welsh results

- Show that crossing probabilities/weights in the easy direction are related to those for the hard direction by a constant factor.
- Introduced for Bernoulli percolation by Russo, Seymour, Welsh in 1978–81.
- Try to glue together easy crossings to get a hard crossing.



RSW for Voronoi percolation (Tassion 2014)

- Tassion proved an RSW result in a weakly correlated ordinary (not first-passage) percolation setting.
- Self-dual model—go from square crossing to hard crossing rather than easy crossing to hard crossing.
- Delicate multi-scale analysis involving inductively controlling both the probability of crossings and the geometry of crossings if they do exist.

Challenges for first-passage percolation

- Goal is now to show there are good probabilities of crossings with certain *weights.*
  - Thus need to choose these weights appropriately, and keep track of them in every construction.
  - At the end, we need to show that the weight we get is not too big—requires our inductive hypothesis.
- We don't have a notion of self-duality
  - Need to go from easy crossing to hard crossing rather than from square crossing to easy crossing.

Keeping track of the weights

- Tassion's analysis is already quite delicate—having a limited "weight budget" makes things substantially trickier in many places.
- Multiscale joining procedure creates paths of weight $\sum c^n w_{s-nk}$, where c is some constant and w_n is the crossing weight at scale n.
  - Need to force k to be large so that this is summable—can do this by skipping over many scales at each joining step.
  - Need the sum we get to be not too large (dominated by the largest term)—apply inductive hypothesis and our *a priori* bound on easy crossings.

Future work

- Investigate properties of limit point metrics.
- Show convergence of the metrics to a limiting metric (eventual goal of current work with Jian Ding and Subhajit Goswami).
- What happens for larger γ?

References

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- Vincent Tassion, *Crossing probabilities for Voronoi percolation*, Ann. Probab. **44** (2016), no. 5, 3385–3398.