The Mirror Model

on the Square Lattice

Yan Dai

University of Arizona

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Background

- Lorentz lattice gas model
- Ehrenfest wind-tree model

**Figure:** A trajectory of a light beam [1].
Walk Model on Square Lattice

- Left and right turns only
- No bound is crossed twice
- No site is visited more than twice
Mirror Model on Square Lattice

- North-west mirror or north-east mirror
- A special case of Lorentz and Ehrenfest models
Mirror Model on Square Lattice

- Kinetic growing trail on oriented square lattice
- Never get trapped, except at the origin
Mirror Model on Square Lattice

- All trajectories are localized w.p.1
- Size distribution of the orbits follows

\[ P(n) \sim a_1 n^{1-\tau} \]

for \( n \to \infty \), where \( \tau = \frac{15}{7} \), \( a_1 \) is a constant
First Theorem
Localization Property

**Theorem**

The walk trajectory of the mirror model on the square lattice \( \mathbb{L}^2 \) is almost surely localized.

Idea of proof by Grimmett [1]:

embedded critical bond percolation
Bond Percolation on $\mathcal{L}$

Figure: The half dual lattice $\mathcal{L}$ [1].
Bond Percolation on $\mathcal{L}$

- NE mirror on vertex $(m, n) \Rightarrow$ open edge from $(m - 1/2, n - 1/2)$ to $(m + 1/2, n + 1/2)$
- NW mirror on vertex $(m, n) \Rightarrow$ open edge from $(m - 1/2, n + 1/2)$ to $(m + 1/2, n - 1/2)$

Remark:
The resulting process is bond percolation on $\mathcal{L}$ with density of open edges $1/2$. 
Properties of Bond Percolation

- The critical probability of bond percolation on square lattice equals $1/2$.
- There exits almost surely no infinite open cluster at the critical point.
- The origin is contained in the interior of a closed circuit of the dual lattice.
Localization Property

**Theorem**

The walk trajectory of the mirror model on the square lattice $\mathbb{L}^2$ is almost surely localized.

**Remark:**

- The origin of $\mathbb{L}^2$ is contained a.s. in the interior of some open circuit of $\mathcal{L}$.
- The circuit corresponds a enclosure of mirrors surrounding the origin.
Walk with Boundary Condition

Boundary condition $\omega_a$: an infinite staircase walk of mirror model ending at $a$.

Figure: A walk with a boundary condition $\omega_a$. 
Walk with Boundary Condition

Properties:

→ The walk trajectory of mirror model with boundary condition $\omega_a$ on $\mathbb{L}^2$ is infinite.
→ The walk hits the boundary $\omega_a$ infinitely many times.
Second Theorem
Theorem
Consider the staircase boundary condition $\omega_a$. Let the walk starts at the endpoint of $\omega_a$. Then the walk hits the boundary $\omega_a$ infinitely many times.

Idea of proof:
Almost open circuits around $a$
Almost open circuit around $a$: a circuit on $\mathcal{L}$ that is consistent with the boundary condition.
Lemma

The endpoint of boundary condition $\omega_a$ is contained a.s. in the interior of almost open circuit of $\mathcal{L}$.
Lemma

Let $(\Omega, \mathcal{F}, P)$ be the probability space for the mirror model, and $(\Omega', \mathcal{F}', P_b)$ be the probability space for the mirror model with boundary condition. Let $\phi : \Omega \rightarrow \Omega'$ be a function mapping a mirror configuration in $\Omega$ to $\Omega'$ satisfying the boundary condition. Then for any $E \in \mathcal{F}'$,

$$P(\phi^{-1}(E)) = P_b(E).$$
Second Theorem

Figure: $\phi$ map.
Second Theorem

Proof of Lemma.
Sample spaces: \( \Omega = \prod_{e \in \mathbb{E}^d} \{0, 1\} \), \( \Omega' = \prod_{e \in \mathbb{E}^d \setminus B} \{0, 1\} \)

Configurations: \( \omega = (\omega(e) : e \in \mathbb{E}^d) \)
\( \omega(e) = 0 \iff e \text{ is closed}, \omega(e) = 1 \iff e \text{ is open} \)

Open cylinders:
\[
C_i(a) = \{ \omega : \omega(e_i) = a \}
\]

Let \( E = C_i(a) \).

- If \( e_i \notin B \), then \( E \) and \( \phi^{-1}(E) \) are the same set. Hence
  \[ P_b(E) = P(\phi^{-1}(E)) = 1/2. \]

- If \( e_i \in B \) and \( a \) is consistent with \( \omega(e_i) \), then
  \[ P_b(E) = 1. \] Since \( \phi^{-1}(E) = \Omega \), then \[ P(\phi^{-1}(E)) = 1. \]

- If \( e_i \in B \) and \( a \) is not consistent with \( \omega(e_i) \), then
  \[ P_b(E) = 0, \text{ and } \phi^{-1}(E) = \emptyset, P(\phi^{-1}(E)) = 0. \]
Theorem

For bond percolation on $\mathcal{L}$, infinitely many annuli contain open circuits around the origin w.p.1.

Idea of proof:

RSW theorem - left and right crossing
Theorem
Consider the staircase boundary condition $\omega_a$. Let the walk starts at the endpoint of $\omega_a$. Then the walk hits the boundary $\omega_a$ infinitely many times.
Figure: The barriers of a typical walk trajectory [2].

- Connected component of mirrors ⇔ bond percolation clusters
- Walk trajectory is the perimeter of the cluster
Model with $\text{SLE}_6$

Figure: Mirror model in the chordal case.
Conjecture

The chordal mirror model on the square lattice converges in distribution to the chordal $\text{SLE}_6$, as the lattice spacing goes to zero.
Numerical Results
Theorem (Schramm, 2001)

Let \( z_0 = x_0 + iy_0 \in \mathbb{H} \), \( E \) be the event that the trace \( \gamma \) of chordal SLE\(_6\) passes to the left of \( z_0 \). Then

\[
P(E) = \frac{1}{2} + \frac{\Gamma(2/3)}{\sqrt{\pi} \Gamma(1/6)} \frac{x_0}{y_0} F_{2,1} \left( \frac{1}{2}, \frac{2}{3}, \frac{3}{2}, -\frac{x_0^2}{y_0^2} \right) .
\]
Numerical Test I

Data information:

- Domain: stair-shaped square
- Lattice: square lattice
- Lattice spacing: defined with respect to the scaled domain
- Samples: 100K
Figure: The difference of passing left probability between mirror model and SLE$_6$. 
Theorem (Lawler, 2001)

Suppose \( \gamma \) is a chordal SLE\(_6\), and \( t_* = \inf\{t : \gamma(t) \in [1, \infty)\} \). Then

\[
P(\gamma(t_*) < 1 + x) = \frac{\Gamma(2/3)}{\Gamma^2(1/3)} \int_0^{x/(1+x)} (u - u^2)^{-2/3} du.
\]
Numerical Test II

Data information:

- Domain: stair-shaped square
- Lattice: square lattice
- Lattice spacing: defined with respect to the scaled domain
- Samples: 100K
Figure: The difference of $x$ probability between mirror model and SLE$_6$. 
Figure: The scaled difference of $x$ probability between mirror model and $SLE_6$. 
Figure: The scaled difference of $x$ probability between mirror model and SLE$_6$. 
The Mirror Model
on the Triangular Lattice
Walk Model on Triangular Lattice

Figure: A 10-step walk on triangular lattice.

- Left and right turns only
- No bound is crossed twice
- No site is visited more than three times
Mirror Model on Triangular Lattice

Figure: The corresponding mirrors on the 10-step walk.

- Left mirror or right mirror
- A generalization of mirror model on triangular lattice
Mirror Model on Triangular Lattice

- Underlying oriented triangular lattice
- Never get trapped, except at the origin
Model with $\text{SLE}_6$

Figure: Mirror model in the chordal case.
Conjecture

The chordal mirror model on the triangular lattice converges in distribution to the chordal $\text{SLE}_6$, as the lattice spacing goes to zero.
Numerical Results
Theorem ([5])

Let \( z_0 = x_0 + i y_0 \in \mathbb{H} \), \( E \) be the event that the trace \( \gamma \) of chordal SLE\(_6\) passes to the left of \( z_0 \). Then

\[
P(E) = \frac{1}{2} + \frac{\Gamma(2/3)}{\sqrt{\pi} \Gamma(1/6)} \frac{x_0}{y_0^2} F_{2,1} \left( \frac{1}{2}, \frac{2}{3}, \frac{3}{2}, -\frac{x_0^2}{y_0^2} \right).
\]
Data information:

- Domain: rhombus
- Lattice: triangular lattice
- Lattice spacing: defined with respect to the scaled domain
- Samples: 100K
Figure: The difference of passing left probability between mirror model and SLE$_6$. 
Figure: The rescaled difference of passing left probability between mirror model and SLE\(_6\).
Theorem ([3])

Suppose $\gamma$ is a chordal SLE$_6$, and $t_* = \inf\{t : \gamma(t) \in [1, \infty)\}$. Then

$$P(\gamma(t_*) < 1 + x) = \frac{\Gamma(2/3)}{\Gamma^2(1/3)} \int_0^{\frac{x}{1+x}} (u - u^2)^{-2/3} du.$$
Data information:

- Domain: rhombus
- Lattice: triangular lattice
- Lattice spacing: defined with respect to the scaled domain
- Samples: 100K
Figure: The difference of $x$ probability between mirror model and SLE$_6$. 
Figure: The scaled difference of $x$ probability between mirror model and $\text{SLE}_{6}$. 
Thank you! Question?