The Mirror Model

on the Square Lattice

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Background

- Lorentz lattice gas model
- Ehrenfest wind-tree model

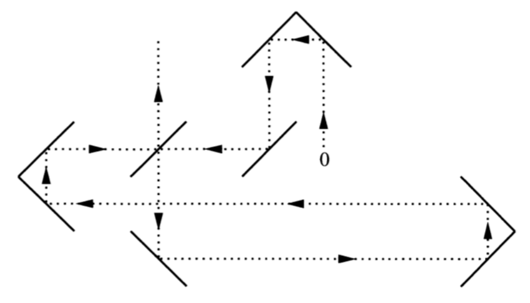
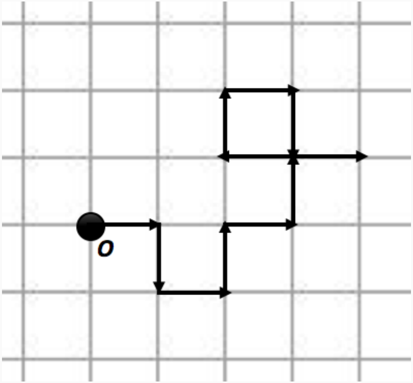


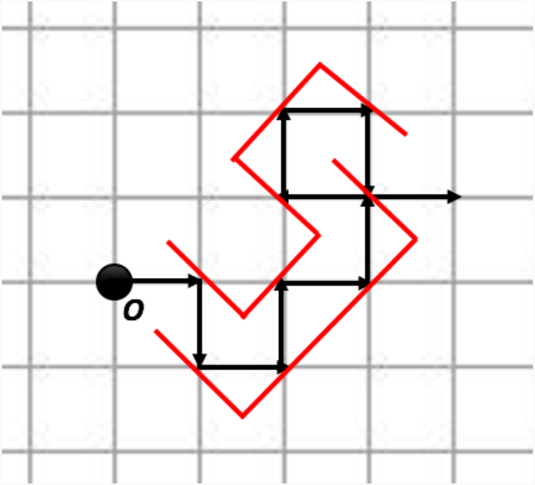
Figure: A trajectory of a light beam [1].

Walk Model on Square Lattice



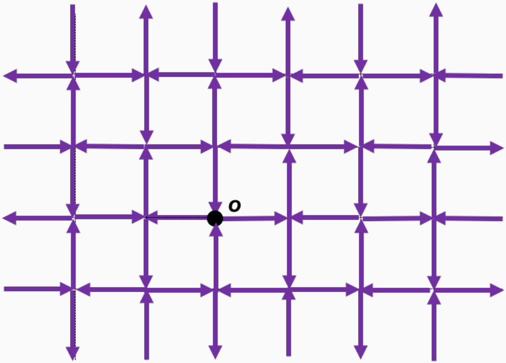
- Left and right turns only
- No bound is crossed twice
- No site is visited more than twice

Mirror Model on Square Lattice



- ▷ North-west mirror or north-east mirror
- ▷ A special case of Lorentz and Ehrenfest models

Mirror Model on Square Lattice



- Kinetic growing trail on oriented square lattice
- Never get trapped, except at the origin

Mirror Model on Square Lattice

- All trajectories are localized w.p.1
- Size distribution of the orbits follows

$$P(n) \sim a_1 n^{1-\tau}$$

for $n \to \infty$, where $\tau = 15/7$, a_1 is a constant

First Theorem

Localization Property

Theorem

The walk trajectory of the mirror model on the square lattice $\mathbb{L}^2$ is almost surely localized.

Idea of proof by Grimmett [1]:
embedded critical bond percolation

Bond Percolation on $\mathcal{L}$

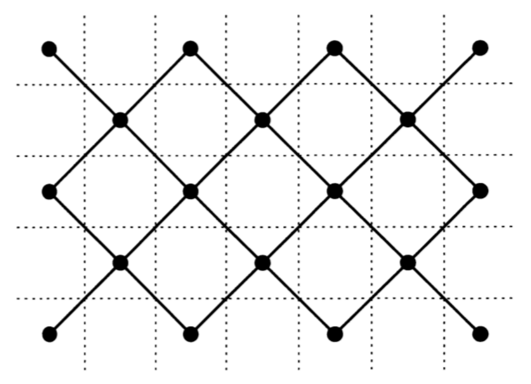


Figure: The half dual lattice $\mathcal{L}$ [1].

Bond Percolation on $\mathcal{L}$

- NE mirror on vertex $(m, n) \Rightarrow$ open edge from $(m - 1/2, n - 1/2)$ to $(m + 1/2, n + 1/2)$
- NW mirror on vertex $(m, n) \Rightarrow$ open edge from $(m - 1/2, n + 1/2)$ to $(m + 1/2, n - 1/2)$

Remark:
The resulting process is bond percolation on $\mathcal{L}$ with density of open edges 1/2.

Properties of Bond Percolation

- The critical probability of bond percolation on square lattice equals 1/2.
- There exits almost surely no infinite open cluster at the critical point.
- The origin is contained in the interior of a closed circuit of the dual lattice.

Localization Property

Theorem

The walk trajectory of the mirror model on the square lattice $\mathbb{L}^2$ is almost surely localized.

Remark:

- → The origin of $\mathbb{L}^2$ is contained a.s. in the interior of some open circuit of $\mathcal{L}$.
- → The circuit corresponds a enclosure of mirrors surrounding the origin.

Walk with Boundary Condition

Boundary condition ω_a:
an infinite staircase walk of mirror model ending at a.

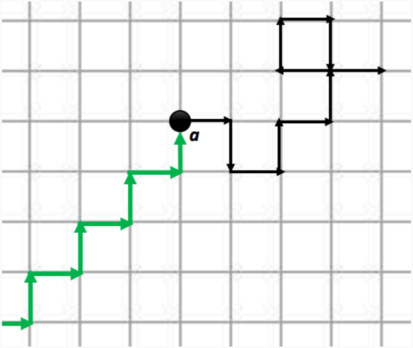


Figure: A walk with a boundary condition ω_a.

Walk with Boundary Condition

Properties:

- → The walk trajectory of mirror model with boundary condition ω_a on $\mathbb{L}^2$ is infinite.
- → The walk hits the boundary ω_a infinitely many times.

Second Theorem

Second Theorem

Theorem

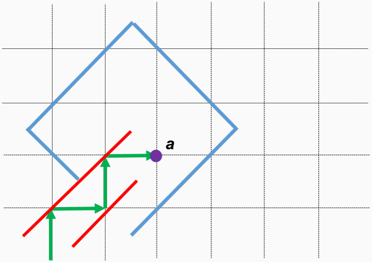
Consider the staircase boundary condition ω_a. Let the walk starts at the endpoint of ω_a. Then the walk hits the boundary ω_a infinitely many times.

Idea of proof:

Almost open circuits around a

Second Theorem

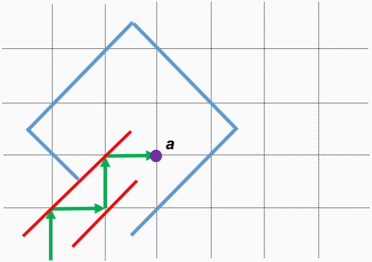
Almost open circuit around a:
a circuit on $\mathcal{L}$ that is consistent with the boundary condition.



Second Theorem

Lemma

The endpoint of boundary condition ω_a is contained a.s. in the interior of almost open circuit of $\mathcal{L}$.



Second Theorem

Lemma

Let $(\Omega, \mathcal{F}, P)$ *be the probability space for the mirror model, and* $(\Omega', \mathcal{F}', P_b)$ *be the probability space for the mirror model with boundary condition. Let* $\phi : \Omega \to \Omega'$ *be a function mapping a mirror configuration in* Ω *to* Ω' *satisfying the boundary condition. Then for any* $E \in \mathcal{F}'$*,*

$$P(\phi^{-1}(E)) = P_b(E).$$

Second Theorem

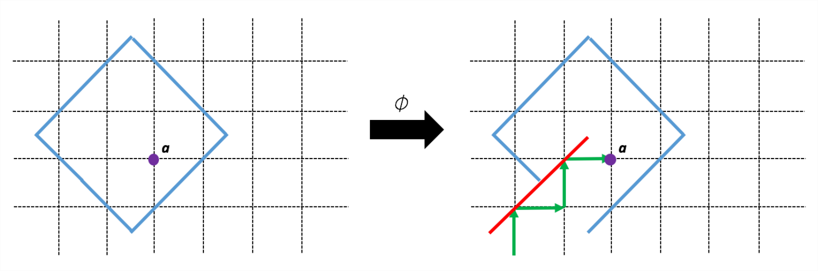


Figure: ϕ map.

Second Theorem

Proof of Lemma.

Sample spaces: $\Omega = \prod_{e \in \mathbb{E}^d} \{0, 1\}$, $\Omega' = \prod_{e \in \mathbb{E}^d \setminus B} \{0, 1\}$
Configurations: $\omega = (\omega(e) : e \in \mathbb{E}^d)$
$\omega(e) = 0 \Leftrightarrow e$ is closed, $\omega(e) = 1 \Leftrightarrow e$ is open
Open cylinders:

$$C_i(a) = \{\omega : \omega(e_i) = a\}$$

Let $E = C_i(a)$.

- If $e_i \notin B$, then E and $\phi^{-1}(E)$ are the same set. Hence $P_b(E) = P(\phi^{-1}(E)) = 1/2$.
- If $e_i \in B$ and a is consistent with $\omega(e_i)$, then $P_b(E) = 1$. Since $\phi^{-1}(E) = \Omega$, then $P(\phi^{-1}(E)) = 1$.
- If $e_i \in B$ and a is not consistent with $w(e_i)$, then $P_b(E) = 0$, and $\phi^{-1}(E) = \emptyset$, $P(\phi^{-1}(E)) = 0$.

Bond Percolation on $\mathcal{L}$

Theorem

For bond percolation on $\mathcal{L}$, infinitely many annuli contain open circuits around the origin w.p.1.

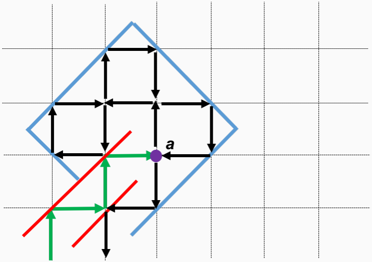
Idea of proof:

RSW theorem - left and right crossing

Second Theorem

Theorem

Consider the staircase boundary condition ω_a. Let the walk starts at the endpoint of ω_a. Then the walk hits the boundary ω_a infinitely many times.



Model with Bond Percolation

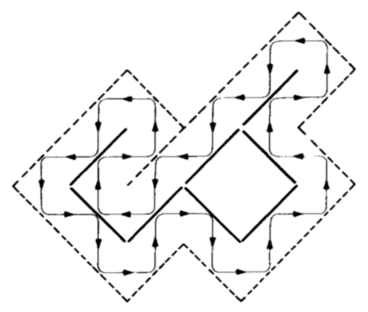


Figure: The barriers of a typical walk trajectory [2].

- ▷ Connected component of mirrors $\Leftrightarrow$ bond percolation clusters
- ▷ Walk trajectory is the perimeter of the cluster

Model with SLE_6

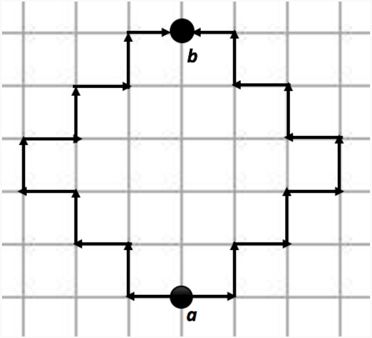


Figure: Mirror model in the chordal case.

Model with SLE_6

Conjecture

The chordal mirror model on the square lattice converges in distribution to the chordal SLE_6*, as the lattice spacing goes to zero.*

Numerical Results

Numerical Test I

Theorem (Schramm, 2001)

Let $z_0 = x_0 + iy_0 \in \mathbb{H}$, E be the event that the trace γ of chordal SLE_6 passes to the left of z_0. Then

$$P(E) = \frac{1}{2} + \frac{\Gamma(2/3)}{\sqrt{\pi}\Gamma(1/6)} \frac{x_0}{y_0} F_{2,1}\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, -\frac{x_0^2}{y_0^2}\right).$$

Numerical Test I

Data information:

- Domain: stair-shaped square
- Lattice: square lattice
- Lattice spacing: defined with respect to the scaled domain
- Samples: 100K

Numerical Test I

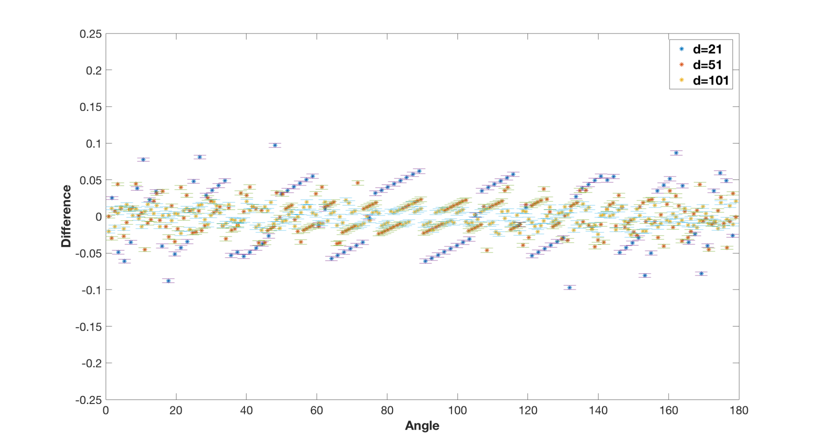


Figure: The difference of passing left probability between mirror model and SLE_6.

Numerical Test II

Theorem (Lawler, 2001)

Suppose γ *is a chordal* SLE_6*, and* $t_* = \inf\{t : \gamma(t) \in [1, \infty)\}$*. Then*

$$P(\gamma(t_*) < 1 + x) = \frac{\Gamma(2/3)}{\Gamma^2(1/3)} \int_0^{\frac{x}{1+x}} (u - u^2)^{-2/3} du.$$

Numerical Test II

Data information:

- Domain: stair-shaped square
- Lattice: square lattice
- Lattice spacing: defined with respect to the scaled domain
- Samples: 100K

Numerical Test II

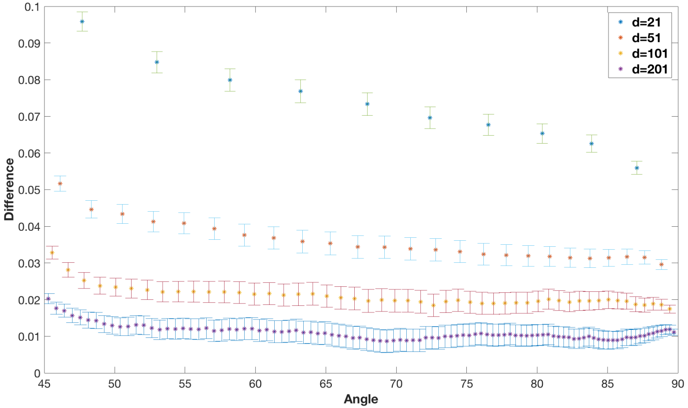


Figure: The difference of *x* probability between mirror model and SLE_6.

Numerical Test II

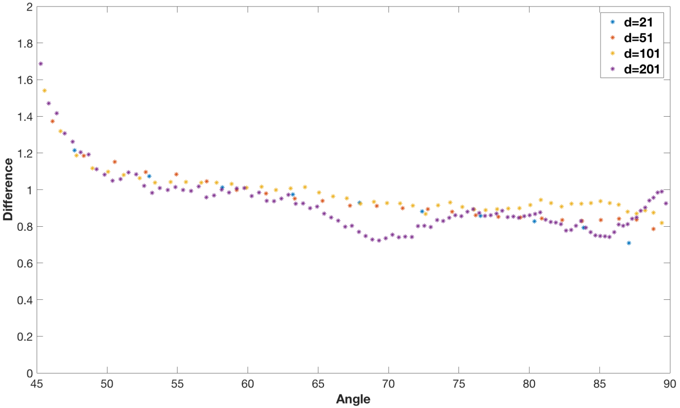


Figure: The scaled difference of *x* probability between mirror model and SLE_6.

Numerical Test II

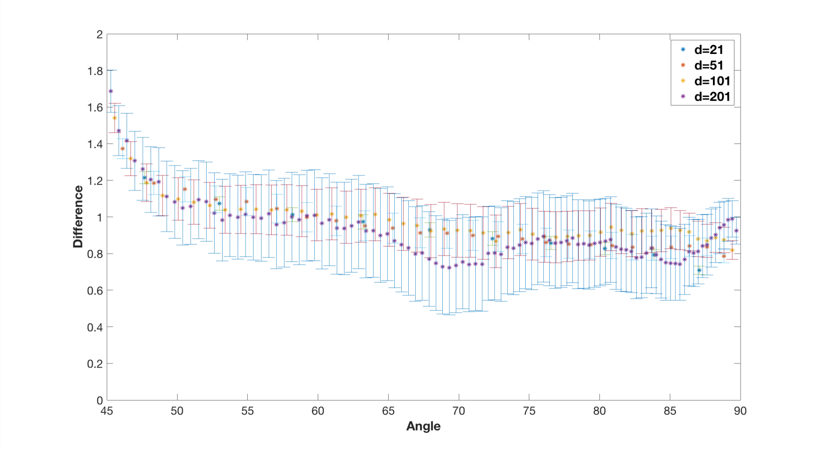


Figure: The scaled difference of x probability between mirror model and SLE_6.

The Mirror Model

on the Triangular Lattice

Walk Model on Triangular Lattice

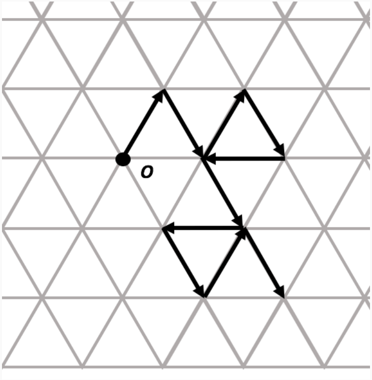


Figure: A 10-step walk on triangular lattice.

- Left and right turns only
- No bound is crossed twice
- No site is visited more than three times

Mirror Model on Triangular Lattice

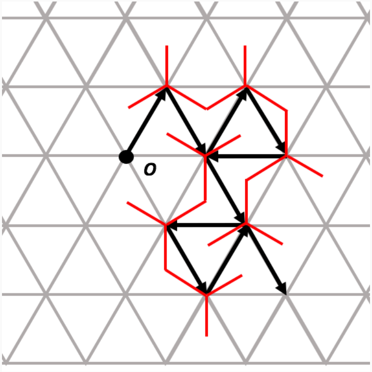
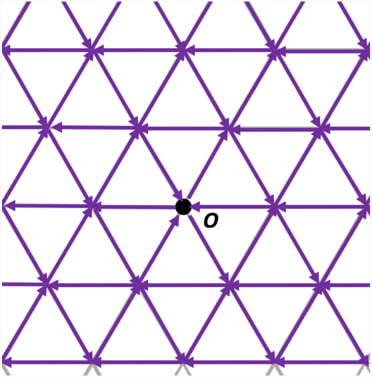


Figure: The corresponding mirrors on the 10-step walk.

- Left mirror or right mirror
- A generalization of mirror model on triangular lattice

Mirror Model on Triangular Lattice



- ▷ Underlying oriented triangular lattice
- ▷ Never get trapped, except at the origin

Model with SLE_6

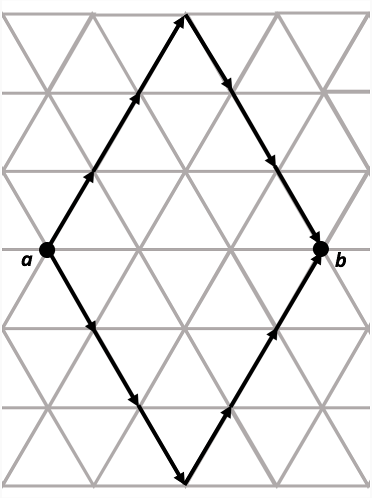


Figure: Mirror model in the chordal case.

Model with SLE_6

Conjecture

The chordal mirror model on the triangular lattice converges in distribution to the chordal SLE_6, as the lattice spacing goes to zero.

Numerical Results

Numerical Test I

Theorem ([5])

Let $z_0 = x_0 + iy_0 \in \mathbb{H}$, E be the event that the trace γ of chordal SLE$_6$ passes to the left of z_0. Then

$$P(E) = \frac{1}{2} + \frac{\Gamma(2/3)}{\sqrt{\pi}\Gamma(1/6)} \frac{x_0}{y_0} F_{2,1}\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, -\frac{x_0^2}{y_0^2}\right).$$

Numerical Test I

Data information:

- Domain: rhombus
- Lattice: triangular lattice
- Lattice spacing: defined with respect to the scaled domain
- Samples: 100K

Numerical Test I

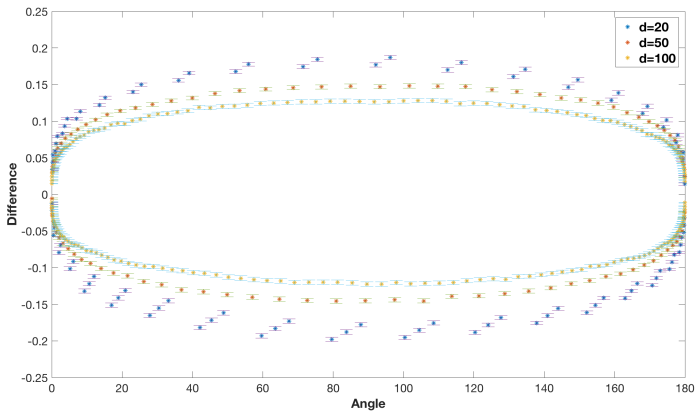


Figure: The difference of passing left probability between mirror model and SLE_6.

Numerical Test I

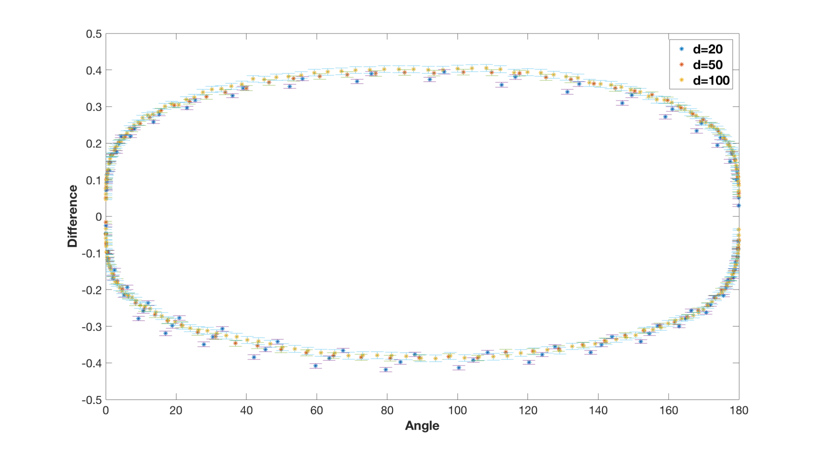


Figure: The rescaled difference of passing left probability between mirror model and SLE_6.

Numerical Test II

Theorem ([3])

Suppose γ is a chordal SLE_6, and $t_* = \inf\{t : \gamma(t) \in [1, \infty)\}$. *Then*

$$P(\gamma(t_*) < 1 + x) = \frac{\Gamma(2/3)}{\Gamma^2(1/3)} \int_0^{\frac{x}{1+x}} (u - u^2)^{-2/3} du.$$

Numerical Test II

Data information:

- Domain: rhombus
- Lattice: triangular lattice
- Lattice spacing: defined with respect to the scaled domain
- Samples: 100K

Numerical Test II

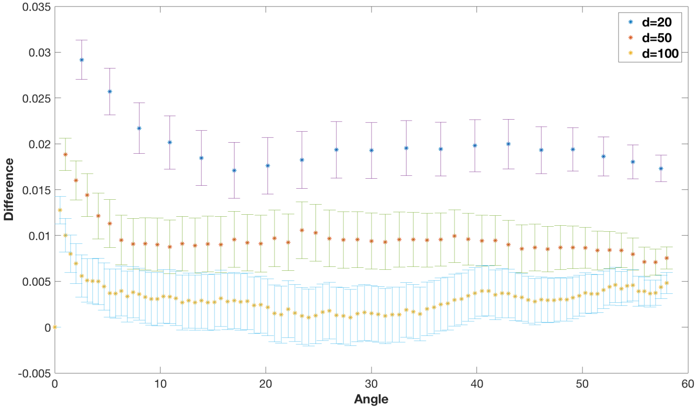


Figure: The difference of *x* probability between mirror model and SLE_6.

Numerical Test II

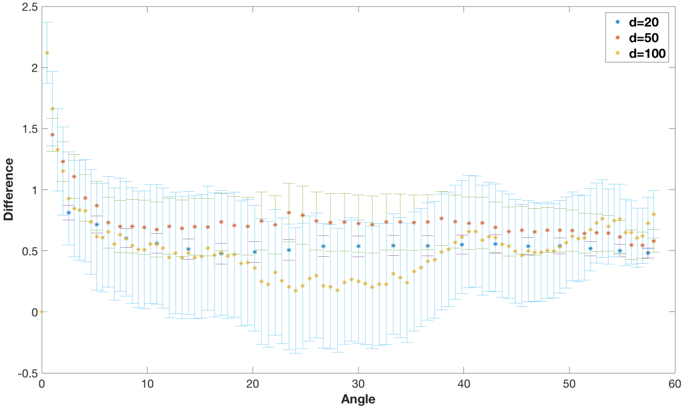


Figure: The scaled difference of *x* probability between mirror model and SLE_6.

Reference

[1] G.R. Grimmett, Percolation, Springer, New York, 1999.
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Thank you!
Question?