Resolvent kernel functions arising from some stochastic partial differential equations

Le Chen

University of Nevada, Las Vegas

Frontier Probability Days

Oregon State University, Corvallis, OR

March 30, 2018

Outline

Stochastic heat equation on $\mathbb{R}$

Stochastic wave equation on $\mathbb{R}$

Further simplifications for SHE on $\mathbb{R}$

Stochastic heat equation on a torus

Plan

Stochastic heat equation on $\mathbb{R}$

Stochastic wave equation on $\mathbb{R}$

Further simplifications for SHE on $\mathbb{R}$

Stochastic heat equation on a torus

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{\nu}{2}\dfrac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x)\dot{W}(t,x), \quad t > 0\ ,\ x \in \mathbb{R} \\ u(0,\cdot) = \mu(\cdot) \end{cases} \tag{SHE}$$

† $\dot{W}$ is space-time white noise, $\lambda \neq 0$, $\nu = 1$.

Mild form:

$$u(t,x) = \int_{\mathbb{R}} G(t, x-y)\mu(\mathrm{d}y) + \lambda \int_0^t \int_{\mathbb{R}} G(t-s, x-y) u(s,y) \dot{W}(\mathrm{d}s, \mathrm{d}y)$$

where $G(t,x) = (2\pi t)^{-1/2} \exp\left(-x^2/(2t)\right)$.

Second moment of SHE

1. Set $J_0(t,x) := \int_{\mathbb{R}} G(t, x-y)\mu(\mathrm{d}y)$.
2. $||u(t,x)||_2^2 = \mathbb{E}\left[u(t,x)^2\right]$.

Then $||u(t,x)||_2^2$ satisfies the following integral equation

$$||u(t,x)||_2^2 = J_0^2(t,x) + \lambda^2 \int_0^t \int_{\mathbb{R}} G(t-s, x-y)^2 \, ||u(s,y)||_2^2 \, \mathrm{d}s\mathrm{d}y.$$

Goal: Solve this integral equation as explicitly as possible!

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Known respresentations of moments

1. Feynman-Kac formula for $p \geq 2$:

$$\mathbb{E}\left[u(t,x)^p\right] = \mathbb{E}^{\mathbf{B}}\left[\left(\prod_{k=1}^{p} u_0(x+B_t^k)\right)\exp\left(\lambda^2 \sum_{1\leq i<j\leq p}\int_0^t \delta_0\left(B_s^i - B_s^j\right)\mathrm{d}s\right)\right]$$

   where B_t^i are independent standard Brownian motion on $\mathbb{R}$.

2. Borodin & Corwin '14 claimed that (see Appendix A.2)

$$\mathbb{E}\left[u(t,x)^p\right] = \frac{1}{(2\pi i)^p}\oint\cdots\oint \prod_{1\leq a<b\leq p}\frac{z_a - z_b}{z_a - z_b - \lambda^2}\prod_{j=1}^{p}\exp\left(\frac{t}{2}z_j^2 + xz_j\right)\mathrm{d}z_j$$

   where $z_j \in \lambda^2\alpha_j + i\mathbb{R}$ with $\alpha_1 > \alpha_2 + 1 > \cdots > \alpha_p + p - 1$.

Goal: Obtain explicit formulas in case of $p = 2$ and for general initial data.

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The delta initial data requires proper interpretation

[C., Hu & Nualart, 17]

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Two ways to explicitly solve the integral equation

$$||u(t,x)||_2^2 = J_0^2(t,x) + \lambda^2 \underbrace{\int_0^t \int_{\mathbb{R}} G(t-s,x-y)^2 \, ||u(s,y)||_2^2 \, \mathrm{d}s\mathrm{d}y}_{:= \left(G^2 \star ||u||_2^2\right)(t,x)}$$

1. Solved through successive approximation.

2. Integral transforms. (Laplace trans. in t and Fourier trans. in x)

Successive approximations

1. Denote $f(t,x) = ||u(t,x)||_2^2$. Recall "$\star$" denotes the space-time convolution. Then

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\mathcal{K} := \sum_{n=1}^{\infty} \lambda^{2n} \underbrace{\left(G^2 \star \cdots \star G^2\right)}_{n\text{'s } G^2}, \quad \text{then} \quad f = J_0^2 + J_0^2 \star \mathcal{K}.
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Successive approximations (cont...)

By induction, one can show that

$$\underbrace{\left(G^2 \star \cdots \star G^2\right)}_{n\text{'s } G^2}(t,x) = G(t/2,x)\frac{t^{n/2-1}}{2^n\Gamma(n/2)}.$$

Indeed, if $n=1$, LHS$=G^2(t,x) = \frac{1}{2\pi t}e^{-\frac{x^2}{t}} = G(t/2,x)\frac{1}{2\sqrt{\pi t}}$ =RHS. Suppose it is true for n, then

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Notice that

$$e^{x^2} = x^{-2} \sum_{n=1}^{\infty} \frac{x^{2n}}{\Gamma(2n/2)} \quad \text{and} \quad e^{x^2} \mathrm{erf}(x) = x^{-2} \sum_{n=1}^{\infty} \frac{x^{2n+1}}{\Gamma((2n+1)/2)},$$

which imply that

$$\sum_{n=1}^{\infty} \frac{(\lambda^2 \sqrt{t}/2)^n}{\Gamma(n/2)} = \frac{\lambda^2 \sqrt{t}}{2\sqrt{\pi}} + \left(\frac{\lambda^2 \sqrt{t}}{2}\right)^2 e^{\left(\frac{\lambda^2 \sqrt{t}}{2}\right)^2} \left[1 + \mathrm{erf}\left(\frac{\lambda^2 \sqrt{t}}{2}\right)\right].$$

Therefore,

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Notice that

$$e^{x^2} = x^{-2} \sum_{n=1}^{\infty} \frac{x^{2n}}{\Gamma(2n/2)} \quad \text{and} \quad e^{x^2} \mathrm{erf}(x) = x^{-2} \sum_{n=1}^{\infty} \frac{x^{2n+1}}{\Gamma((2n+1)/2)},$$

which imply that

$$\sum_{n=1}^{\infty} \frac{(\lambda^2 \sqrt{t}/2)^n}{\Gamma(n/2)} = \frac{\lambda^2 \sqrt{t}}{2\sqrt{\pi}} + \left(\frac{\lambda^2 \sqrt{t}}{2}\right)^2 e^{\left(\frac{\lambda^2 \sqrt{t}}{2}\right)^2} \left[1 + \mathrm{erf}\left(\frac{\lambda^2 \sqrt{t}}{2}\right)\right].$$

Therefore,

Hence,

$$\mathcal{K}(t,x) = \sum_{n=1}^{\infty} \lambda^{2n} \underbrace{\left(G^2 \star \cdots \star G^2\right)}_{n\text{'s } G^2}(t,x) = G(t/2,x) \sum_{n=1}^{\infty} \frac{\lambda^{2n} t^{n/2-1}}{2^n \Gamma(n/2)}$$

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Successive approximations (cont...)

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Therefore,

$$\mathcal{K}(t,x) = G(t/2,x) \left(\frac{\lambda^2}{\sqrt{4\pi t}} + \frac{\lambda^4}{4} e^{\lambda^4 t/4} \left[1 + \operatorname{erf}\left(\frac{\lambda^2 \sqrt{t}}{2}\right)\right]\right).$$

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Therefore,

$$\mathcal{K}(t,x) = G(t/2,x) \left(\frac{\lambda^2}{\sqrt{4\pi t}} + \frac{\lambda^4}{2} e^{\lambda^4 t/4} \Phi\left(\lambda^2 \sqrt{\frac{t}{2}}\right)\right).$$

Successive approximations (cont...)

In summary,

$$\boxed{\mathbb{E}[u(t,x)^2] = J_0^2(t,x) + (J_0^2 \star \mathcal{K})(t,x)}$$

- $J_0(t,x) = \int_{\mathbb{R}} G(t, x-y)\mu(\mathrm{d}y)$.
- $\mathcal{K}(t,x) = G(t/2,x)\left(\frac{\lambda^2}{\sqrt{4\pi t}} + \frac{\lambda^4}{2} e^{\lambda^4 t/4}\Phi\left(\lambda^2\sqrt{\frac{t}{2}}\right)\right)$.
- Initial data μ can be any measure such that

$$J_0(t,x) < \infty \quad \forall t > 0, x \in \mathbb{R} \qquad \Longleftrightarrow \qquad \int_{\mathbb{R}} e^{-a|x|^2}\mu(\mathrm{d}x) < \infty \quad \forall a > 0.$$

Successive approximations (cont...)

In summary,

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Laplace-Fourier transform

1. Recall tha $f(t,x) = J_0^2(t,x) + \lambda^2 (G^2 \star f)(t,x)$.

2. Let $\mathcal{L}$ (resp. $\mathcal{F}$) denote the Laplace (resp. Fourier) transform in t (resp. x) variable with new variable z (resp. ξ).

3. Then

$$\mathcal{L}\mathcal{F}[f](z,\xi) = \mathcal{L}\mathcal{F}[J_0^2](z,\xi) + \lambda^2 \mathcal{L}\mathcal{F}[G^2](z,\xi)\, \mathcal{L}\mathcal{F}[f](z,\xi),$$

   and so

$$\mathcal{L}\mathcal{F}[f](z,\xi) = \mathcal{L}\mathcal{F}[J_0^2](z,\xi) + \frac{\lambda^2 \mathcal{L}\mathcal{F}[G^2](z,\xi)}{1-\lambda^2 \mathcal{L}\mathcal{F}[G^2](z,\xi)} \mathcal{L}\mathcal{F}[J_0^2](z,\xi).$$

4. Notice that $\mathcal{F}[G^2(t,\cdot)](\xi) = \frac{1}{\sqrt{4\pi t}} \mathcal{F}[G(t/2,\cdot)](\xi) = \frac{1}{\sqrt{4\pi t}} e^{-|\xi|^2 t/4}$.

5. Notice also that $\mathcal{L}[t^\gamma](z) = \frac{\Gamma(\gamma+1)}{z^{\gamma+1}}$, $\gamma > -1$. So $\mathcal{L}[t^{-1/2}](z) = \frac{\Gamma(1/2)}{z^{1/2}}$ and

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4. Notice that $\mathcal{F}[G^2(t,\cdot)](\xi) = \frac{1}{\sqrt{4\pi t}} \mathcal{F}[G(t/2,\cdot)](\xi) = \frac{1}{\sqrt{4\pi t}} e^{-|\xi|^2 t/4}$.
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2. Let $\mathcal{L}$ (resp. $\mathcal{F}$) denote the Laplace (resp. Fourier) transform in t (resp. x) variable with new variable z (resp. ξ).
3. Then

$$\mathcal{LF}[f](z,\xi) = \mathcal{LF}[J_0^2](z,\xi) + \lambda^2 \mathcal{LF}[G^2](z,\xi)\, \mathcal{LF}[f](z,\xi),$$

and so

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6. From 5, we see that

$$\frac{\lambda^2 \mathcal{L}\mathcal{F}[G^2](z,\xi)}{1-\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)} = \frac{\lambda^2}{\sqrt{4z+|\xi|^2}-\lambda^2}.$$

7. First assume $\xi = 0$. Notice that

$$\mathcal{L}^{-1}\left[\frac{\lambda^2}{2\sqrt{z}-\lambda^2}\right](t) = \mathrm{Res}_{z=\frac{\lambda^4}{4}} \frac{\lambda^2 e^{zt}}{2\sqrt{z}-\lambda^2} + \frac{1}{2\pi i}\int_{-\infty}^{0+} \frac{\lambda^2 e^{zt}}{2\sqrt{z}-\lambda^2} \mathrm{d}z.$$

The residue part is equal to $\frac{\lambda^4}{2} e^{\frac{\lambda^4 t}{4}}$. The contour integral is equal to

$$\begin{aligned} -\frac{1}{2\pi i}\int_0^\infty \left(\frac{\lambda^2}{2\sqrt{ui}-\lambda^2} + \frac{\lambda^2}{2\sqrt{ui}+\lambda^2}\right) e^{-ut}\mathrm{d}u \\ = \frac{\lambda^2}{2\pi}\int_0^\infty \frac{\sqrt{u}e^{-ut}}{u+\lambda^4/4}\mathrm{d}u \\ = \frac{\lambda^2}{\sqrt{4\pi t}} - \frac{\lambda^4}{4} e^{\frac{\lambda^4 t}{4}} \mathrm{erfc}\left(\frac{\sqrt{t}\lambda^2}{2}\right). \end{aligned} \quad (\&)$$

Hence

$$\mathcal{L}^{-1}\left[\frac{\lambda^2}{2\sqrt{z}-\lambda^2}\right](t) = \frac{\lambda^2}{\sqrt{4\pi t}} + \frac{\lambda^4}{4} e^{\frac{\lambda^4 t}{4}} \left[1+\mathrm{erf}\left(\frac{\sqrt{t}\lambda^2}{2}\right)\right]$$

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$$= \frac{\lambda^2}{2\pi}\int_0^\infty \frac{\sqrt{u}e^{-ut}}{u+\lambda^4/4}\mathrm{d}u$$

$$= \frac{\lambda^2}{\sqrt{4\pi t}} - \frac{\lambda^4}{4}e^{\frac{\lambda^4 t}{4}}\mathrm{erfc}\left(\frac{\sqrt{t}\lambda^2}{2}\right). \qquad (\&)$$

Hence $$\mathcal{L}^{-1}\left[\frac{\lambda^2}{2\sqrt{z}-\lambda^2}\right](t) = \frac{\lambda^2}{\sqrt{4\pi t}} + \frac{\lambda^4}{4}e^{\frac{\lambda^4 t}{4}}\left[1+\mathrm{erf}\left(\frac{\sqrt{t}\lambda^2}{2}\right)\right]$$

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Hence

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Laplace-Fourier transform (cont...)

6. From 5, we see that

$$\frac{\lambda^2 \mathcal{L}\mathcal{F}[G^2](z,\xi)}{1-\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)} = \frac{\lambda^2}{\sqrt{4z+|\xi|^2}-\lambda^2}.$$

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&= \frac{\lambda^2}{2\pi}\int_0^{\infty}\frac{\sqrt{u}e^{-ut}}{u+\lambda^4/4}\mathrm{d}u \\
&= \frac{\lambda^2}{\sqrt{4\pi t}} - \frac{\lambda^4}{4} e^{\frac{\lambda^4 t}{4}} \mathrm{erfc}\left(\frac{\sqrt{t}\lambda^2}{2}\right). \qquad (\&)
\end{aligned}$$

Hence $\quad \mathcal{L}^{-1}\left[\frac{\lambda^2}{2\sqrt{z}-\lambda^2}\right](t) = \frac{\lambda^2}{\sqrt{4\pi t}} + \frac{\lambda^4}{4} e^{\frac{\lambda^4 t}{4}}\left[1+\mathrm{erf}\left(\frac{\sqrt{t}\lambda^2}{2}\right)\right]$

6. From 5, we see that

$$\frac{\lambda^2 \mathcal{L}\mathcal{F}[G^2](z,\xi)}{1-\lambda^2 \mathcal{L}\mathcal{F}[G^2](z,\xi)} = \frac{\lambda^2}{\sqrt{4z+|\xi|^2}-\lambda^2}.$$

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Hence $$\mathcal{L}^{-1}\left[\frac{\lambda^2}{2\sqrt{z}-\lambda^2}\right](t) = \frac{\lambda^2}{\sqrt{4\pi t}} + \frac{\lambda^4}{4} e^{\frac{\lambda^4 t}{4}}\left[1+\mathrm{erf}\left(\frac{\sqrt{t}\lambda^2}{2}\right)\right]$$

8. Hence, if $\xi \neq 0$,

$$\mathcal{L}^{-1}\left[\frac{\lambda^2}{\sqrt{4z+|\xi|^2}-\lambda^2}\right](t) = e^{-\frac{|\xi|^2}{4}}\left(\frac{\lambda^2}{\sqrt{4\pi t}} + \frac{\lambda^4}{4}e^{\frac{\lambda^4 t}{4}}\left[1+\mathrm{erf}\left(\frac{\sqrt{t}\lambda^2}{2}\right)\right]\right)$$

9. The inverse Fourier transform of RHD of 8 is straightforward:

$$G(t/2,x)\left(\frac{\lambda^2}{\sqrt{4\pi t}} + \frac{\lambda^4}{4}e^{\frac{\lambda^4 t}{4}}\left[1+\mathrm{erf}\left(\frac{\sqrt{t}\lambda^2}{2}\right)\right]\right).$$

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10. Therefore,

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Laplace-Fourier transform (cont...)

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1. Erdelyi's *Tables of Integral Transforms*, Vol. I, p. 136, Eq. (23).
2. Otherwise, we have that

$$\begin{aligned}\int_0^\infty \frac{\sqrt{u}}{u+a^2} e^{-ut}\mathrm{d}u &= \int_0^\infty \frac{2w^2}{w^2+a^2} e^{-w^2 t}\mathrm{d}w \\ &= 2\int_0^\infty e^{-w^2 t}\mathrm{d}w - 2a^2\int_0^\infty \frac{1}{w^2+a^2} e^{-w^2 t}\mathrm{d}w \\ &= \frac{\sqrt{\pi}}{\sqrt{t}} - 2a^2 I(t).\end{aligned}$$

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Summary: two ways to compute the resolvent kernel function

$$\boxed{\mathbb{E}\left[u(t,x)^2\right] = J_0^2(t,x) + \left(J_0^2 \star \mathcal{K}\right)(t,x)}$$

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$$\mathcal{K}(t,x) = \sum_{n=1}^{\infty} \lambda^n \underbrace{\left(G^2 \star \cdots \star G^2\right)}_{n\text{'s } G^2}(t,x).$$

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Plan

Stochastic heat equation on $\mathbb{R}$

Stochastic wave equation on $\mathbb{R}$

Further simplifications for SHE on $\mathbb{R}$

Stochastic heat equation on a torus

Stochastic Wave Equation on $\mathbb{R}$

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \kappa^2 \frac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x) \dot{W}(t,x), \quad t > 0 \ , \ x \in \mathbb{R}, \ (\kappa > 0) \\ u(0,\cdot) = g(\cdot), \quad \frac{\partial u}{\partial t}(0,\cdot) = \mu(\cdot) \end{cases} \tag{SWE}$$

1. $G(t,x) = \frac{1}{2} 1_{[-\kappa t, \kappa t]}(x), \quad$ for $t \geq 0$, $x \in \mathbb{R}$.
2. Set $J_0(t,x)$ to be the solution to the homogeneous equation, i.e.,

$$J_0(t,x) = \frac{1}{2}\left[g(x+\kappa t) + g(x-\kappa t)\right] + \int_{\mathbb{R}} G(t, x-y)\mu(\mathrm{d}y).$$

3. (SWE) is interpreted in the integral form:

$$u(t,x) = J_0(t,x) + \lambda \int_0^t \int_{\mathbb{R}} G(t-s, x-y) u(s,y) W(\mathrm{d}s, \mathrm{d}y).$$

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Two ways to solve the second moment for SWE

$$\boxed{||u(t,x)||_2^2 = J_0^2(t,x) + (J_0^2 \star \mathcal{K})(t,x)}$$

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1. We claim that

$$\underbrace{\left(G^2 \star \cdots \star G^2\right)}_{n\text{'s } G^2}(t,x) = \frac{\left[(\kappa t)^2 - x^2\right]^{n-1}}{2^{3n-1}\Gamma(n)^2 \kappa^{n-1}} 1_{[-\kappa t, \kappa t]}(x). \qquad (*)$$

2. Notice that the *hyperbolic Bessel function* $I_n(x)$ has the expansion:

$$I_n(x) = \sum_{k=1}^{\infty} \frac{(x^2/4)^k}{\Gamma(k)\Gamma(n+k)}.$$

3. Hence,

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2. Suppose it is true for n, then

$$\Big[\underbrace{\left(G^2 \star \cdots \star G^2\right)}_{n\text{'s } G^2} \star G^2\Big](t,x) = \frac{1}{2^{3n-1}\Gamma(n)^2\kappa^{n-1}} \frac{1}{4}\int_0^t \int_{\mathbb{R}} \left[(\kappa s)^2 - y^2\right]^{n-1}$$

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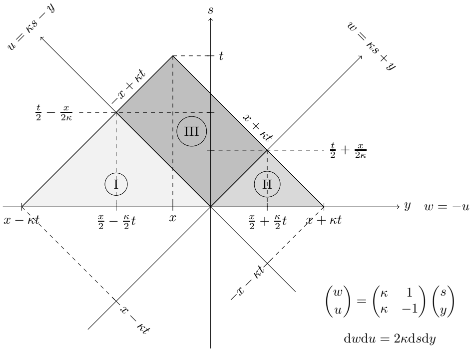
Successive approximation (the proof of $(*)$)

3. By the change of variables $(s, y) \longrightarrow (w, u)$:



the RHD of eq. in step 2 becomes

$$\frac{1}{2^{3n+1}\Gamma(n)^2\kappa^{n-1}} \frac{1}{2\kappa} \int_0^{\kappa t - x} \mathrm{d}u \, u^{n-1} \int_0^{\kappa t + x} \mathrm{d}w \, w^{n-1} = \frac{\left[(\kappa t)^2 - x^2\right]^n}{2^{3n+2}\Gamma(n+1)^2\kappa^n}$$

if $x \in [-\kappa t, \kappa t]$, and 0 otherwise.

Integral transforms

$$\mathcal{K}(t,x) = \mathcal{F}^{-1}\mathcal{L}^{-1}\left[\frac{\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)}{1-\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)}\right](t,x).$$

1. First notice that

$$\mathcal{F}[G^2(t,\cdot)](\xi) = 4^{-1}\int_{-\kappa t}^{\kappa t} e^{-ix\xi}\mathrm{d}x = \frac{\sin(t\kappa\xi)}{2\xi}, \qquad \xi\in\mathbb{R},$$

$$\mathcal{L}\mathcal{F}[G^2](z,\xi) = \int_0^\infty \frac{\sin(t\kappa\xi)}{2\xi}e^{-zt}\mathrm{d}t = \frac{\kappa}{2(z^2+\kappa^2\xi^2)}, \qquad \Re(z)>0.$$

2. Hence,

$$\begin{aligned}\frac{\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)}{1-\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)} &= \frac{\kappa\lambda^2}{2z^2+2\kappa^2\left(\xi^2-\frac{\lambda^2}{2\kappa}\right)} \\ &= \frac{\lambda^2}{4i\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}}\left[\frac{1}{z-i\kappa\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}}-\frac{1}{z+i\kappa\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}}\right].\end{aligned}$$

$G(t,x) = 1/2 \times 1_{[-\kappa t,\kappa t]}(x)$.

$$\mathcal{K}(t,x) = \mathcal{F}^{-1}\mathcal{L}^{-1}\left[\frac{\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)}{1-\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)}\right](t,x).$$

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2. Hence,

$$\frac{\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)}{1-\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)} = \frac{\kappa\lambda^2}{2z^2+2\kappa^2\left(\xi^2-\frac{\lambda^2}{2\kappa}\right)}$$
$$= \frac{\lambda^2}{4i\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}}\left[\frac{1}{z-i\kappa\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}}-\frac{1}{z+i\kappa\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}}\right].$$

$G(t,x) = 1/2\times 1_{[-\kappa t,\kappa t]}(x)$.

Integral transforms

$$\mathcal{K}(t,x) = \mathcal{F}^{-1}\mathcal{L}^{-1}\left[\frac{\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)}{1-\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)}\right](t,x).$$

1. First notice that

$$\mathcal{F}[G^2(t,\cdot)](\xi) = 4^{-1}\int_{-\kappa t}^{\kappa t} e^{-ix\xi}\mathrm{d}x = \frac{\sin(t\kappa\xi)}{2\xi}, \qquad \xi\in\mathbb{R},$$

$$\mathcal{L}\mathcal{F}[G^2](z,\xi) = \int_0^\infty \frac{\sin(t\kappa\xi)}{2\xi}e^{-zt}\mathrm{d}t = \frac{\kappa}{2(z^2+\kappa^2\xi^2)}, \qquad \Re(z)>0.$$

2. Hence,

$$\frac{\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)}{1-\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)} = \frac{\kappa\lambda^2}{2z^2+2\kappa^2\left(\xi^2-\frac{\lambda^2}{2\kappa}\right)}$$

$$= \frac{\lambda^2}{4i\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}}\left[\frac{1}{z-i\kappa\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}}-\frac{1}{z+i\kappa\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}}\right].$$

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$$\begin{aligned}\frac{\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)}{1-\lambda^2\mathcal{L}\mathcal{F}[G^2](z,\xi)} &= \frac{\kappa\lambda^2}{2z^2+2\kappa^2\left(\xi^2-\frac{\lambda^2}{2\kappa}\right)}\\ &= \frac{\lambda^2}{4i\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}}\left[\frac{1}{z-i\kappa\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}}-\frac{1}{z+i\kappa\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}}\right].\end{aligned}$$

$G(t,x) = 1/2\times 1_{[-\kappa t,\kappa t]}(x)$.

Integral Transforms (cont...)

3. Inverse Laplace transform can be easily computed as

$$\frac{\lambda^2}{2}\frac{\sin\left(\kappa t\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}\right)}{\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}}$$

4. The following inverse Fourier transform is more involved:

$$\frac{\lambda^2}{2\pi}\int_0^\infty \frac{\sin\left(\kappa t\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}\right)}{\sqrt{\xi^2-\frac{\lambda^2}{2\kappa}}}\cos(x\xi)\mathrm{d}\xi$$

5. Thanks to [Erdelyi's *Table of Int. Trans.*, I, p. 26, (30)], for $x > 0$ and $a \in \mathbb{C}$,

$$\int_0^\infty \frac{\sin\left(b\sqrt{\xi^2+a^2}\right)}{\sqrt{\xi^2+a^2}}\cos(\xi x)\mathrm{d}\xi = \frac{\pi}{2}J_0\left(a\sqrt{b^2-x^2}\right)1_{[0,b]}(x)$$

Hence: $$\mathcal{K}(t,x) = \frac{\lambda^2}{4}I_0\left(\sqrt{\frac{\lambda^2\left[(\kappa t)^2-x^2\right]}{2\kappa}}\right)1_{[-\kappa t,\kappa t]}(x)$$

Integral Transforms (cont...)

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Integral Transforms (cont...)

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Plan

Stochastic heat equation on $\mathbb{R}$

Stochastic wave equation on $\mathbb{R}$

Further simplifications for SHE on $\mathbb{R}$

Stochastic heat equation on a torus

Further simplifications for

$$||u(t,x)||_2^2 = J_0^2(t,x) + (J_0^2 \star \mathcal{K})(t,x)$$

$$\begin{aligned}\mathbb{E}\left[u(t,x_1)u(t,x_2)\right] = \; & J_0(t,x_1)J_0(t,x_2) + \frac{\lambda^2}{2\nu}\iint_{\mathbb{R}^2}\mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2) \\ & \times G_{\nu/2}\left(t, \frac{x_1+x_2}{2} - \frac{z_1+z_2}{2}\right)\Phi\left(\frac{\lambda^2 t - |x_1-x_2| - |z_1-z_2|}{\sqrt{2\nu t}}\right) \\ & \times \exp\left(\frac{\lambda^4 t}{4\nu} - \frac{\lambda^2}{2\nu}(|x_1-x_2| + |z_1-z_2|)\right)\end{aligned}$$

Explicitly worked-out examples:

1. $\mu = \delta_0$
2. $\mu(\mathrm{d}x) = \mathrm{d}x$
3. $\mu(\mathrm{d}x) = e^{\alpha x}\mathrm{d}x$

   $\mu(\mathrm{d}x) = (e^{\alpha x} \pm e^{-\alpha x})\mathrm{d}x$, $\alpha \in \mathbb{R}$.

$$\left(\frac{\partial}{\partial t} - \frac{\nu}{2}\frac{\partial^2}{\partial x^2}\right)u(t,x) = \lambda u(t,x)\dot{W}(t,x)$$

$$G_\nu(t,x) = (2\pi\nu t)^{-1/2}\exp(-x^2/(2\nu t))$$

Further simplifications for

$||u(t,x)||_2^2 = J_0^2(t,x) + (J_0^2 \star \mathcal{K})(t,x)$

$$\mathbb{E}\left[u(t,x_1)u(t,x_2)\right] = J_0(t,x_1)J_0(t,x_2) + \frac{\lambda^2}{2\nu}\iint_{\mathbb{R}^2}\mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)$$
$$\times\, G_{\nu/2}\left(t, \frac{x_1+x_2}{2} - \frac{z_1+z_2}{2}\right)\Phi\left(\frac{\lambda^2 t - |x_1-x_2| - |z_1-z_2|}{\sqrt{2\nu t}}\right)$$
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Example 1: Delta initial data $\mu = \delta_0$

$$\mathbb{E}\left[u(t,x_1)u(t,x_2)\right] = G_\nu(t,x_1)G_\nu(t,x_2) + \frac{\lambda^2}{2\nu} G_{\nu/2}\left(t, \frac{x_1+x_2}{2}\right) \times \exp\left(\frac{\lambda^4 t}{4\nu} - \frac{\lambda^2}{2\nu}|x_1-x_2|\right) \Phi\left(\frac{\lambda^2 t - |x_1-x_2|}{\sqrt{2\nu t}}\right)$$

$$\mathbb{E}\left[u(t,x)^2\right] = G_\nu(t,x)^2 + \frac{\lambda^2}{2\nu} G_{\nu/2}(t,x) \exp\left(\frac{\lambda^4 t}{4\nu}\right) \Phi\left(\frac{\lambda^2 t}{\sqrt{2\nu t}}\right) = \lambda^{-2}\mathcal{K}(t,x)$$

$G_\nu(t,x) = (2\pi\nu t)^{-1/2}\exp(-x^2/(2\nu t))$

Example 2: Leb. measure $\mu(\mathrm{d}x) \equiv \mathrm{d}x$

$$
\begin{aligned}
\mathbb{E}\left[u(t,x_1)u(t,x_2)\right] =& 2\exp\left(\frac{\lambda^4 t}{4\nu} - \frac{\lambda^2}{2\nu}|x_1 - x_2|\right)\Phi\left(\frac{\lambda^2 t - |x_1 - x_2|}{\sqrt{2\nu t}}\right) \\
&+ 2\Phi\left(\frac{|x_1 - x_2|}{\sqrt{2\nu t}}\right) - 1
\end{aligned}
$$

$$
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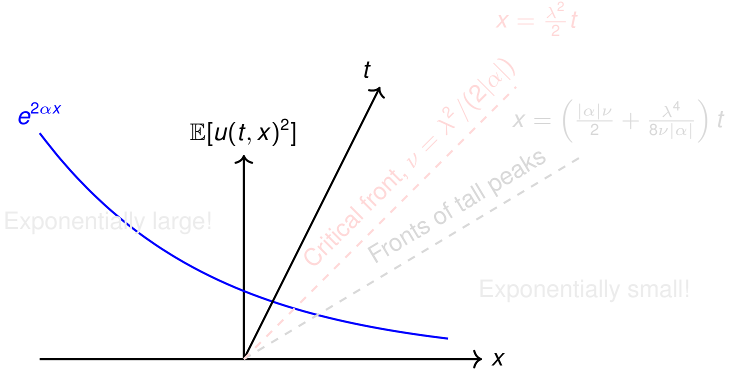
&+ 2\Phi\left(\frac{|x_1-x_2|}{\sqrt{2\nu t}}\right) - 1 \\
\asymp& 1 + \frac{2\sqrt{\nu t}\left(2\left|x_1-x_2\right| - \lambda^2 t\right)}{\sqrt{\pi}\left((x_1-x_2)^2 - \lambda^2 t\left|x_1-x_2\right|\right)}\exp\left(-\frac{(x_1-x_2)^2}{4\nu t}\right) \\
&\text{as } |x_1-x_2| \to \infty.
\end{aligned}
$$

$$
\mathbb{E}\left[u(t,x)^2\right] = 2\exp\left(\frac{\lambda^4 t}{4\nu}\right)\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)
$$

Special case 3: Exponential initial data $\mu(\mathrm{d}x) = e^{\alpha x}\mathrm{d}x$

$$\mathbb{E}\left[u(t,x)^2\right] = 2\exp\left(2\alpha x + \alpha^2 \nu t + \frac{\lambda^4 t}{4\nu}\right)\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)$$

$$\frac{\partial}{\partial t}u = \frac{\nu}{2}\Delta u + \lambda u \dot{W}, \quad u_0 = e^{\alpha x}$$



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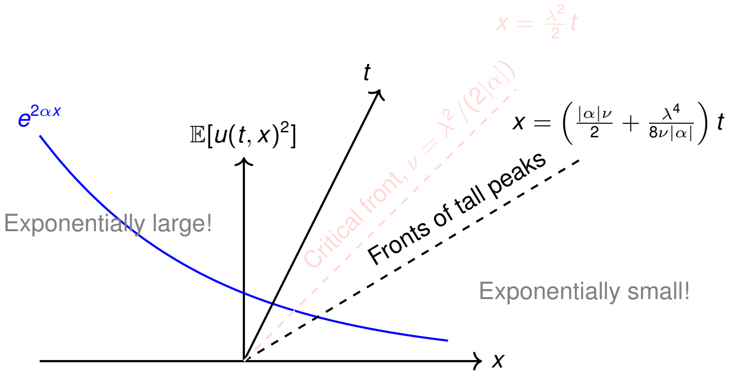

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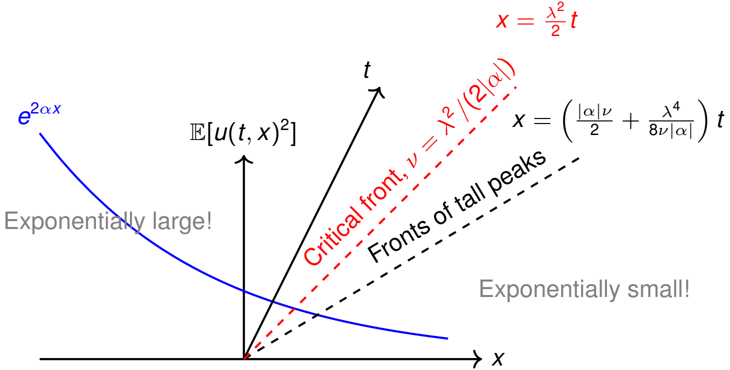

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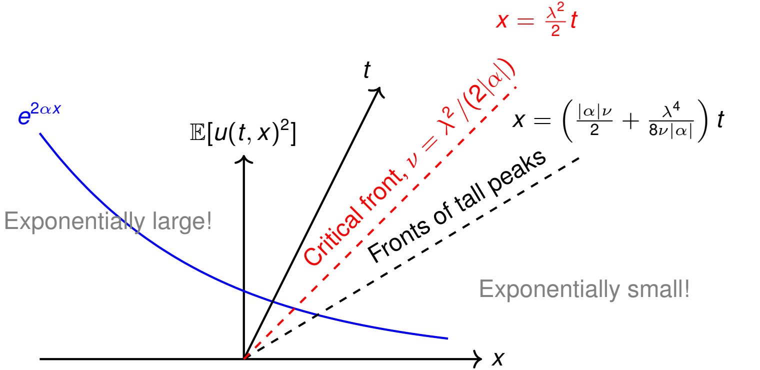
$$\frac{\partial}{\partial t}u = \frac{\nu}{2}\Delta u + \lambda u\dot{W}, \quad u_0 = e^{\alpha x}$$

1. $\nu < \lambda^2/(2|\alpha|)$: Noise dominates.
2. $\nu > \lambda^2/(2|\alpha|)$: Migration dominates.



Example 3: Exponential initial data $\mu(\mathrm{d}x) = (e^{\alpha x} \pm e^{-\alpha x})\mathrm{d}x$

$$
\begin{aligned}
\mathbb{E}\left[u(t,x)^2\right] =& 2\exp\left(\alpha^2\nu t + \frac{\lambda^4 t}{4\nu}\right)\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)\left(e^{2\alpha x} + e^{-2\alpha x}\right) \\
&\pm \frac{4\alpha\nu}{\lambda^2 + 2\alpha\nu} e^{\alpha^2\nu t} \\
&\pm \frac{4\lambda^2}{\lambda^4 - 4\alpha^2\nu^2}\left[\lambda^2 e^{\frac{\lambda^4 t}{4\nu}}\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right) - 2\alpha\nu e^{\alpha^2\nu t}\Phi\left(\alpha\sqrt{2\nu t}\right)\right]
\end{aligned}
$$

"$+$" : as $\alpha \to 0$, $\mathbb{E}[u(t,x)^2]$ converges to the Lebesgue case $\mu(\mathrm{d}x) = 2\mathrm{d}x$.
"$-$" : as $\alpha \to 0$, $\mathbb{E}[u(t,x)^2] \to 0$.

- Critical diffussion: $\nu = \lambda^2/(2|\alpha|)$.

Final remark
$m(t, x_1, x_2) := \mathbb{E}[u(t, x_1)u(t, x_2)]$

$$\begin{cases} \dfrac{\partial}{\partial t} m(t, x_1, x_2) = \left[\dfrac{\nu}{2}\left(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}\right) + \lambda \delta_0(x_1 - x_2)\right] m(t, x_1, x_2), \quad x_1, x_2 \in \mathbb{R} \\ m(0, \mathrm{d}x_1, \mathrm{d}x_2) = \mu(\mathrm{d}x_1)\mu(\mathrm{d}x_2) \end{cases}$$

$\Downarrow$ explicitly solved!

$$\begin{aligned} m(t, x_1, x_2) = \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2) \Bigg[& G_\nu(t, x_1 - y) G_\nu(t, x_2 - y) \\ & + \frac{\lambda^2}{2\nu} G_{\nu/2}\left(t, \frac{x_1 + x_2}{2} - \frac{z_1 + z_2}{2}\right) \Phi\left(\frac{\lambda^2 t - |x_1 - x_2| - |z_1 - z_2|}{\sqrt{2\nu t}}\right) \\ & \times \exp\left(\frac{\lambda^4 t}{4\nu} - \frac{\lambda^2}{2\nu}(|x_1 - x_2| + |z_1 - z_2|)\right) \Bigg] \end{aligned}$$

Final remark

$$m(t,x_1,x_2) := \mathbb{E}[u(t,x_1)u(t,x_2)]$$

$$\begin{cases} \dfrac{\partial}{\partial t} m(t,x_1,x_2) = \left[\dfrac{\nu}{2}\left(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}\right) + \lambda\delta_0(x_1 - x_2)\right] m(t,x_1,x_2), \quad x_1, x_2 \in \mathbb{R} \\ m(0,\mathrm{d}x_1,\mathrm{d}x_2) = \mu(\mathrm{d}x_1)\mu(\mathrm{d}x_2) \end{cases}$$

$$\Downarrow$$

explicitly solved!

$$\begin{aligned} m(t,x_1,x_2) = \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2) \Bigg[& G_\nu(t,x_1-y)G_\nu(t,x_2-y) \\ & + \frac{\lambda^2}{2\nu} G_{\nu/2}\left(t, \frac{x_1+x_2}{2} - \frac{z_1+z_2}{2}\right) \Phi\left(\frac{\lambda^2 t - |x_1-x_2| - |z_1-z_2|}{\sqrt{2\nu t}}\right) \\ & \times \exp\left(\frac{\lambda^4 t}{4\nu} - \frac{\lambda^2}{2\nu}(|x_1-x_2| + |z_1-z_2|)\right)\Bigg] \end{aligned}$$

Plan

Stochastic heat equation on $\mathbb{R}$

Stochastic wave equation on $\mathbb{R}$

Further simplifications for SHE on $\mathbb{R}$

Stochastic heat equation on a torus

Stochastic heat equation on a torus

– Ongoing project with Axel Saenz

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x)\dot{W}(t,x), & t > 0\ ,\ x \in [0,L], \\ u(t,0) = u(t,L), & t > 0 \\ u(0,\cdot) = \mu. \end{cases}$$

- $Q(t,x) = \sum_{n\in\mathbb{Z}} G(t,x+nL), \qquad Q(t,x,y) := Q(t,x-y).$

Goal: Evaluate, explicitly,

$$\mathcal{K}(t,x) := \sum_{n=1}^{\infty} \lambda^{2n} \Big(\underbrace{Q^2 \star \cdots \star Q^2}_{n\text{'s } Q^2} \Big)(t,x)$$

where $\quad (f \star g)(t,x) := \int_0^t \mathrm{d}s \int_0^L \mathrm{d}y\, f(t-s,x-y) g(s,y).$

$G(t,x) = (2\pi t)^{-1/2} \exp(-x^2/(2t))$

Stochastic heat equation on a torus

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The resolvent kernel function

$$\mathcal{K}(t,x) = \begin{pmatrix}\mathcal{H}_1(t) & \mathcal{H}_2(t)\end{pmatrix} \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \begin{pmatrix}Q\left(\frac{t}{2},x\right) & Q\left(\frac{t}{2},x+\frac{L}{2}\right)\end{pmatrix}^T,$$

for all $t > 0$ and $x \in [0, L]$, with

$$\mathcal{H}_j(t) = \alpha_{j,0} \exp\left(y_{j,0} t\right) + \sum_{n=1}^{\infty} \alpha_{j,n} \exp\left(-y_{j,n} t\right), \quad j = 1, 2,$$

where all $y_{j,n} \geq 0$ and $\alpha_{j,n}$ can be determined explicitly. In particular,

$$y_{1,0} = \lim_{t\to\infty} \frac{1}{t} \log \mathcal{K}(t,x),$$

where $y_{1,0}$ is the unique strictly positive sol. to $\coth(\sqrt{y}L) = 2\lambda^{-2}\sqrt{y}$.
Moreover, for any $\epsilon \in (0, 1)$, there exists $L_0 = L_0(\lambda, \epsilon) > 0$ s.t.,

$$\frac{\lambda^4}{4}\left(1 + 2(1-\epsilon)e^{-L\lambda^2}\right)^2 \leq y_{1,0} \leq \frac{\lambda^4}{4}\left(1 + 2(1+\epsilon)e^{-L\lambda^2}\right)^2, \quad \forall L > L_0,$$

i.e., $\quad y_{1,0} \asymp \frac{\lambda^4}{4}\left(1 + 2e^{-L\lambda^2}\right)^2$ as $L \to \infty$.

The resolvent kernel function

$$\mathcal{K}(t,x) = \begin{pmatrix} \mathcal{H}_1(t) & \mathcal{H}_2(t) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} Q\left(\frac{t}{2}, x\right) & Q\left(\frac{t}{2}, x+\frac{L}{2}\right) \end{pmatrix}^T,$$

for all $t > 0$ and $x \in [0, L]$, with

$$\mathcal{H}_j(t) = \alpha_{j,0} \exp\left(y_{j,0} t\right) + \sum_{n=1}^{\infty} \alpha_{j,n} \exp\left(-y_{j,n} t\right), \quad j = 1, 2,$$

where all $y_{j,n} \geq 0$ and $\alpha_{j,n}$ can be determined explicitly. In particular,

$$y_{1,0} = \lim_{t\to\infty} \frac{1}{t} \log \mathcal{K}(t,x),$$

where $y_{1,0}$ is the unique strictly positive sol. to $\coth(\sqrt{y}L) = 2\lambda^{-2}\sqrt{y}$.
Moreover, for any $\epsilon \in (0,1)$, there exists $L_0 = L_0(\lambda, \epsilon) > 0$ s.t.,

$$\frac{\lambda^4}{4}\left(1 + 2(1-\epsilon)e^{-L\lambda^2}\right)^2 \leq y_{1,0} \leq \frac{\lambda^4}{4}\left(1 + 2(1+\epsilon)e^{-L\lambda^2}\right)^2, \quad \forall L > L_0,$$

i.e., $y_{1,0} \asymp \frac{\lambda^4}{4}\left(1 + 2e^{-L\lambda^2}\right)^2$ as $L \to \infty$.

The resolvent kernel function

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Thank you!