Yang–Mills for probabilists

Sourav Chatterjee

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  - Gravity: General relativity (GR). Einstein.

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- Even from the point of view of theoretical physicists, there are very important unsolved theoretical problems — quark confinement, mass gap, etc.

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- Both gravity and quantum effects manifest themselves in black holes. Small black holes can potentially form when particles collide with each other at very high speeds, as in particle accelerators.

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- The most promising approach: String theory.
- Roughly speaking, strings moving randomly trace out random surfaces. Higher dimensional strings, known as branes, trace out higher dimensional random manifolds.

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- The principle of going to one dimension higher is known as the holographic principle.

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- In the remaining part of the talk, I will present some concrete mathematical problem for probabilists.
- The physics connections will not be discussed in any great depth due to time constraints. I will only say one or two sentences for each problem, connecting the math problems with the physics problems mentioned before.

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- Euclidean Yang–Mills theories are supposed to be scaling limits of lattice gauge theories.

Constructive field theory

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- The constructive field theory program waged a valiant battle for more than thirty years (1960–1990), making sense of various quantum field theories in two and three dimensions, but never quite reached its ultimate goal of constructing 4D quantum Yang–Mills theories. May be revival possible?

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- Then $\mathfrak{g}$ is a subspace of the space of all $N \times N$ skew-Hermitian matrices.

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- This means that at each x, $F(x)$ is an $n \times n$ array of skew-Hermitian matrices of order N, whose $(j,k)^{\mathrm{th}}$ entry is the matrix

$$F_{jk}(x) = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} + [A_j(x), A_k(x)].$$

The Yang–Mills action

- Let $\mathcal{A}$ be the space of all smooth G connection forms on $\mathbb{R}^n$. The Yang–Mills action on this space is the function

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- Explicitly, this is

$$S_{\mathrm{YM}}(A) = -\int_{\mathbb{R}^n} \sum_{j,k=1}^{n} \mathrm{Tr}(F_{jk}(x)^2)dx.$$

Formal definition of Euclidean YM theories

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  - Z is the normalizing constant that makes this a probability measure.

Problem

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- While it has been possible to give rigorous meanings to similar descriptions of Brownian motion and various quantum field theories in dimensions two and three, 4D Euclidean Yang–Mills theories have so far remained largely intractable.

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- The Wilson action of U is defined as

$$S_{\mathrm{W}}(U) := \sum_{p \in P(\Lambda)} \mathrm{Re}(\mathrm{Tr}(I - U_p)).$$

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Definition of lattice gauge theory

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- The uniqueness (or non-uniqueness) is in general unknown for lattice gauge theories in dimension four when β is large.

From LGT to Euclidean YM theory: Wilson's heuristic

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- This defines a configuration of unitary matrices assigned to directed edges of $\epsilon\mathbb{Z}^n$.

Wilson's heuristic, continued

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- The above heuristic was used by Wilson to justify the approximation of Euclidean Yang–Mills theory by lattice gauge theory, scaling the inverse coupling strength β like ϵ^{4-n} as the lattice spacing $\epsilon \to 0$.

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- In the above formulation, β does not scale with ϵ at all when $n = 4$.
- Currently, however, the general belief in the physics community is that β should scale like some multiple of $\log(1/\epsilon)$ in dimension four.

Wilson loops

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- Given a loop γ with directed edges $e_1, \ldots, e_m$, the Wilson loop variable W_γ is defined as

$$W_\gamma := \mathrm{Tr}(U(e_1)U(e_2)\cdots U(e_m)).$$

Open problem #1: Yang–Mills existence

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- The description of the limit is part of the problem.
- The most important groups are $SU(2)$ and $SU(3)$.
- Large body of work in 2D. Less in 3D. Almost none in 4D, except for a very long series of papers by Bałaban that people find very difficult to understand. May be someone can take off from where Bałaban stopped? Or revive the project using different ideas?

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- Even physicists do not think they have a proof of mass gap.
- A solution of this problem will explain, roughly speaking, why mass exists in the universe.

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- Disproof at large β for 4D $U(1)$ theory by Guth (1980) and Fröhlich & Spencer (1982).
- Proof of this conjecture will explain why we do not observe free quarks in nature. This is one of the biggest mysteries of particle physics.

Open problem #4: Gauge-string duality

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- Technically speaking, this problem can be discussed only after solving the problem of YM existence.
- The main step is to show that Wilson loop expectations in a continuum Yang–Mills theory can be expressed as integrals over trajectories of strings in a string theory, where the trajectories are in one dimension higher.

Some results about gauge-string duality

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- This proves a version of gauge-string duality and the holographic principle. Possibly the first rigorous result.
- The expansion was proved only at small β (strong coupling). Will be a very important breakthrough to prove something similar at large β.
- In 2D, the terms were explicitly evaluated by Basu & Ganguly (2016) using combinatorial techniques. May be the techniques can extend to higher dimensions?

The master loop equation

The following is a generalization of what are called Makeenko–Migdal equations or master loop equations. They hold at all β, and give the starting point for the proof of the $1/N$ expansion and gauge-string duality.

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Theorem (C., 2015)

Consider $SO(N)$ LGT on $\mathbb{Z}^n$. For a collection of loops $s = (\ell_1, \ldots, \ell_m)$, define

$$\phi(s) := \frac{\langle W_{\ell_1} W_{\ell_2} \cdots W_{\ell_m} \rangle}{N^m}.$$

Let $|s|$ be the total number of edges in s. Then

$$\begin{aligned}(N-1)|s|\phi(s) &= \sum_{s' \in \mathbb{T}^-(s)} \phi(s') - \sum_{s' \in \mathbb{T}^+(s)} \phi(s') + N \sum_{s' \in \mathbb{S}^-(s)} \phi(s') \\ &\quad - N \sum_{s' \in \mathbb{S}^+(s)} \phi(s') + \frac{1}{N} \sum_{s' \in \mathbb{M}^-(s)} \phi(s') - \frac{1}{N} \sum_{s' \in \mathbb{M}^+(s)} \phi(s') \\ &\quad + N\beta \sum_{s' \in \mathbb{D}^-(s)} \phi(s') - N\beta \sum_{s' \in \mathbb{D}^+(s)} \phi(s'),\end{aligned}$$

where $\mathbb{T}^\pm$, $\mathbb{S}^\pm$, $\mathbb{M}^\pm$ and $\mathbb{D}^\pm$ are certain operations that produce new collections of loops from old.

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- The preprint also has an extensive review of the mathematical literature on these topics, which I did not cover in this talk.
- Special thanks to David Brydges, Erhard Seiler and Steve Shenker for teaching me most of what I know about Yang–Mills theories, lattice gauge theories and quantum field theories.