Yang–Mills for probabilists

Sourav Chatterjee
The four fundamental forces:

- The electromagnetic force: Interaction of light and matter.
- The weak force: Interactions between sub-atomic particles.
- The strong force: Force that holds together quarks that form protons and neutrons.
- Gravity.
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  - Gravity: **General relativity (GR).** Einstein.
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➤ Even from the point of view of theoretical physicists, there are very important unsolved theoretical problems — quark confinement, mass gap, etc.
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More about quantum gravity

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- In quantum physics, the trajectories are probabilistic (although the notion of probability is replaced by complex probability amplitudes).

- General relativity is a classical theory, in the sense that the structure of curved spacetime is deterministic.

- A quantum theory of gravity would replace the fixed spacetime by a randomly fluctuating spacetime (random Riemannian manifold).

- The most promising approach: String theory.

  - Roughly speaking, strings moving randomly trace out random surfaces. Higher dimensional strings, known as branes, trace out higher dimensional random manifolds.
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Gauge-string duality

- There is a theoretical result of Weinberg and Witten (1980) that it is impossible to generate gravity using quantum field theories in the traditional sense (hard to explain without going into details).

- However, Weinberg and Witten have the unstated assumption that one is looking for both theories in the same dimension.

- In 1997, Maldacena made the remarkable discovery that certain quantum field theories are ‘dual’ to certain string theories in one dimension higher!

- Duality means that any calculation in one theory corresponds to some calculation in the other theory.

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The physics connections will not be discussed in any great depth due to time constraints. I will only say one or two sentences for each problem, connecting the math problems with the physics problems mentioned before.
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- Euclidean Yang–Mills theories are supposed to be scaling limits of lattice gauge theories.
The problem of rigorously constructing Euclidean Yang–Mills theories, and then using them to construct quantum Yang–Mills theories, is the problem of Yang–Mills existence.
Constructive field theory

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The constructive field theory program waged a valiant battle for more than thirty years (1960–1990), making sense of various quantum field theories in two and three dimensions, but never quite reached its ultimate goal of constructing 4D quantum Yang–Mills theories. May be revival possible?
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- Let $\mathfrak{g}$ be the Lie algebra of $G$.
- Then $\mathfrak{g}$ is a subspace of the space of all $N \times N$ skew-Hermitian matrices.
Connections and curvature

- A smooth $G$ connection form on $\mathbb{R}^n$ is a smooth map from $\mathbb{R}^n$ into $\mathfrak{g}^n$. 
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If $A$ is a $G$ connection form, its value $A(x)$ at $x$ is an $n$-tuple $(A_1(x), \ldots, A_n(x))$ of skew-Hermitian matrices. In the language of differential forms,

$$A = \sum_{j=1}^{n} A_j \ dx_j.$$
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This means that at each $x$, $F(x)$ is an $n \times n$ array of skew-Hermitian matrices of order $N$, whose $(j, k)^{th}$ entry is the matrix

$$F_{jk}(x) = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} + [A_j(x), A_k(x)].$$
Let $\mathcal{A}$ be the space of all smooth $G$ connection forms on $\mathbb{R}^n$. The Yang–Mills action on this space is the function

$$S_{YM}(A) := - \int_{\mathbb{R}^n} \text{Tr}(F \wedge \ast F),$$

where $F$ is the curvature form of $A$ and $\ast$ denotes the Hodge star operator, assuming that this integral is finite.
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$$S_{YM}(A) = - \int_{\mathbb{R}^n} \sum_{j,k=1}^{n} \text{Tr}(F_{jk}(x)^2)dx.$$
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$$d\mu(A) = \frac{1}{Z} \exp \left( - \frac{1}{4g^2} S_{YM}(A) \right) dA,$$

where:

- $A \in A$, the space of all smooth $G$ connection forms on $\mathbb{R}^n$,
- $S_{YM}$ is the Yang–Mills action,
- $dA = \prod_{j=1}^{\infty} \prod_{x \in \mathbb{R}^n} d(A_j(x))$ is 'infinite-dimensional Lebesgue measure' on $A$,
- $g$ is a parameter called the coupling strength, and
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The above description of Euclidean Yang–Mills theory with gauge group $G$ is not directly mathematically meaningful because of the problems associated with the definition Lebesgue measure on $A$. 

While it has been possible to give rigorous meanings to similar descriptions of Brownian motion and various quantum field theories in dimensions two and three, 4D Euclidean Yang–Mills theories have so far remained largely intractable.
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The lattice gauge theory with gauge group $G$ on a finite set $\Lambda \subseteq \mathbb{Z}^n$ is defined as follows.

Suppose that for any two adjacent vertices $x, y \in \Lambda$, we have a group element $U(x, y) \in G$, with $U(y, x) = U(x, y)^{-1}$.

Let $G(\Lambda)$ denote the set of all such configurations.

A square bounded by four edges is called a plaquette. Let $P(\Lambda)$ denote the set of all plaquettes in $\Lambda$.

For a plaquette $p \in P(\Lambda)$ with vertices $x_1, x_2, x_3, x_4$ in anti-clockwise order, and a configuration $U \in G(\Lambda)$, define $U_p := U(x_1, x_2)U(x_2, x_3)U(x_3, x_4)U(x_4, x_1)$.

The Wilson action of $U$ is defined as $S_W(U) := \sum_{p \in P(\Lambda)} \Re(\text{Tr}(I - U_p))$. 

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Lattice gauge theories


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Wilson (1974) proposed a discretization of Euclidean Yang–Mills theories, now known as lattice gauge theories. The lattice gauge theory with gauge group $G$ on a finite set $\Lambda \subseteq \mathbb{Z}^n$ is defined as follows. Suppose that for any two adjacent vertices $x, y \in \Lambda$, we have a group element $U(x, y) \in G$, with $U(y, x) = U(x, y)^{-1}$.
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Lattice gauge theories

- The lattice gauge theory with gauge group $G$ on a finite set $\Lambda \subseteq \mathbb{Z}^n$ is defined as follows.
- Suppose that for any two adjacent vertices $x, y \in \Lambda$, we have a group element $U(x, y) \in G$, with $U(y, x) = U(x, y)^{-1}$.
- Let $G(\Lambda)$ denote the set of all such configurations.
- A square bounded by four edges is called a plaquette. Let $P(\Lambda)$ denote the set of all plaquettes in $\Lambda$.
- For a plaquette $p \in P(\Lambda)$ with vertices $x_1, x_2, x_3, x_4$ in anti-clockwise order, and a configuration $U \in G(\Lambda)$, define
  \[ U_p := U(x_1, x_2) U(x_2, x_3) U(x_3, x_4) U(x_4, x_1). \]
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    \[
    U_p := U(x_1, x_2)U(x_2, x_3)U(x_3, x_4)U(x_4, x_1).
    \]
- The Wilson action of $U$ is defined as
  \[
  S_W(U) := \sum_{p \in P(\Lambda)} \text{Re}(\text{Tr}(I - U_p)).
  \]
Definition of lattice gauge theory

Let $\sigma_\Lambda$ be the product Haar measure on $G(\Lambda)$. 

Given $\beta > 0$, let $\mu_{\Lambda,\beta}$ be the probability measure on $G(\Lambda)$ defined as

$$d\mu_{\Lambda,\beta}(U) := \frac{1}{Z} e^{-\beta S_W(U)} d\sigma_{\Lambda}(U),$$

where $Z$ is the normalizing constant.

This probability measure is called the lattice gauge theory on $\Lambda$ for the gauge group $G$, with inverse coupling strength $\beta$.

An infinite volume limit of the theory is a weak limit of the above probability measures as $\Lambda \uparrow \mathbb{Z}^n$.

The infinite volume limit may or may not be unique.

The uniqueness (or non-uniqueness) is in general unknown for lattice gauge theories in dimension four when $\beta$ is large.
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From LGT to Euclidean YM theory: Wilson’s heuristic

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Let $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$.

For a directed edge $(x, x + \varepsilon e_j)$ of $\varepsilon \mathbb{Z}^n$, define

\[ U(x, x + \varepsilon e_j) := e^{\varepsilon A_j(x)}, \]

and let $U(x + \varepsilon e_j, x) := U(x, x + \varepsilon e_j)^{-1}$. 
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This defines a configuration of unitary matrices assigned to directed edges of \( \epsilon \mathbb{Z}^n \).
Wilson’s heuristic, continued

By the Baker–Campbell–Hausdorff formula for products of matrix exponentials, one can derive the formal approximation

$$S_W(U) \approx -\frac{\epsilon^{4-n}}{4} S_{YM}(A).$$
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The above heuristic was used by Wilson to justify the approximation of Euclidean Yang–Mills theory by lattice gauge theory, scaling the inverse coupling strength \( \beta \) like \( \epsilon^{4-n} \) as the lattice spacing \( \epsilon \to 0 \).
The most important dimension is $n = 4$, because spacetime is four-dimensional.
Scaling in dimension four

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In the above formulation, \( \beta \) does not scale with \( \epsilon \) at all when \( n = 4 \).

Currently, however, the general belief in the physics community is that \( \beta \) should scale like some multiple of \( \log(1/\epsilon) \) in dimension four.
Suppose that we have a lattice gauge theory on $\Lambda \subseteq \mathbb{Z}^n$ with gauge group $G$. 
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Given a loop $\gamma$ with directed edges $e_1, \ldots, e_m$, the Wilson loop variable $W_\gamma$ is defined as

$$W_\gamma := \text{Tr}(U(e_1)U(e_2)\cdots U(e_m)).$$
Open problem #1: Yang–Mills existence

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- The description of the limit is part of the problem.
- The most important groups are $SU(2)$ and $SU(3)$.
- Large body of work in 2D. Less in 3D. Almost none in 4D, except for a very long series of papers by Bałaban that people find very difficult to understand. May be someone can take off from where Bałaban stopped? Or revive the project using different ideas?
Open problem #2: Yang–Mills mass gap

- Again, the problem has many parts, but the main step is to show that 4D non-Abelian lattice gauge theories have exponential decay of correlations at any $\beta$. 

- There are standard techniques for showing exponential decay of correlations at small $\beta$ (e.g. by Dobrushin's condition).

- Showing exponential decay at large $\beta$ is conjectured for many models in statistical physics, but most of these problems, including the YM mass gap, are open.

- Even physicists do not think they have a proof of mass gap. 

- A solution of this problem will explain, roughly speaking, why mass exists in the universe.
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- A solution of this problem will explain, roughly speaking, why mass exists in the universe.
Open problem #3: Quark confinement

Suppose that we are given a 4D non-Abelian lattice gauge theory.

Show that for any $\beta$, there are constants $C(\beta)$ and $c(\beta)$ such that for any loop $\gamma$, $|\langle W_\gamma \rangle| \leq C(\beta) e^{-c(\beta) \text{area}(\gamma)}$, where $\langle W_\gamma \rangle$ is the expected value of the Wilson loop variable $W_\gamma$ and area($\gamma$) is the minimal surface area enclosed by $\gamma$.

Showing for rectangles is good enough.

There is a proof at small $\beta$ by Osterwalder & Seiler (1978).

Proof at large $\beta$ for 3D $U(1)$ theory by G"opfert and Mack (1982).

Disproof at large $\beta$ for 4D $U(1)$ theory by Guth (1980) and Fr"ohlich & Spencer (1982).

Proof of this conjecture will explain why we do not observe free quarks in nature. This is one of the biggest mysteries of particle physics.
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Open problem #4: Gauge-string duality

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- Technically speaking, this problem can be discussed only after solving the problem of YM existence.
- The **main step** is to show that Wilson loop expectations in a continuum Yang–Mills theory can be expressed as integrals over trajectories of strings in a string theory, where the trajectories are in one dimension higher.
Some results about gauge-string duality

- Consider $SO(N)$ lattice gauge theory on $\mathbb{Z}^n$, $n$ arbitrary.

In recent work (C., 2015 and C. & Jafarov, 2016), we gave a formula for Wilson loop expectations in this theory as asymptotic series expansions in $1/N$, where each coefficient in the series arises as a sum over trajectories in a certain lattice string theory, where the trajectories are in $\mathbb{Z}^{n+1}$.

This proves a version of gauge-string duality and the holographic principle. Possibly the first rigorous result.

The expansion was proved only at small $\beta$ (strong coupling). Will be a very important breakthrough to prove something similar at large $\beta$.

In 2D, the terms were explicitly evaluated by Basu & Ganguly (2016) using combinatorial techniques. May be the techniques can extend to higher dimensions?
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The master loop equation

The following is a generalization of what are called Makeenko–Migdal equations or master loop equations. They hold at all $\beta$, and give the starting point for the proof of the $1/N$ expansion and gauge-string duality.
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**Theorem (C., 2015)**

Consider $SO(N)$ LGT on $\mathbb{Z}^n$. For a collection of loops $s = (\ell_1, \ldots, \ell_m)$, define

$$\phi(s) := \frac{\langle W_{\ell_1} W_{\ell_2} \cdots W_{\ell_m} \rangle}{N^m}.$$ 

Let $|s|$ be the total number of edges in $s$. Then

$$(N - 1)|s|\phi(s) = \sum_{s' \in T^-(s)} \phi(s') - \sum_{s' \in T^+(s)} \phi(s') + N \sum_{s' \in S^-(s)} \phi(s')$$

$$- N \sum_{s' \in S^+(s)} \phi(s') + \frac{1}{N} \sum_{s' \in M^-(s)} \phi(s') - \frac{1}{N} \sum_{s' \in M^+(s)} \phi(s')$$

$$+ N\beta \sum_{s' \in D^-(s)} \phi(s') - N\beta \sum_{s' \in D^+(s)} \phi(s'),$$

where $T^\pm$, $S^\pm$, $M^\pm$ and $D^\pm$ are certain operations that produce new collections of loops from old.

Sourav Chatterjee  Yang–Mills for probabilists
Final remarks

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Special thanks to David Brydges, Erhard Seiler and Steve Shenker for teaching me most of what I know about Yang–Mills theories, lattice gauge theories and quantum field theories.
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