Averaging results for non-autonomous slow-fast systems of SPDEs

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jointly with Alessandra Lunardi

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Consider the perturbed system

\[
\begin{aligned}
X'_\epsilon(t) &= \epsilon f_1(X_\epsilon(t), Y_\epsilon(t)), \quad X_\epsilon(0) = x \in \mathbb{R}^n, \\
Y'_\epsilon(t) &= f_2(X_\epsilon(t), Y_\epsilon(t)), \quad Y_\epsilon(0) = y \in \mathbb{R}^m,
\end{aligned}
\]

where \(0 < \epsilon \ll 1\).
Averaging principle

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\begin{align*}
X'_\epsilon(t) &= \epsilon f_1(X_\epsilon(t), Y_\epsilon(t)), \quad X_\epsilon(0) = x \in \mathbb{R}^n, \\
Y''_\epsilon(t) &= f_2(X_\epsilon(t), Y_\epsilon(t)), \quad Y_\epsilon(0) = y \in \mathbb{R}^m,
\end{align*}
\]

(1)

where \(0 < \epsilon << 1\).

Under reasonable assumptions on \(f_1\) and \(f_2\), for any fixed \(T > 0\)

\[
\lim_{\epsilon \to 0} \sup_{t \in [0, T]} |X_\epsilon(t) - x| = 0.
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The behavior of the slow variable \(X_{\epsilon}\) on time intervals of order \(\epsilon^{-1}\) is of interest, because on such time scales significant changes take place.
For any frozen slow component $x \in \mathbb{R}^n$, consider the fast equation

$$Y'_{x,y}(t) = f_2(x, Y_x(t)), \quad Y_{x,y}(0) = y,$$

and assume that the limit

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f_1(x, Y_{x,y}(t)) \, dt =: \bar{f}(x)$$

exists, for some $\bar{f} : \mathbb{R}^n \to \mathbb{R}^n$, independent of $y \in \mathbb{R}^m$. 

The averaging principle says that in the time interval $[0, T/\epsilon]$ the slow motion $X_{\epsilon}$ can be approximated by the trajectories of the averaged system

$$\bar{X}'(t) = \bar{f}(\bar{X}(t)), \quad \bar{X}(0) = x.$$

That is

$$\lim_{\epsilon \to 0} \sup_{t \in [0, T/\epsilon]} |X_{\epsilon}(t) - \bar{X}(t)|_{\mathbb{R}^n} = 0.$$
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A further development concerns the case of random perturbations of dynamical systems.
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For example, in system (1) the coefficient \( f_2 \) may be assumed to depend also on a parameter \( \omega \in \Omega \), (so that the fast variable is a random process), or even the perturbing coefficient \( f_1 \) may be taken random.
A further development concerns the case of random perturbations of dynamical systems.

For example, in system (1) the coefficient $f_2$ may be assumed to depend also on a parameter $\omega \in \Omega$, (so that the fast variable is a random process), or even the perturbing coefficient $f_1$ may be taken random.

One has to reinterpret condition

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f_1(x, Y_{x,y}(t)) \, dt =: \bar{f}(x)$$

and the type of convergence of $X_\epsilon$ to $\bar{X}$. 
In 1968 Khasminskii has proved that averaging holds for the following system of stochastic differential equations

\[
\begin{align*}
\frac{dX_\epsilon(t)}{d\epsilon} &= f_1(X_\epsilon(t), Y_\epsilon(t)) \, dt + g_1(X_\epsilon(t), Y_\epsilon(t)) \, dw(t), \\
\frac{dY_\epsilon(t)}{d\epsilon} &= \frac{1}{\epsilon} f_2(X_\epsilon(t), Y_\epsilon(t)) \, dt + \frac{1}{\sqrt{\epsilon}} g_2(X_\epsilon(t), Y_\epsilon(t)) \, dw(t),
\end{align*}
\]

with initial conditions \(X_\epsilon(0) = x \in \mathbb{R}^n\) and \(Y_\epsilon(0) = y \in \mathbb{R}^m\), for some \(k\)-dimensional Brownian motion \(w(t)\).
In 1968 Khasminskii has proved that averaging holds for the following system of stochastic differential equations

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\begin{align*}
\dot{X}_\epsilon(t) &= f_1(X_\epsilon(t), Y_\epsilon(t)) \, dt + g_1(X_\epsilon(t), Y_\epsilon(t)) \, dw(t), \\
\dot{Y}_\epsilon(t) &= \frac{1}{\epsilon} f_2(X_\epsilon(t), Y_\epsilon(t)) \, dt + \frac{1}{\sqrt{\epsilon}} g_2(X_\epsilon(t), Y_\epsilon(t)) \, dw(t),
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with initial conditions \(X_\epsilon(0) = x \in \mathbb{R}^n\) and \(Y_\epsilon(0) = y \in \mathbb{R}^m\), for some \(k\)-dimensional Brownian motion \(w(t)\).

In this case the perturbation in the slow motion is given by the sum of a deterministic part and a stochastic part

\[ \epsilon f_1(x, y) \, dt + \sqrt{\epsilon} g_1(x, y) dw(t), \]

and the fast motion is described by a stochastic differential equation.
Under reasonable assumptions on the coefficients $f_2$ and $g_2$, the fast equation with frozen slow component $x \in \mathbb{R}^n$

$$
\begin{cases}
  dY^{x,y}(t) = f_2(x, Y^{x,y}(t)) \, dt + g_2(x, Y^{x,y}(t)) \, dw(t), \\
  Y^{x,y}(0) = y \in \mathbb{R}^m,
\end{cases}
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is well posed.
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\end{cases}
\]

is well posed.

Then, for every fixed $x \in \mathbb{R}^n$, we can introduce the transition semigroup

\[
P^x_t \varphi(y) = \mathbb{E} \varphi(Y^{x,y}(t)),
\]

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is Borel bounded.
Main assumptions

We assume that there exists $\bar{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that for every $t \geq 0$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$

$$\mathbb{E} \left| \frac{1}{T} \int_t^{t+T} f_1(x, Y^{x,y}(s)) \, ds - \bar{f}(x) \right| \leq \alpha(T),$$

where $\alpha(T) \to 0$ as $T \to \infty$. 
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where $\alpha(T) \to 0$ as $T \to \infty$.

We also assume that there exists $\bar{a} : \mathbb{R}^n \to \mathbb{R}^{k \times n}$ such that for every $t \geq 0$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$

$$\max_{i,j} \mathbb{E} \left| \frac{1}{T} \int_t^{t+T} g_{1,i,k}^1(x, Y_{x,y}^x(s)) g_{1,k,j}^1(x, Y_{x,y}^x(s)) \, ds - \bar{a}_{i,j}^1(x) \right| \leq \alpha(T),$$

where $\alpha(T) \to 0$ as $T \to \infty$. 
The convergence result

Under reasonable conditions on the coefficients the averaged equation

$$d\bar{X}(t) = \bar{b}(\bar{X}(t)) \, dt + \sqrt{\bar{a}}(\bar{X}(t)) \, dw(t), \quad \bar{X}(0) = x,$$

is well posed.
The convergence result

Under reasonable conditions on the coefficients the averaged equation

\[ d\tilde{X}(t) = \bar{b}(\tilde{X}(t)) \, dt + \sqrt{\bar{a}(\tilde{X}(t))} \, dw(t), \quad \tilde{X}(0) = x, \]

is well posed.

The averaging principle says that

the slow component \( X_\epsilon(\cdot) \) converges weakly in the space of continuous trajectories \( C([0, T]; \mathbb{R}^n) \) to the solution \( \tilde{X}(\cdot) \) of the averaged equation.

Moreover, if \( g_1 \) does not depend on the fast variable, the convergence is stronger.
How to verify the assumptions?

Assume that

the semigroup $P_t^x$ associated with the fast equation admits a unique invariant measure $\mu^x$

and for any $x, y \in H$ and $\varphi \in \text{Lip}(H)$

$$\left| P_t^x \varphi(y) - \int_H \varphi(z) \mu^x(dz) \right| \leq c \left( 1 + |x|_H + |y|_H \right) e^{-\delta t} [\varphi]_{\text{Lip}(H)}.$$
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\]

Then, the two assumptions are satisfied if we define

\[
\bar{b}(x) = \int_{\mathbb{R}^m} b_1(x, y) d\mu^x(y),
\]

and

\[
\bar{a}^{i,j}(x) = \int_{\mathbb{R}^m} g^{i,k}_1(x, y) g^{k,j}_1(x, y) d\mu^x(y).
\]
Averaging for SPDEs

In a series of papers, also together with M. Freidlin, we have considered an infinite dimensional analogue of (2) in a bounded domain $D \subset \mathbb{R}^d$, $d \geq 1$,

$$
\begin{align*}
\frac{\partial u_\epsilon}{\partial t}(t, \xi) &= A_1 u_\epsilon(t, \xi) + f_1(\xi, u_\epsilon(t, \xi), v_\epsilon(t, \xi)) \\
&\quad + g_1(\xi, u_\epsilon(t, \xi), v_\epsilon(t, \xi)) \frac{\partial w^{Q_1}}{\partial t}(t, \xi), \\
\frac{\partial v_\epsilon}{\partial t}(t, \xi) &= \frac{1}{\epsilon} \left[ (A_2 - \lambda) v_\epsilon(t, \xi) + f_2(\xi, u_\epsilon(t, \xi), v_\epsilon(t, \xi)) \right] \\
&\quad + \frac{1}{\sqrt{\epsilon}} g_2(\xi, u_\epsilon(t, \xi), v_\epsilon(t, \xi)) \frac{\partial w^{Q_2}}{\partial t}(t, \xi),
\end{align*}
$$

(3)

with initial conditions $u_\epsilon(0, \xi) = x(\xi)$, $v_\epsilon(0, \xi) = y(\xi)$ and suitable boundary conditions.
Well-posedness of the system

Here, we assume

- $A_1$ and $A_2$ are second order uniformly elliptic operators.
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- $w^{Q_1}(t, \xi)$ and $w^{Q_2}(t, \xi)$ are cylindrical Wiener processes in $H := L^2(D)$, defined on a complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, with covariance $Q_1$ and $Q_2$. 

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- \( Q_1 \) and \( Q_2 \) are bounded linear operators in \( H \), fulfilling suitable assumptions and not Hilbert-Schmidt, in general. When \( d = 1 \), we could take \( Q_i = I \).
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- the mappings $f_i, g_i : D \times \mathbb{R}^2 \to \mathbb{R}$ are measurable;
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- the mappings \( f_i(\xi, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( g_i(\xi, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{R} \) are Lipschitz-continuous, or more general.
Under the hypotheses above, the stochastic system admits a unique adapted mild solution

\[(u_\epsilon, v_\epsilon) \in L^p(\Omega; C([0, T]; H)) \times L^p(\Omega; C([0, T]; H)),\]

for any \(p \geq 1\) and \(T > 0\), and for any \(\epsilon > 0\).
By adapting to this infinite dimensional situations the arguments described above, we can average the coefficients $f_1$ and $g_1$ of the slow equation, and obtain the averaged equation

$$
  du(t) = \left[ A_1 u(t) + \bar{F}(u(t)) \right] dt + \bar{G}(u(t)) \, dw^{Q_1}(t), \quad u(0) = x.
$$

Then,

we show that it admits a unique mild solution

$$
  \bar{u} \in L^p(\Omega, C([0, T]; H)), \text{ for any } p \geq 1 \text{ and } T > 0.
$$
Therefore, we prove that under the conditions above, for any $T > 0$ we have

$$\mathcal{L}(u_\epsilon) \rightharpoonup \mathcal{L}(\bar{u}), \quad \text{in } C([0, T]; H), \quad \text{as } \epsilon \downarrow 0.$$

If

$$g_1(\xi, \sigma_1, \sigma_2) = g_1(\xi, \sigma_1), \quad (\xi, \sigma_1, \sigma_2) \in D \times \mathbb{R}^2,$$

then, for any $\eta > 0$

$$\lim_{\epsilon \to 0} \mathbb{P} \left( |u_\epsilon - \bar{u}|_{C([0, T]; H)} > \eta \right) = 0,$$

or, even more,

$$\lim_{\epsilon \to 0} \mathbb{E} \sup_{t \in [0, T]} |u_\epsilon(t) - \bar{u}(t)|_H^p = 0,$$
In the proof of the averaging limit, we have used
- the Khasminskii method of localization in time,
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In recent years, many different other models of slow-fast systems of SPDEs have been considered. So, now the literature on the validity of the averaging principle for SPDEs is quite large.
The case of non-autonomous systems

Together with A. Lunardi, we dealt with

non-autonomous systems of reaction-diffusion equations of Hodgkin-Huxley or Ginzburg-Landau type, perturbed by a Gaussian noise of multiplicative type.
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The classical Hodgkin-Huxley model has time-independent coefficients, but (see Wainrib 2013)

systems with time-dependent coefficients are particularly important to study models of learning in neuronal activity.
The system

We are dealing here with the following class of equations

\[
\begin{align*}
\frac{\partial u_\epsilon}{\partial t}(t) &= \Delta u_\epsilon(t) + b_1(\xi, u_\epsilon(t), v_\epsilon(t)) + g_1(\xi, u_\epsilon(t)) \frac{\partial w^{Q_1}}{\partial t}(t, \xi), \\
\frac{\partial v_\epsilon}{\partial t}(t) &= \frac{1}{\epsilon} [(\gamma(t/\epsilon)\Delta - \alpha)v_\epsilon(t) + b_2(t/\epsilon, \xi, u_\epsilon(t), v_\epsilon(t))] + \frac{1}{\sqrt{\epsilon}} g_2(t/\epsilon, \xi, v_\epsilon(t)) \frac{\partial w^{Q_2}}{\partial t}(t, \xi), \\
u_\epsilon(0, \xi) &= x(\xi), \quad v_\epsilon(0, \xi) = y(\xi), \quad \xi \in D, \\
N_1 u_\epsilon(t, \xi) = N_2 v_\epsilon(t, \xi) &= 0, \quad t \geq 0, \quad \xi \in \partial D.
\end{align*}
\]

In fact, we considered more general differential operators.
The noise

The noises $w^{Q_1}(t)$ and $w^{Q_2}(t)$ are cylindrical Wiener processes in $H$, with covariance $Q_1$ and $Q_2$. That is,

$$w^{Q_i}(t, \xi) = \sum_{k=1}^{\infty} Q_i e_k(\xi) \beta_k(t), \quad i = 1, 2,$$

where $\{e_k\}_{k \in \mathbb{N}}$ is the orthonormal basis in $H$ that diagonalizes $\Delta$, with eigenvalues $\{-\alpha_k\}_{k \in \mathbb{N}}$, and $\{\beta_k(t)\}_{k \in \mathbb{N}}$ is a sequence of independent Brownian motions.
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We assume \( Q_i e_k = \lambda_{i,k} e_k \), for every \( k \geq 1 \) and \( i = 1, 2 \), and

\[
  \kappa_i := \sum_{k=1}^{\infty} \lambda_{i,k}^{\rho_i} |e_k|_\infty^2 < \infty, \quad \zeta := \sum_{k=1}^{\infty} \alpha_k^{-\beta} |e_k|_\infty^2 < \infty,
\]

for some constants \( \rho_i \in (2, +\infty] \) and \( \beta \in (0, +\infty) \) such that

\[
  \frac{\beta (\rho_i - 2)}{\rho_i} < 1.
\]
Notice that when

$$\alpha_k \sim k^{2/d}, \quad \sup_{k \in \mathbb{N}} |e_k|_\infty < \infty,$$

the condition above on the eigenvalues $\lambda_{i,k}$ of the operators $Q_i$ becomes

$$\kappa_i = \sum_{k=1}^{\infty} \lambda_{i,k}^{\rho_i} < \infty,$$

for some

$$\rho_i < \frac{2d}{d - 2}.$
The coefficients $b_i$ and $g_i$:

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Just to simplify our presentation, we assume that the diffusion coefficients $g_1$ and $g_2$ are two bounded Lipschitz-continuous functions. Moreover,

$$b_1(\xi, u, v) = -\alpha(\xi) u^{2n+1} + \sum_{j=0}^{2n} \alpha_j(\xi) u^j + h_1(\xi, u, v),$$

and

$$b_2(t, \xi, u, v) = -\beta(t, \xi) v^{2m+1} + \sum_{j=1}^{2m} \beta_j(t, \xi) v^j + h_2(t, \xi, u, v),$$

where $h_1$ and $h_2$ are locally Lipschitz functions with linear growth. All coefficients $\alpha, \beta, \alpha_j$ and $\beta_j$ are continuous, and

$$\inf_{\xi \in \bar{D}} \alpha(\xi) > 0, \quad \inf_{(t,\xi) \in \mathbb{R}^+ \times \bar{D}} \beta(t, \xi) > 0.$$
For every $x, y \in C(\bar{D})$, we set
\[
B_1(x, y)(\xi) := b_1(\xi, x(\xi), y(\xi)), \quad \xi \in D,
\]
and
\[
B_2(t, x, y)(\xi) := b_2(t, \xi, x(\xi), y(\xi)), \quad t \geq 0, \quad \xi \in D,
\]
For every \( x, y \in C(\bar{D}) \), we set

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\]

Moreover, for every \( x, z \in C(\bar{D}) \), we set

\[
[G_1(x)z](\xi) := g_1(\xi, x(\xi))z(\xi), \quad \xi \in D,
\]

and

\[
[G_2(t, x)z](\xi) := g_2(t, \xi, x(\xi))z(\xi), \quad t \geq 0, \quad \xi \in D.
\]
The evolution family generated by $\gamma(t)\Delta$

We assume

$$0 < \gamma_0 \leq \gamma(t) \leq \gamma_1, \quad t \geq 0,$$

and we define

$$\gamma(t, s) := \int_s^t \gamma(r) \, dr, \quad s < t.$$  

We denote by $A$ the realization of $\Delta$, endowed with the given boundary conditions, in all spaces $L^p(D)$, $1 < p < \infty$, and in $C(\bar{D})$.  

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We denote by $A$ the realization of $\Delta$, endowed with the given boundary conditions, in all spaces $L^p(D)$, $1 < p < \infty$, and in $C(\bar{D})$.

For any $\epsilon > 0$ we set

$$U_\epsilon(t, s) = \exp \left( \frac{\gamma(r, \rho)}{\epsilon} A - \frac{\alpha}{\epsilon} (t - s) \right), \quad s < t.$$
Clearly, for every initial condition $x$, we have that

$$u(t) = U_\epsilon(t, s)x, \quad t \geq s,$$

is the unique mild solution to the linear problem

$$\partial_t u(t) = \frac{1}{\epsilon}(\gamma(t)\Delta - \alpha)u(t), \quad t > s, \quad u(s) = x,$$

endowed with the given boundary conditions.
The slow-fast system

With the notations introduced above, our system can be rewritten in the following abstract form

\[
\begin{align*}
    du_\varepsilon(t) &= [Au_\varepsilon(t) + B_1(u_\varepsilon(t), v_\varepsilon(t))] \, dt + G_1(u_\varepsilon(t)) \, dw^{Q_1}(t), \\
    dv_\varepsilon(t) &= \frac{1}{\varepsilon} \left[ (\gamma(t/\varepsilon)\Delta - \alpha) v_\varepsilon(t) + B_2(t/\varepsilon, u_\varepsilon(t), v_\varepsilon(t)) \right] \, dt \\
    &\quad + \frac{1}{\sqrt{\varepsilon}} G_2(t/\varepsilon, v_\varepsilon(t)) \, dw^{Q_2}(t), \\
    u_\varepsilon(0) &= x, \quad v_\varepsilon(0) = y.
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    dv_\epsilon(t) &= \frac{1}{\epsilon} \left[ (\gamma(t/\epsilon) \Delta - \alpha) v_\epsilon(t) + B_2(t/\epsilon, u_\epsilon(t), v_\epsilon(t)) \right] \, dt \\
    &\quad + \frac{1}{\sqrt{\epsilon}} G_2(t/\epsilon, v_\epsilon(t)) \, dw^{Q_2}(t), \\
    u_\epsilon(0) &= x, \quad v_\epsilon(0) = y.
\end{align*}
\]

In what follows, we shall denote

\[
H = L^2(D), \quad E = C(\bar{D}).
\]
We show that for any $\epsilon > 0$ and $x, y \in E$ there exists a unique adapted mild solution to the problem above in $L^p(\Omega; C_b((s, T]; E \times E))$, with $s < T$ and $p \geq 1$.

This means that there exist two unique adapted processes $u_\epsilon$ and $v_\epsilon$ in $L^p(\Omega; C_b((s, T]; E))$ such that

\[ u_\epsilon(t) = e^{tA}x + \int_s^t e^{(t-r)A}B_1(u_\epsilon(r), v_\epsilon(r)) \, ds \]

\[ + \int_s^t e^{(t-s)A}G_1(u_\epsilon(r)) \, dw_{Q_1}(r), \]

and

\[ v_\epsilon(t) = U_\epsilon(t, s)y + \frac{1}{\epsilon} \int_s^t U_\epsilon(t, r)B_2(r, u_\epsilon(r), v_\epsilon(r)) \, dr \]

\[ + \frac{1}{\sqrt{\epsilon}} \int_s^t U_\epsilon(t, r)G_2(r, v_\epsilon(r)) \, dw_{Q_2}(r). \]
Some bounds

We show that for any $p \geq 1$ and $s < T$ there exists a constant $c_{p,s,T} > 0$ such that for any $x, y \in E$ and $\epsilon \in (0, 1]$

$$
\mathbb{E} \sup_{t \in [s, T]} |u_\epsilon(t)|_E^p \leq c_{p,s,T} \left(1 + |x|_E^p + |y|_E^p \right),
$$

and

$$
\mathbb{E} \int_s^T |v_\epsilon(t)|_E^p \, dt \leq c_{p,s,T} \left(1 + |x|_E^p + |y|_E^p \right).
$$
Some bounds

We show that for any \( p \geq 1 \) and \( s < T \) there exists a constant \( c_{p,s,T} > 0 \) such that for any \( x, y \in E \) and \( \epsilon \in (0, 1] \)

\[
\mathbb{E} \sup_{t \in [s,T]} |u_\epsilon(t)|_E^p \leq c_{p,s,T} \left(1 + |x|_E^p + |y|_E^p \right),
\]

and

\[
\mathbb{E} \int_s^T |v_\epsilon(t)|_E^p \, dt \leq c_{p,s,T} \left(1 + |x|_E^p + |y|_E^p \right).
\]

Moreover, we show that there exists \( \bar{\theta} > 0 \) such that for any \( \theta \in [0, \bar{\theta}) \), \( x \in C^\theta(\bar{D}) \), \( y \in E \) and \( s < T \)

\[
\sup_{\epsilon \in (0,1]} \mathbb{E} \left| u_\epsilon \right|_{L^\infty(s,T;C^\theta(\bar{D}))} \leq c_{s,T} \left(1 + |x|_{C^\theta(\bar{D})} + |y|_E \right).
\]
Finally, we prove that for any $\theta > 0$ there exists $\gamma(\theta) > 0$ such that for any $T > 0$, $p \geq 2$, $x \in C^\theta(\bar{D})$, $y \in E$ and $r_1, r_2 \in [s, t]$:

$$\sup_{\epsilon \in (0, 1)} \mathbb{E} |u_\epsilon(r_1) - u_\epsilon(r_2)|_E^p \leq c_p(T) \left(1 + \|x\|_{C^\theta(\bar{D})}^{pm_1} + \|y\|_E^p\right) |r_1 - r_2|^{\gamma(\theta)p}.$$
Finally, we prove that for any \( \theta > 0 \) there exists \( \gamma(\theta) > 0 \) such that for any \( T > 0, p \geq 2, x \in C^\theta(\bar{D}), y \in E \) and \( r_1, r_2 \in [s, t] \)

\[
\sup_{\epsilon \in (0,1)} \mathbb{E} |u_\epsilon(r_1) - u_\epsilon(r_2)|_E^p \leq c_p(T) \left( 1 + |x|_{C^\theta(\bar{D})}^{pm_1} + |y|_E^p \right) |r_1 - r_2|^{\gamma(\theta)p}.
\]

This implies that the family \( \{\mathcal{L}(u_\epsilon)\}_{\epsilon \in (0,1)} \) is tight in \( C([s, T]; E) \), for any \( x \in C^\theta(\bar{D}) \), with \( \theta > 0 \), and for any \( y \in E \).
The fast equation

For any frozen slow component \( x \in E \), any initial condition \( y \in E \) and any \( s \in \mathbb{R} \), we introduce the problem

\[
dv(t) = \left[(\gamma(t)A - \alpha)v(t) + B_2(t, x, v(t))\right] \, dt + G_2(t, v(t)) \, d\bar{w}^Q_2(t),
\]

with \( v(s) = y \), where

\[
\bar{w}^Q_2(t) = \begin{cases} 
    w_1^Q(t), & \text{if } t \geq 0, \\
    w_2^Q(-t), & \text{if } t < 0,
\end{cases}
\]

for two independent \( Q_2 \)-Wiener processes, \( w_1^Q(t) \) and \( w_2^Q(t) \).
The fast equation

For any frozen slow component $x \in E$, any initial condition $y \in E$ and any $s \in \mathbb{R}$, we introduce the problem

$$
dv(t) = [(\gamma(t)A - \alpha)v(t) + B_2(t, x, v(t))] \, dt + G_2(t, v(t)) \, d\tilde{w}^{Q_2}(t),$$

with $v(s) = y$, where

$$
\tilde{w}^{Q_2}(t) = \begin{cases} 
  w_1^{Q_2}(t), & \text{if } t \geq 0, \\
  w_2^{Q_2}(-t), & \text{if } t < 0,
\end{cases}
$$

for two independent $Q_2$-Wiener processes, $w_1^{Q_2}(t)$ and $w_2^{Q_2}(t)$. The process $v^x(\cdot; s, y) \in L^p(\Omega; C([s, T]; E))$ is a mild solution if

$$
v^x(t; s, y) = U_\alpha(t, s)y + \int_s^t U_\alpha(t, r) \, B_2(r, x, v^x(r; s, y)) \, dr
$$

$$
+ \int_s^t U_\alpha(t, r) \, G_2(r, v^x(r; s, y)) \, d\tilde{w}^{Q_2}(r).
$$
We prove that

for any \( x, y \in \) and for any \( p \geq 1 \) and \( s < T \),
there exists a unique mild solution
\[
\nu^x(\cdot; s, y) \in L^p(\Omega; C((s, T]; E) \cap L^\infty((s, T); E)).
\]
We prove that

for any $x, y \in E$ and for any $p \geq 1$ and $s < T$, there exists a unique mild solution

$$v^x(\cdot; s, y) \in L^p(\Omega; C((s, T]; E) \cap L^\infty((s, T); E))$$.

We prove also that there exists $\delta > 0$ such that for any $x, y \in E$ and $p \geq 1$

$$\mathbb{E} \left| v^x(t; s, y) \right|^p_E \leq c_p \left( 1 + e^{-\delta p(t-s)} \left| y \right|^p_E + \left| x \right|^p_E \right), \quad s < t.$$
The fast equation in $\mathbb{R}$

An adapted process $v^x \in L^p(\Omega; C(\mathbb{R}; E))$ is a *mild solution* of the equation above in $\mathbb{R}$ if, for every $s < t$,

$$v^x(t) = U_\alpha(t, s)v^x(s) + \int_s^t U_\alpha(t, r) B_2(r, x, v^x(r)) \, dr$$

$$+ \int_s^t U_\alpha(t, r) G_2(r, v^x(r)) \, d\tilde{w}^Q_2(r).$$
We prove that if $\alpha > 0$ is large enough and/or $L_{g_2}$ is small enough, for any $t \in \mathbb{R}$ and $x \in E$

there exists $\eta^x(t) \in L^p(\Omega; E)$, for all $p \geq 1$,

such that

$$\lim_{s \to -\infty} \mathbb{E} |v^x(t; s, y) - \eta^x(t)|_E^p = 0,$$

for any $y \in E$ and $t \in \mathbb{R}$.

Moreover, for every $p \geq 1$ there exists some $\delta_p > 0$ such that

$$\mathbb{E} |v^x(t; s, y) - \eta^x(t)|_E^p \leq c_p e^{-\delta_p(t-s)} (1 + |x|_E^p + |y|_E^p).$$

Finally, $\eta^x$ is a mild solution in $\mathbb{R}$ of the fast equation.
The evolution system of probabilities

For any fixed $x \in E$, we define the transition evolution operator

$$P^x_{s,t} \varphi(y) = \mathbb{E} \varphi(v^x(t; s, y)), \quad s < t, \quad y \in E,$$

where $\varphi \in B_b(E)$.
The evolution system of probabilities

For any fixed $x \in E$, we define the transition evolution operator

$$P^x_{s,t} \varphi(y) = \mathbb{E} \varphi(v^x(t; s, y)), \quad s < t, \quad y \in E,$$

where $\varphi \in B_b(E)$.

For any $t \in \mathbb{R}$ and $x \in E$, we set

$$\mu^x_t := \mathcal{L}(\eta^x(t)).$$
The evolution system of probabilities

For any fixed $x \in E$, we define the transition evolution operator

$$P_{s,t}^x \varphi(y) = \mathbb{E} \varphi(v^x(t; s, y)), \quad s < t, \quad y \in E,$$

where $\varphi \in B_b(E)$.

For any $t \in \mathbb{R}$ and $x \in E$, we set

$$\mu_t^x := \mathcal{L}(\eta^x(t)).$$

We show that the family $\{\mu_t^x\}_{t \in \mathbb{R}}$ defines an evolution system of probability measures on $E$ for the fast equation.

This means that $\mu_t^x$ is a probability measure on $E$, for any $t \in \mathbb{R}$, and for every $\varphi \in C_b(E)$

$$\int_E P_{s,t}^x \varphi(y) \mu_s^x(dy) = \int_E \varphi(y) \mu_t^x(dy), \quad s < t.$$
Notice that, due to the previous estimates, for any $p \geq 1$ we have

$$\sup_{t \in \mathbb{R}} \mathbb{E} |\eta^x(t)|_E^p \leq c_p \left( 1 + |x|_E^p \right), \quad x \in E,$$

so that

$$\sup_{t \in \mathbb{R}} \int_E |y|_E^p \mu^x_t(dy) \leq c_p \left( 1 + |x|_E^p \right).$$
Notice that, due to the previous estimates, for any $p \geq 1$ we have

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so that

$$\sup_{t \in \mathbb{R}} \int_E |y|_E^p \mu_t^x(dy) \leq c_p \left(1 + |x|_E^p\right).$$

Moreover, we prove that for any $R > 0$ there exists $c_R > 0$ such that

$$x_1, x_2 \in B_E(R) \implies \sup_{t \in \mathbb{R}} \mathbb{E} |\eta^{x_1}(t) - \eta^{x_2}(t)|_E^2 \leq c_R |x_1 - x_2|_E^2.$$
The key limiting result

Under the conditions above,

$$\lim_{s \to -\infty} P_{s,t}^x \varphi(y) = \int_E \varphi(y) \mu_t^x(dy),$$

for any $\varphi \in C_b(E)$.
The key limiting result

Under the conditions above,

\[
\lim_{s \to -\infty} P_{s,t}^x \varphi(y) = \int_E \varphi(y) \mu_t^x(dy),
\]

for any \( \varphi \in C_b(E) \).

Moreover, if \( \varphi \in C^1_b(E) \),

\[
\left| P_{s,t}^x \varphi(y) - \int_E \varphi(z) \mu_t^x(dz) \right| \leq \| \varphi \|_{C^1_b(E)} e^{-\delta_1(t-s)} (1 + |x|_E + |y|_E).
\]
The key limiting result

Under the conditions above,

\[ \lim_{s \to -\infty} P_{s,t}^x \varphi(y) = \int_E \varphi(y) \mu_t^x (dy), \]

for any \( \varphi \in C_b(E) \).

Moreover, if \( \varphi \in C^1_b(E) \),

\[ \left| P_{s,t}^x \varphi(y) - \int_E \varphi(z) \mu_t^x (dz) \right| \leq \| \varphi \|_{C^1_b(E)} e^{-\delta_1(t-s)} (1 + |x|_E + |y|_E). \]

Finally, if \( \{ \nu_t^x \}_{t \in \mathbb{R}} \) is another evolution family of measures for the same equation, such that

\[ \sup_{t \in \mathbb{R}} \int_E |y|_E \nu_t^x (dy) < \infty, \]

then

\[ \nu_t^x = \mu_t^x, \quad t \in \mathbb{R}, \ x \in E. \]
Almost periodic functions

Let \((X, d_X)\) and \((Y, d_Y)\) be two complete metric spaces. For any bounded function \(f : \mathbb{R} \to Y\) and \(\epsilon > 0\), we define

\[
T(f, \epsilon) = \{\tau \in \mathbb{R} : d_Y(f(t + \tau), f(t)) < \epsilon, \text{ for all } t \in \mathbb{R}\}.
\]
Almost periodic functions

Let \((X, d_X)\) and \((Y, d_Y)\) be two complete metric spaces. For any bounded function \(f : \mathbb{R} \to Y\) and \(\epsilon > 0\), we define

\[
T(f, \epsilon) = \{\tau \in \mathbb{R} : d_Y(f(t + \tau), f(t)) < \epsilon, \text{ for all } t \in \mathbb{R}\}.
\]

- A continuous function \(f : \mathbb{R} \to Y\) is said to be almost periodic if, for all \(\epsilon > 0\) there exists a number \(l_\epsilon > 0\) such that

\[
T(f, \epsilon) \cap [a, a + l_\epsilon] \neq \emptyset, \quad a \in \mathbb{R}.
\]
Almost periodic functions

Let \((X, d_X)\) and \((Y, d_Y)\) be two complete metric spaces. For any bounded function \(f : \mathbb{R} \rightarrow Y\) and \(\epsilon > 0\), we define

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\]

- A continuous function \(f : \mathbb{R} \rightarrow Y\) is said to be \textit{almost periodic} if, for all \(\epsilon > 0\) there exists a number \(l_\epsilon > 0\) such that

\[
T(f, \epsilon) \cap [a, a + l_\epsilon] \neq \emptyset, \quad a \in \mathbb{R}.
\]

- Let \(F \subset X\) and, for any \(x \in F\), let \(f(\cdot, x) : \mathbb{R} \rightarrow Y\) be a continuous function. The family of functions \(\{f(\cdot, x)\}_{x \in F}\) is said \textit{uniformly almost periodic} if for any \(\epsilon > 0\) there exists \(l_\epsilon > 0\) such that

\[
\bigcap_{x \in F} T(f(\cdot, x), \epsilon) \cap [a, a + l_\epsilon] \neq \emptyset, \quad a \in \mathbb{R}.
\]
The important consequence of almost periodicity

- There exists the mean value in $Y$ of any almost periodic function $f : \mathbb{R} \to Y$, that is

$$\exists \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(s) \, ds \in Y.$$ 

Moreover, for every $t \in \mathbb{R}$

$$\lim_{T \to \infty} \frac{1}{T} \int_{t}^{t+T} f(s) \, ds = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(s) \, ds,$$

uniformly with respect to $t \in \mathbb{R}$.

- If $\{f(\cdot, x)\}_{x \in F}$ is a uniformly almost periodic family of functions, with $F \subset X$, then

$$\exists \lim_{T \to \infty} \frac{1}{T} \int_{t}^{t+T} f(s, x) \, ds = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(s, x) \, ds,$$

uniformly with respect to $t \in \mathbb{R}$ and $x \in F$. 

Almost periodicity of the evolution family of measures

We assume the following conditions on $\gamma(t)$ and the coefficients $b_2(t, \xi, \sigma)$ and $g_2(t, \xi, \sigma)$.

- The function $\gamma : \mathbb{R} \rightarrow (0, \infty)$ is periodic.
- For every $R > 0$, the families of functions $B_R := \left\{ b_2(\cdot, \xi, \sigma) : \xi \in \bar{D}, \sigma \in B_{\mathbb{R}^2}(R) \right\}$, $G_R := \left\{ g_2(\cdot, \xi, \sigma) : \xi \in \bar{D}, \sigma \in B_{\mathbb{R}}(R) \right\}$

are both uniformly almost periodic.

Under these conditions, it is easy to check that for any $R > 0$ the family of functions

$$\{B_2(\cdot, x, y) : (x, y) \in B_{E \times E}(R)\}, \quad \{G_2(\cdot, y) : y \in B_E(R)\},$$

are both uniformly almost periodic.
In Da Prato-Tudor (1995), SPDEs with periodic and almost periodic coefficients are studied and it is proven that if

- $\gamma(\cdot)$ is periodic,
- the family of functions

$$\{B_2(\cdot, x, y) : (x, y) \in B_{E \times E}(R)\}, \quad \{G_2(\cdot, y) : y \in B_E(R)\},$$

are both uniformly almost periodic, for any $R > 0$,
- the family of measures $\{\mu_t^x\}_{t \in \mathbb{R}}$ is tight in $\mathcal{P}(E)$,

then the mapping

$$t \in \mathbb{R} \mapsto \mu_t^x \in \mathcal{P}(E)$$

is almost periodic.
In fact, we prove that if $\alpha$ is sufficiently large and/or $L_{g_2}$ is sufficiently small, there exists $\theta > 0$ such that for any $p \geq 1$ and for any $x \in E$

$$\sup_{t \in \mathbb{R}} \mathbb{E} |\eta^x(t)|_C^p \leq c_p \left(1 + |x|^p_E\right).$$

In particular, the family of measures

$$\Lambda_R := \{\mu^x_t; t \in \mathbb{R}, x \in B_E(R)\},$$

is tight in $\mathcal{P}(E)$, for any $R > 0$. 
In fact, we prove that if $\alpha$ is sufficiently large and/or $L_{g_2}$ is sufficiently small, there exists $\theta > 0$ such that for any $p \geq 1$ and for any $x \in E$

$$
\sup_{t \in \mathbb{R}} \mathbb{E} |\eta^x(t)|^p_{C^0}\bar{D})) \leq c_p \left(1 + |x|^p_E\right).
$$

In particular, the family of measures

$$
\Lambda_R := \{\mu^x_t; t \in \mathbb{R}, x \in B_E(R)\},
$$

is tight in $\mathcal{P}(E)$, for any $R > 0$.

In view of Da Prato-Tudor result, this implies that the mapping

$$
t \in \mathbb{R} \mapsto \mu^x_t \in \mathcal{P}(E),
$$

is almost periodic, for any fixed $x \in E$. 

S. Cerrai
Averaging for non-autonomous slow-fast systems of SPDEs
The averaged equation

We first show that for every compact set \( K \subset E \), the family of functions

\[
\left\{ t \in \mathbb{R} \mapsto \int_E B_1(x, z) \mu_t^x(dz) \in E : x \in K \right\}
\]

is uniformly almost periodic.
The averaged equation

We first show that for every compact set $K \subset E$, the family of functions

$$\left\{ t \in \mathbb{R} \mapsto \int_E B_1(x, z) \mu_t^x(dz) \in E : x \in K \right\}$$

is uniformly almost periodic.

Due to the almost periodicity of the family above, we can define

$$\bar{B}(x) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_E B_1(x, y) \mu_t^x(dy) dt, \quad x \in E.$$
The averaged equation

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Due to the almost periodicity of the family above, we can define

\[
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\]

Notice that this, together with the estimates we have for \( B_1 \), yields

\[
|\bar{B}(x)|_E \leq c \left( 1 + |x|_{E^1}^{m_1} \right).
\]
The fundamental limit

If $\alpha$ is sufficiently large and/or $L_{g_2}$ is sufficiently small, there exist some constants $\kappa_1, \kappa_2 \geq 0$ such that for any $T > 0$, $s \in \mathbb{R}$ and $x, y \in E$

$$E \left| \frac{1}{T} \int_{s}^{s+T} B_1(x, v^x(t; s, y)) dt - \bar{B}(x) \right|_E^2$$

$$\leq \frac{c}{T} \left( 1 + |x|^{\kappa_1}_E + |y|^{\kappa_2}_E \right) + \alpha(T, x),$$

for some mapping $\alpha : [0, \infty) \times E \to [0, +\infty)$ such that

$$\sup_{T > 0} \alpha(T, x) \leq c \left( 1 + |x|^{m_1}_E \right), \quad x \in E,$$

and, for any compact set $K \subset E$,

$$\lim_{T \to \infty} \sup_{x \in K} \alpha(T, x) = 0.$$
The mapping $\bar{B} : E \rightarrow E$ is locally Lipschitz-continuous. Moreover, for any $x, h \in E$ and $\delta \in \mathcal{M}_h$

$$\langle \bar{B}(x + h) - \bar{B}(x), \delta \rangle_E \leq c (1 + |h|_E + |x|_E).$$

Here $\mathcal{M}_h$ denotes a suitable subset of the subdifferential of the norm of $h$. 

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Averaging for non-autonomous slow-fast systems of SPDEs
The mapping $\bar{B} : E \to E$ is locally Lipschitz-continuous. Moreover, for any $x, h \in E$ and $\delta \in \mathcal{M}_h$

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Here $\mathcal{M}_h$ denotes a suitable subset of the subdifferential of the norm of $h$.

Thus, we can introduce the averaged equation

$$du(t) = [Au(t) + \bar{B}(u(t))] \, dt + G(u(t)) \, dw^{Q_1}(t), \quad u(0) = x \in E.$$
The mapping $\bar{B} : E \rightarrow E$ is locally Lipschitz-continuous. Moreover, for any $x, h \in E$ and $\delta \in \mathcal{M}_h$

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Here $\mathcal{M}_h$ denotes a suitable subset of the subdifferential of the norm of $h$.

Thus, we can introduce the averaged equation

$$du(t) = [Au(t) + \bar{B}(u(t))] \, dt + G(u(t)) \, dw^{Q_1}(t), \quad u(0) = x \in E.$$

In view of the nice properties of $\bar{B}$, the equation above admits a unique mild solution

$$\bar{u} \in L^p(\Omega; C_b((0, T]; E)).$$
The averaging limit

Fix $x \in C^\theta(\bar{D})$, for some $\theta > 0$, and $y \in E$. Then, if $\alpha$ is large enough and/or $L_{g_2}$ is small enough, for any $T > 0$ and $\eta > 0$ we have

$$\lim_{\epsilon \to 0} \mathbb{P} \left( \sup_{t \in [0, T]} |u_\epsilon(t) - \bar{u}(t)|_E > \eta \right) = 0,$$

where $\bar{u}$ is the solution of the averaged equation.
A few comments about the proof

For any \( h \in D(A) \) and \( \epsilon > 0 \), we have

\[
\begin{align*}
\int_{D} u_{\epsilon}(t, \xi)h(\xi)\, d\xi &= \int_{D} x(\xi)h(\xi)\, d\xi + \int_{0}^{t} \int_{D} u_{\epsilon}(s, \xi)Ah(\xi)\, d\xi\, ds \\
+ \int_{0}^{t} \int_{D} \bar{B}(u_{\epsilon}(s, \cdot))(\xi)h(\xi)\, d\xi\, ds + \int_{0}^{t} \int_{D} [G_{1}(u_{\epsilon}(s)h](\xi)dw^{Q_{2}}(s, \xi) \\
+ \int_{0}^{t} \int_{D} (B_{1}(u_{\epsilon}(s), v_{\epsilon}(s))(\xi) − \bar{B}(u_{\epsilon}(s))(\xi)) \, h(\xi)\, d\xi\, ds.
\end{align*}
\]
Therefore, due to the tightness of the family \( \{ \mathcal{L}(u_\epsilon) \} \in \mathcal{P}(C([0, T]; E)) \), we have to prove

\[
\lim_{\epsilon \to 0} \mathbb{E} \sup_{t \in [0, T]} |R_\epsilon(t)|_E = 0,
\]

where

\[
R_\epsilon(t) := \int_0^t \int_D (B_1(u_\epsilon(s), v_\epsilon(s))(\xi) - \bar{B}(u_\epsilon(s))(\xi)) h(\xi) \, d\xi \, ds.
\]
Clearly, the limit
\[
\lim_{\epsilon \to 0} \mathbb{E} \sup_{t \in [0,T]} |R_\epsilon(t)|_E = 0,
\]
(4)
is a consequence of the fundamental result
\[
\mathbb{E} \left| \frac{1}{T} \int_s^{s+T} B_1(x, \nu^x(t; s, y)) \, dt - \bar{B}(x) \right|_E^2 \\
\leq \frac{c}{T} \left( 1 + |x|^{\kappa_1}_E + |y|^{\kappa_2}_E \right) + \alpha(T, x) \to 0, \quad \text{as } T \to \infty.
\]
(5)
Clearly, the limit

\[
\lim_{\epsilon \to 0} \mathbb{E} \sup_{t \in [0,T]} |R_\epsilon(t)|_E = 0, \quad (4)
\]

is a consequence of the fundamental result

\[
\mathbb{E} \left| \frac{1}{T} \int_s^{s+T} B_1(x, \nu^x(t; s, y)) \, dt - \bar{B}(x) \right|^2_E 
\leq \frac{c}{T} \left( 1 + |x|^\kappa_1_E + |y|^\kappa_2_E \right) + \alpha(T, x) \to 0, \quad \text{as } T \to \infty. \quad (5)
\]

But going from (5) to (4), is not painless. We use here the Khasminskii method of localization in time, but this requires first a truncation procedure for the coefficients and some uniform estimates.
Thank You