Averaging results for non-autonomous slow-fast systems of SPDEs

Sandra Cerrai

jointly with Alessandra Lunardi

Frontier Probability Days
Corvallis, Oregon

March 29-31, 2018

Averaging principle

Consider the perturbed system

$$\left\{ \begin{array}{ll} X_\epsilon'(t) = \epsilon\, f_1(X_\epsilon(t), Y_\epsilon(t)), & X_\epsilon(0) = x \in \mathbb{R}^n, \\ \\ Y_\epsilon'(t) = f_2(X_\epsilon(t), Y_\epsilon(t)), & Y_\epsilon(0) = y \in \mathbb{R}^m, \end{array} \right. \tag{1}$$

where $0 < \epsilon << 1$.

Averaging principle

Consider the perturbed system

$$\begin{cases} X_\epsilon'(t) = \epsilon\, f_1(X_\epsilon(t), Y_\epsilon(t)), & X_\epsilon(0) = x \in \mathbb{R}^n, \\ \\ Y_\epsilon'(t) = f_2(X_\epsilon(t), Y_\epsilon(t)), & Y_\epsilon(0) = y \in \mathbb{R}^m, \end{cases} \tag{1}$$

where $0 < \epsilon << 1$.

Under reasonable assumptions on f_1 and f_2, for any fixed $T > 0$

$$\lim_{\epsilon \to 0} \sup_{t \in [0,T]} |X_\epsilon(t) - x| = 0.$$

Averaging principle

Consider the perturbed system

$$\begin{cases} X_\epsilon'(t) = \epsilon\, f_1(X_\epsilon(t), Y_\epsilon(t)), \quad X_\epsilon(0) = x \in \mathbb{R}^n, \\ \\ Y_\epsilon'(t) = f_2(X_\epsilon(t), Y_\epsilon(t)), \quad Y_\epsilon(0) = y \in \mathbb{R}^m, \end{cases} \tag{1}$$

where $0 < \epsilon << 1$.

Under reasonable assumptions on f_1 and f_2, for any fixed $T > 0$

$$\lim_{\epsilon \to 0} \sup_{t \in [0,T]} |X_\epsilon(t) - x| = 0.$$

The behavior of the slow variable X_ϵ on time intervals of order ϵ^{-1} is of interest, because on such time scales significant changes take place.

For any frozen slow component $x \in \mathbb{R}^n$, consider the fast equation

$$Y'_{x,y}(t) = f_2(x, Y_x(t)), \qquad Y_{x,y}(0) = y,$$

and assume that the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f_1(x, Y_{x,y}(t))\, dt =: \bar{f}(x)$$

exists, for some $\bar{f} : \mathbb{R}^n \to \mathbb{R}^n$, independent of $y \in \mathbb{R}^m$.

For any frozen slow component $x \in \mathbb{R}^n$, consider the fast equation

$$Y'_{x,y}(t) = f_2(x, Y_x(t)), \qquad Y_{x,y}(0) = y,$$

and assume that the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f_1(x, Y_{x,y}(t))\, dt =: \bar{f}(x)$$

exists, for some $\bar{f} : \mathbb{R}^n \to \mathbb{R}^n$, independent of $y \in \mathbb{R}^m$.

The averaging principle says that in the time interval $[0, T/\epsilon]$ the slow motion X_ϵ can be approximated by the trajectories of the averaged system

$$\bar{X}'(t) = \bar{f}(\bar{X}(t)), \qquad \bar{X}(0) = x.$$

That is

$$\lim_{\epsilon\to 0} \sup_{t\in[0,T/\epsilon]} |X_\epsilon(t) - \bar{X}(t)|_{\mathbb{R}^n} = 0.$$

Averaging principle for randomly perturbed systems

A further development concerns the case of random perturbations of dynamical systems.

Averaging principle for randomly perturbed systems

A further development concerns the case of random perturbations of dynamical systems.

For example, in system (1) the coefficient f_2 may be assumed to depend also on a parameter $\omega \in \Omega$, (so that the fast variable is a random process), or even the perturbing coefficient f_1 may be taken random.

Averaging principle for randomly perturbed systems

A further development concerns the case of random perturbations of dynamical systems.

For example, in system (1) the coefficient f_2 may be assumed to depend also on a parameter $\omega \in \Omega$, (so that the fast variable is a random process), or even the perturbing coefficient f_1 may be taken random.

One has to reinterpret condition

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f_1(x, Y_{x,y}(t))\, dt =: \bar{f}(x)$$

and the type of convergence of X_ϵ to $\bar{X}$.

In 1968 Khasminskii has proved that averaging holds for the following system of stochastic differential equations

$$\begin{cases} dX_\epsilon(t) = f_1(X_\epsilon(t), Y_\epsilon(t))\,dt + g_1(X_\epsilon(t), Y_\epsilon(t))\,dw(t), \\ \\ dY_\epsilon(t) = \dfrac{1}{\epsilon} f_2(X_\epsilon(t), Y_\epsilon(t))\,dt + \dfrac{1}{\sqrt{\epsilon}}\,g_2(X_\epsilon(t), Y_\epsilon(t))\,dw(t), \end{cases} \tag{2}$$

with initial conditions $X_\epsilon(0) = x \in \mathbb{R}^n$ and $Y_\epsilon(0) = y \in \mathbb{R}^m$, for some k-dimensional Brownian motion $w(t)$.

In 1968 Khasminskii has proved that averaging holds for the following system of stochastic differential equations

$$\left\{\begin{array}{l} dX_\epsilon(t) = f_1(X_\epsilon(t), Y_\epsilon(t))\,dt + g_1(X_\epsilon(t), Y_\epsilon(t))\,dw(t), \\ \\ dY_\epsilon(t) = \dfrac{1}{\epsilon} f_2(X_\epsilon(t), Y_\epsilon(t))\,dt + \dfrac{1}{\sqrt{\epsilon}}\,g_2(X_\epsilon(t), Y_\epsilon(t))\,dw(t), \end{array}\right. \tag{2}$$

with initial conditions $X_\epsilon(0) = x \in \mathbb{R}^n$ and $Y_\epsilon(0) = y \in \mathbb{R}^m$, for some k-dimensional Brownian motion $w(t)$.

In this case the perturbation in the slow motion is given by the sum of a deterministic part and a stochastic part

$$\epsilon f_1(x, y)\,dt + \sqrt{\epsilon}\,g_1(x, y) dw(t),$$

and the fast motion is described by a stochastic differential equation.

Under reasonable assumptions on the coefficients f_2 and g_2, the fast equation with frozen slow component $x \in \mathbb{R}^n$

$$\begin{cases} dY^{x,y}(t) = f_2(x, Y^{x,y}(t))\, dt + g_2(x, Y^{x,y}(t))\, dw(t), \\ \\ Y^{x,y}(0) = y \in \mathbb{R}^m, \end{cases}$$

is well posed.

Under reasonable assumptions on the coefficients f_2 and g_2, the fast equation with frozen slow component $x \in \mathbb{R}^n$

$$\left\{\begin{array}{l} dY^{x,y}(t) = f_2(x, Y^{x,y}(t))\, dt + g_2(x, Y^{x,y}(t))\, dw(t), \\ \\ Y^{x,y}(0) = y \in \mathbb{R}^m, \end{array}\right.$$

is well posed.

Then, for every fixed $x \in \mathbb{R}^n$, we can introduce the transition semigroup

$$P_t^x \varphi(y) = \mathbb{E}\, \varphi(Y^{x,y}(t)),$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is Borel bounded.

Main assumptions

We assume that there exists $\bar{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that for every $t \geq 0$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$

$$\mathbb{E}\left|\frac{1}{T}\int_t^{t+T} f_1(x, Y^{x,y}(s))\, ds - \bar{f}(x)\right| \leq \alpha(T),$$

where $\alpha(T) \to 0$ as $T \to \infty$.

Main assumptions

We assume that there exists $\bar{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that for every $t \geq 0$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$

$$\mathbb{E}\left|\frac{1}{T}\int_t^{t+T} f_1(x, Y^{x,y}(s))\, ds - \bar{f}(x)\right| \leq \alpha(T),$$

where $\alpha(T) \to 0$ as $T \to \infty$.

We also assume that there exists $\bar{a} : \mathbb{R}^n \to \mathbb{R}^{k\times n}$ such that for every $t \geq 0$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$

$$\max_{i,j} \mathbb{E}\left|\frac{1}{T}\int_t^{t+T} g_1^{i,k}(x, Y^{x,y}(s)) g_1^{k,j}(x, Y^{x,y}(s))\, ds - \bar{a}^{i,j}(x)\right| \leq \alpha(T),$$

where $\alpha(T) \to 0$ as $T \to \infty$.

The convergence result

Under reasonable conditions on the coefficients the averaged equation

$$d\bar{X}(t) = \bar{b}(\bar{X}(t))\,dt + \sqrt{\bar{a}}(\bar{X}(t))\,dw(t), \qquad \bar{X}(0) = x,$$

is well posed.

The convergence result

Under reasonable conditions on the coefficients the averaged equation

$$d\bar{X}(t) = \bar{b}(\bar{X}(t))\,dt + \sqrt{\bar{a}}(\bar{X}(t))\,dw(t), \qquad \bar{X}(0) = x,$$

is well posed.

The averaging principle says that

the slow component $X_\epsilon(\cdot)$ converges weakly in the space of continuous trajectories $C([0,T];\mathbb{R}^n)$ to the solution $\bar{X}(\cdot)$ of the averaged equation.

Moreover, if g_1 does not depend on the fast variable, the convergence is stronger.

How to verify the assumptions?

Assume that

the semigroup P_t^x associated with the fast equation admits a unique invariant measure μ^x

and for any $x, y \in H$ and $\varphi \in \mathrm{Lip}(H)$

$$\left| P_t^x \varphi(y) - \int_H \varphi(z)\, \mu^x(dz) \right| \leq c\, (1 + |x|_H + |y|_H)\, e^{-\delta t}\, [\varphi]_{\mathrm{Lip}(H)}.$$

How to verify the assumptions?

Assume that

the semigroup P_t^x associated with the fast equation admits a unique invariant measure μ^x

and for any $x, y \in H$ and $\varphi \in \mathrm{Lip}(H)$

$$\left| P_t^x \varphi(y) - \int_H \varphi(z)\, \mu^x(dz) \right| \leq c\, (1 + |x|_H + |y|_H)\, e^{-\delta t}\, [\varphi]_{\mathrm{Lip}(H)}.$$

Then, the two assumptions are satisfied if we define

$$\bar{b}(x) = \int_{\mathbb{R}^m} b_1(x, y)\, d\mu^x(y),$$

and

$$\bar{a}^{i,j}(x) = \int_{\mathbb{R}^m} g_1^{i,k}(x, y) g_1^{k,j}(x, y)\, d\mu^x(y).$$

Averaging for SPDEs

In a series of papers, also together with M. Freidlin, we have considered an infinite dimensional analogue of (2) in a bounded domain $D \subset \mathbb{R}^d$, $d \geq 1$,

$$\left\{\begin{array}{rl} \dfrac{\partial u_\epsilon}{\partial t}(t,\xi) = & \mathcal{A}_1 u_\epsilon(t,\xi) + f_1(\xi, u_\epsilon(t,\xi), v_\epsilon(t,\xi)) \\ & +g_1(\xi, u_\epsilon(t,\xi), v_\epsilon(t,\xi))\, \dfrac{\partial w^{Q_1}}{\partial t}(t,\xi), \\ & \\ \dfrac{\partial v_\epsilon}{\partial t}(t,\xi) = & \dfrac{1}{\epsilon}\left[(\mathcal{A}_2 - \lambda) v_\epsilon(t,\xi) + f_2(\xi, u_\epsilon(t,\xi), v_\epsilon(t,\xi))\right] \\ & \\ & +\dfrac{1}{\sqrt{\epsilon}}\, g_2(\xi, u_\epsilon(t,\xi), v_\epsilon(t,\xi))\, \dfrac{\partial w^{Q_2}}{\partial t}(t,\xi), \end{array}\right. \tag{3}$$

with initial conditions $u_\epsilon(0,\xi) = x(\xi),\ v_\epsilon(0,\xi) = y(\xi)$ and suitable boundary conditions.

Well-posedness of the system

Here, we assume

- $\mathcal{A}_1$ and $\mathcal{A}_2$ are second order uniformly elliptic operators.

Well-posedness of the system

Here, we assume

- $\mathcal{A}_1$ and $\mathcal{A}_2$ are second order uniformly elliptic operators.
- $w^{Q_1}(t, \xi)$ and $w^{Q_2}(t, \xi)$ are cylindrical Wiener processes in $H := L^2(D)$, defined on a complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, with covariance Q_1 and Q_2.

Well-posedness of the system

Here, we assume

- $\mathcal{A}_1$ and $\mathcal{A}_2$ are second order uniformly elliptic operators.
- $w^{Q_1}(t,\xi)$ and $w^{Q_2}(t,\xi)$ are cylindrical Wiener processes in $H := L^2(D)$, defined on a complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, with covariance Q_1 and Q_2.
- Q_1 and Q_2 are bounded linear operators in H, fulfilling suitable assumptions and not Hilbert-Schmidt, in general. When $d = 1$, we could take $Q_i = I$.

Well-posedness of the system

Here, we assume

- $\mathcal{A}_1$ and $\mathcal{A}_2$ are second order uniformly elliptic operators.
- $w^{Q_1}(t,\xi)$ and $w^{Q_2}(t,\xi)$ are cylindrical Wiener processes in $H := L^2(D)$, defined on a complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, with covariance Q_1 and Q_2.
- Q_1 and Q_2 are bounded linear operators in H, fulfilling suitable assumptions and not Hilbert-Schmidt, in general. When $d = 1$, we could take $Q_i = I$.
- the mappings $f_i, g_i : D \times \mathbb{R}^2 \to \mathbb{R}$ are measurable;

Well-posedness of the system

Here, we assume

- $\mathcal{A}_1$ and $\mathcal{A}_2$ are second order uniformly elliptic operators.
- $w^{Q_1}(t,\xi)$ and $w^{Q_2}(t,\xi)$ are cylindrical Wiener processes in $H := L^2(D)$, defined on a complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, with covariance Q_1 and Q_2.
- Q_1 and Q_2 are bounded linear operators in H, fulfilling suitable assumptions and not Hilbert-Schmidt, in general. When $d = 1$, we could take $Q_i = I$.
- the mappings $f_i, g_i : D \times \mathbb{R}^2 \to \mathbb{R}$ are measurable;
- the mappings $f_i(\xi,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ and $g_i(\xi,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ are Lipschitz-continuous, or more general.

Under the hypotheses above, the stochastic system admits a unique adapted mild solution

$$(u_\epsilon, v_\epsilon) \in L^p(\Omega; C([0,T]; H)) \times L^p(\Omega; C([0,T]; H)),$$

for any $p \geq 1$ and $T > 0$, and for any $\epsilon > 0$.

By adapting to this infinite dimensional situations the arguments described above, we can average the coefficients f_1 and g_1 of the slow equation, and obtain the averaged equation

$$du(t) = \left[A_1 u(t) + \bar{F}(u(t))\right]\, dt + \bar{G}(u(t))\, dw^{Q_1}(t), \qquad u(0) = x.$$

Then,

we show that it admits a unique mild solution $\bar{u} \in L^p(\Omega, C([0,T];H))$, for any $p \geq 1$ and $T > 0$.

Therefore, we prove that under the conditions above, for any $T > 0$ we have

$$\mathcal{L}(u_\epsilon) \rightharpoonup \mathcal{L}(\bar{u}), \quad \text{in } C([0,T];H), \qquad \text{as } \epsilon \downarrow 0.$$

If

$$g_1(\xi,\sigma_1,\sigma_2) = g_1(\xi,\sigma_1), \quad (\xi,\sigma_1,\sigma_2) \in D \times \mathbb{R}^2,$$

then, for any $\eta > 0$

$$\lim_{\epsilon\to 0} \mathbb{P}\left(|u_\epsilon - \bar{u}|_{C([0,T];H)} > \eta\right) = 0,$$

or, even more,

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} |u_\epsilon(t) - \bar{u}(t)|_H^p = 0,$$

In the proof of the averaging limit, we have used

- the Khasminskii method of localization in time,
- the method of corrector functions and elliptic equations in Hilbert spaces.

In the proof of the averaging limit, we have used

- the Khasminskii method of localization in time,
- the method of corrector functions and elliptic equations in Hilbert spaces.

In recent years, many different other models of slow-fast systems of SPDEs have been considered. So, now the literature on the validity of the averaging principle for SPDEs is quite large.

The case of non-autonomous systems

Together with A. Lunardi, we dealt with

non-autonomous systems of reaction-diffusion equations of Hodgkin-Huxley or Ginzburg -Landau type, perturbed by a Gaussian noise of multiplicative type.

The case of non-autonomous systems

Together with A. Lunardi, we dealt with

non-autonomous systems of reaction-diffusion equations of Hodgkin-Huxley or Ginzburg -Landau type, perturbed by a Gaussian noise of multiplicative type.

In neurophysiology the Hodgkin-Huxley model, and its simplified version given by the Fitzhugh-Nagumo system, are used to describe the activation and deactivation dynamics of a spiking neuron.

The case of non-autonomous systems

Together with A. Lunardi, we dealt with

non-autonomous systems of reaction-diffusion equations of Hodgkin-Huxley or Ginzburg -Landau type, perturbed by a Gaussian noise of multiplicative type.

In neurophysiology the Hodgkin-Huxley model, and its simplified version given by the Fitzhugh-Nagumo system, are used to describe the activation and deactivation dynamics of a spiking neuron.

The classical Hodgkin-Huxley model has time-independent coefficients, but (see Wainrib 2013)

systems with time-dependent coefficients are particularly important to study models of learning in neuronal activity.

The system

We are dealing here with the following class of equations

$$\left\{\begin{array}{l}
\displaystyle \frac{\partial u_\epsilon}{\partial t}(t) = \Delta u_\epsilon(t) + b_1(\xi, u_\epsilon(t), v_\epsilon(t)) + \ g_1(\xi, u_\epsilon(t))\,\frac{\partial w^{Q_1}}{\partial t}(t,\xi),\\[10pt]
\displaystyle \frac{\partial v_\epsilon}{\partial t}(t) = \frac{1}{\epsilon}\left[(\gamma(t/\epsilon)\Delta - \alpha)v_\epsilon(t) + b_2(t/\epsilon, \xi, u_\epsilon(t), v_\epsilon(t))\right]\\[10pt]
\displaystyle \qquad\qquad\qquad + \frac{1}{\sqrt{\epsilon}}\, g_2(t/\epsilon, \xi, v_\epsilon(t))\,\frac{\partial w^{Q_2}}{\partial t}(t,\xi),\\[10pt]
u_\epsilon(0,\xi) = x(\xi), \qquad v_\epsilon(0,\xi) = y(\xi), \qquad \xi \in \ D,\\[10pt]
\mathcal{N}_1 u_\epsilon\,(t,\xi) = \mathcal{N}_2 v_\epsilon\,(t,\xi) = 0, \qquad t \geq 0, \qquad \xi \in \ \partial D.
\end{array}\right.$$

In fact, we considered more general differential operators.

The noise

The noises $w^{Q_1}(t)$ and $w^{Q_2}(t)$ are cylindrical Wiener processes in H, with covariance Q_1 and Q_2. That is,

$$w^{Q_i}(t,\xi) = \sum_{k=1}^{\infty} Q_i e_k(\xi)\,\beta_k(t), \qquad i = 1, 2,$$

where $\{e_k\}_{k\in\mathbb{N}}$ is the orthonormal basis in H that diagonalizes Δ, with eigenvalues $\{-\alpha_k\}_{k\in\mathbb{N}}$, and $\{\beta_k(t)\}_{k\in\mathbb{N}}$ is a sequence of independent Brownian motions.

The noise

The noises $w^{Q_1}(t)$ and $w^{Q_2}(t)$ are cylindrical Wiener processes in H, with covariance Q_1 and Q_2. That is,

$$w^{Q_i}(t,\xi) = \sum_{k=1}^{\infty} Q_i e_k(\xi)\,\beta_k(t), \qquad i = 1,2,$$

where $\{e_k\}_{k\in\mathbb{N}}$ is the orthonormal basis in H that diagonalizes Δ, with eigenvalues $\{-\alpha_k\}_{k\in\mathbb{N}}$, and $\{\beta_k(t)\}_{k\in\mathbb{N}}$ is a sequence of independent Brownian motions.

We assume $Q_i e_k = \lambda_{i,k} e_k$, for every $k \geq 1$ and $i = 1,2$, and

$$\kappa_i := \sum_{k=1}^{\infty} \lambda_{i,k}^{\rho_i}\,|e_k|_\infty^2 < \infty, \qquad \zeta := \sum_{k=1}^{\infty} \alpha_k^{-\beta}\,|e_k|_\infty^2 < \infty,$$

for some constants $\rho_i \in (2,+\infty]$ and $\beta \in (0,+\infty)$ such that

$$\frac{\beta(\rho_i - 2)}{\rho_i} < 1.$$

Notice that when

$$\alpha_k \sim k^{2/d}, \qquad \sup_{k\in\mathbb{N}} |e_k|_\infty < \infty,$$

the condition above on the eigenvalues $\lambda_{i,k}$ of the operators Q_i becomes

$$\kappa_i = \sum_{k=1}^{\infty} \lambda_{i,k}^{\rho_i} < \infty,$$

for some

$$\rho_i < \frac{2d}{d-2}.$$

The coefficients b_i and g_i

Just to simplify our presentation, we assume that the diffusion coefficients g_1 and g_2 are two bounded Lipschitz-continuous functions.

The coefficients b_i and g_i

Just to simplify our presentation, we assume that the diffusion coefficients g_1 and g_2 are two bounded Lipschitz-continuous functions. Moreover,

$$b_1(\xi, u, v) = -\alpha(\xi)\, u^{2n+1} + \sum_{j=0}^{2n} \alpha_j(\xi) u^j + h_1(\xi, u, v),$$

and

$$b_2(t, \xi, u, v) = -\beta(t, \xi) v^{2m+1} + \sum_{j=1}^{2m} \beta_j(t, \xi) v^j + h_2(t, \xi, u, v),$$

where h_1 and h_2 are locally Lipschitz functions with linear growth. All coefficients α, β, α_j and β_j are continuous, and

$$\inf_{\xi \in \bar{D}} \alpha(\xi) > 0, \qquad \inf_{(t,\xi) \in \mathbb{R}^+ \times \bar{D}} \beta(t, \xi) > 0.$$

For every $x, y \in C(\bar{D})$, we set

$$B_1(x,y)(\xi) := b_1(\xi, x(\xi), y(\xi)), \qquad \xi \in D,$$

and

$$B_2(t,x,y)(\xi) := b_2(t, \xi, x(\xi), y(\xi)), \qquad t \geq 0, \quad \xi \in D,$$

For every $x, y \in C(\bar{D})$, we set

$$B_1(x,y)(\xi) := b_1(\xi, x(\xi), y(\xi)), \qquad \xi \in D,$$

and

$$B_2(t,x,y)(\xi) := b_2(t,\xi, x(\xi), y(\xi)), \qquad t \geq 0, \quad \xi \in D,$$

Moreover, for every $x, z \in C(\bar{D})$, we set

$$[G_1(x)z](\xi) := g_1(\xi, x(\xi))z(\xi), \qquad \xi \in D,$$

and

$$[G_2(t,x)z](\xi) := g_2(t,\xi, x(\xi))z(\xi), \qquad t \geq 0, \quad \xi \in D.$$

The evolution family generated by $\gamma(t)\Delta$

We assume

$$0 < \gamma_0 \leq \gamma(t) \leq \gamma_1, \qquad t \geq 0,$$

and we define

$$\gamma(t,s) := \int_s^t \gamma(r)\, dr, \qquad s < t.$$

We denote by A the realization of Δ, endowed with the given boundary conditions, in all spaces $L^p(D)$, $1 < p < \infty$, and in $C(\bar{D})$.

The evolution family generated by $\gamma(t)\Delta$

We assume

$$0 < \gamma_0 \leq \gamma(t) \leq \gamma_1, \qquad t \geq 0,$$

and we define

$$\gamma(t,s) := \int_s^t \gamma(r)\,dr, \qquad s < t.$$

We denote by A the realization of Δ, endowed with the given boundary conditions, in all spaces $L^p(D)$, $1 < p < \infty$, and in $C(\bar{D})$.

For any $\epsilon > 0$ we set

$$U_\epsilon(t,s) = \exp\left(\frac{\gamma(r,\rho)}{\epsilon}\,A - \frac{\alpha}{\epsilon}(t-s)\right), \qquad s < t.$$

Clearly, for every initial condition x, we have that

$$u(t) = U_\epsilon(t,s)x, \qquad t \geq s,$$

is the unique mild solution to the linear problem

$$\partial_t u(t) = \frac{1}{\epsilon}(\gamma(t)\Delta - \alpha)u(t), \quad t > s, \quad u(s) = x,$$

endowed with the given boundary conditions.

The slow-fast system

With the notations introduced above, our system can be rewritten in the following abstract form

$$\left\{\begin{array}{l} du_\epsilon(t) = [Au_\epsilon(t) + B_1(u_\epsilon(t), v_\epsilon(t))]\, dt + G_1(u_\epsilon(t))\, dw^{Q_1}(t), \\[2ex] dv_\epsilon(t) = \dfrac{1}{\epsilon}\, [(\gamma(t/\epsilon)\Delta - \alpha)v_\epsilon(t) + B_2(t/\epsilon, u_\epsilon(t), v_\epsilon(t))]\, dt \\[2ex] \qquad\qquad + \dfrac{1}{\sqrt{\epsilon}}\, G_2(t/\epsilon, v_\epsilon(t))\, dw^{Q_2}(t), \\[2ex] u_\epsilon(0) = x, \qquad v_\epsilon(0) = y. \end{array}\right.$$

The slow-fast system

With the notations introduced above, our system can be rewritten in the following abstract form

$$\begin{cases} du_\epsilon(t) = [Au_\epsilon(t) + B_1(u_\epsilon(t), v_\epsilon(t))]\, dt + G_1(u_\epsilon(t))\, dw^{Q_1}(t), \\ \\ dv_\epsilon(t) = \dfrac{1}{\epsilon}\,[(\gamma(t/\epsilon)\Delta - \alpha)v_\epsilon(t) + B_2(t/\epsilon, u_\epsilon(t), v_\epsilon(t))]\, dt \\ \\ \qquad\qquad + \dfrac{1}{\sqrt{\epsilon}}\, G_2(t/\epsilon, v_\epsilon(t))\, dw^{Q_2}(t), \\ \\ u_\epsilon(0) = x, \qquad v_\epsilon(0) = y. \end{cases}$$

In what follows, we shall denote

$$H = L^2(D), \qquad E = C(\bar{D}).$$

We show that for any $\epsilon > 0$ and $x, y \in E$ there exists a unique adapted mild solution to the problem above in $L^p(\Omega; C_b((s, T]; E \times E))$, with $s < T$ and $p \geq 1$.

This means that there exist two unique adapted processes u_ϵ and v_ϵ in $L^p(\Omega; C_b((s, T]; E))$ such that

$$u_\epsilon(t) = e^{tA}x + \int_s^t e^{(t-r)A} B_1(u_\epsilon(r), v_\epsilon(r))\, ds$$

$$+ \int_s^t e^{(t-s)A} G_1(u_\epsilon(r))\, dw^{Q_1}(r),$$

and

$$v_\epsilon(t) = U_\epsilon(t,s)y + \frac{1}{\epsilon} \int_s^t U_\epsilon(t,r) B_2(r, u_\epsilon(r), v_\epsilon(r))\, dr$$

$$+ \frac{1}{\sqrt{\epsilon}} \int_s^t U_\epsilon(t,r) G_2(r, v_\epsilon(r))\, dw^{Q_2}(r).$$

Some bounds

We show that for any $p \geq 1$ and $s < T$ there exists a constant $c_{p,s,T} > 0$ such that for any $x, y \in E$ and $\epsilon \in (0,1]$

$$\mathbb{E} \sup_{t \in [s,T]} |u_\epsilon(t)|_E^p \leq c_{p,s,T} \left(1 + |x|_E^p + |y|_E^p\right),$$

and

$$\mathbb{E} \int_s^T |v_\epsilon(t)|_E^p \, dt \leq c_{p,s,T} \left(1 + |x|_E^p + |y|_E^p\right).$$

Some bounds

We show that for any $p \geq 1$ and $s < T$ there exists a constant $c_{p,s,T} > 0$ such that for any $x, y \in E$ and $\epsilon \in (0,1]$

$$\mathbb{E} \sup_{t \in [s,T]} |u_\epsilon(t)|_E^p \leq c_{p,s,T} \left(1 + |x|_E^p + |y|_E^p\right),$$

and

$$\mathbb{E} \int_s^T |v_\epsilon(t)|_E^p \, dt \leq c_{p,s,T} \left(1 + |x|_E^p + |y|_E^p\right).$$

Moreover, we show that there exists $\bar{\theta} > 0$ such that for any $\theta \in [0, \bar{\theta})$, $x \in C^\theta(\bar{D})$, $y \in E$ and $s < T$

$$\sup_{\epsilon \in (0,1]} \mathbb{E} \, |u_\epsilon|_{L^\infty(s,T;C^\theta(\bar{D}))} \leq c_{s,T} \left(1 + |x|_{C^\theta(\bar{D})} + |y|_E\right).$$

Finally, we prove that for any $\theta > 0$ there exists $\gamma(\theta) > 0$ such that for any $T > 0$, $p \geq 2$, $x \in C^\theta(\bar{D})$, $y \in E$ and $r_1, r_2 \in [s,t]$

$$\sup_{\epsilon \in (0,1)} \mathbb{E}\,|u_\epsilon(r_1) - u_\epsilon(r_2)|_E^p \leq c_p(T)\left(1 + |x|_{C^\theta(\bar{D})}^{pm_1} + |y|_E^p\right)|r_1 - r_2|^{\gamma(\theta)p}.$$

Finally, we prove that for any $\theta > 0$ there exists $\gamma(\theta) > 0$ such that for any $T > 0$, $p \geq 2$, $x \in C^\theta(\bar{D})$, $y \in E$ and $r_1, r_2 \in [s, t]$

$$\sup_{\epsilon \in (0,1)} \mathbb{E}\,|u_\epsilon(r_1) - u_\epsilon(r_2)|_E^p \leq c_p(T)\left(1 + |x|^{pm_1}_{C^\theta(\bar{D})} + |y|_E^p\right)|r_1 - r_2|^{\gamma(\theta)p}.$$

This implies that

the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon \in (0,1]}$ is tight in $C([s, T]; E)$, for any $x \in C^\theta(\bar{D})$, with $\theta > 0$, and for any $y \in E$.

The fast equation

For any frozen slow component $x \in E$, any initial condition $y \in E$ and any $s \in \mathbb{R}$, we introduce the problem

$$dv(t) = [(\gamma(t)A - \alpha)v(t) + B_2(t, x, v(t))]\, dt + G_2(t, v(t))\, d\bar{w}^{Q_2}(t),$$

with $v(s) = y$, where

$$\bar{w}^{Q_2}(t) = \begin{cases} w_1^{Q_2}(t), & \text{if } t \geq 0, \\ w_2^{Q_2}(-t), & \text{if } t < 0, \end{cases}$$

for two independent Q_2-Wiener processes, $w_1^{Q_2}(t)$ and $w_2^{Q_2}(t)$.

The fast equation

For any frozen slow component $x \in E$, any initial condition $y \in E$ and any $s \in \mathbb{R}$, we introduce the problem

$$dv(t) = [(\gamma(t)A - \alpha)v(t) + B_2(t, x, v(t))]\, dt + G_2(t, v(t))\, d\bar{w}^{Q_2}(t),$$

with $v(s) = y$, where

$$\bar{w}^{Q_2}(t) = \begin{cases} w_1^{Q_2}(t), & \text{if } t \geq 0, \\ w_2^{Q_2}(-t), & \text{if } t < 0, \end{cases}$$

for two independent Q_2-Wiener processes, $w_1^{Q_2}(t)$ and $w_2^{Q_2}(t)$.

The process $v^x(\cdot; s, y) \in L^p(\Omega; C([s, T]; E))$ is a *mild solution* if

$$v^x(t; s, y) = U_\alpha(t, s)y + \int_s^t U_\alpha(t, r)\, B_2(r, x, v^x(r; s, y))\, dr$$

$$+ \int_s^t U_\alpha(t, r)\, G_2(r, v^x(r; s, y))\, d\bar{w}^{Q_2}(r).$$

We prove that

for any $x, y \in$ and for any $p \geq 1$ and $s < T$,
there exists a unique mild solution
$v^x(\cdot; s, y) \in L^p(\Omega; C((s, T]; E) \cap L^\infty((s, T); E))$.

We prove that

$$\text{for any } x, y \in \text{ and for any } p \geq 1 \text{ and } s < T,$$
$$\text{there exists a unique mild solution}$$
$$v^x(\cdot; s, y) \in L^p(\Omega; C((s, T]; E) \cap L^\infty((s, T); E)).$$

We prove also that there exists $\delta > 0$ such that for any $x, y \in E$ and $p \geq 1$

$$\mathbb{E}\,|v^x(t; s, y)|_E^p \leq c_p\,\left(1 + e^{-\delta p(t-s)}\,|y|_E^p + |x|_E^p\right), \qquad s < t.$$

The fast equation in $\mathbb{R}$

An adapted process $v^x \in L^p(\Omega; C(\mathbb{R}; E))$ is a *mild solution* of the equation above in $\mathbb{R}$ if, for every $s < t$,

$$v^x(t) = U_\alpha(t,s)v^x(s) + \int_s^t U_\alpha(t,r)\, B_2(r,x,v^x(r))\, dr$$

$$+ \int_s^t U_\alpha(t,r)\, G_2(r,v^x(r))\, d\bar{w}^{Q_2}(r).$$

We prove that if $\alpha > 0$ is large enough and/or L_{g_2} is small enough, for any $t \in \mathbb{R}$ and $x \in E$

$$\text{there exists } \eta^x(t) \in L^p(\Omega; E), \text{ for all } p \geq 1,$$

such that

$$\lim_{s \to -\infty} \mathbb{E}\,|v^x(t;s,y) - \eta^x(t)|_E^p = 0,$$

for any $y \in E$ and $t \in \mathbb{R}$.

Moreover, for every $p \geq 1$ there exists some $\delta_p > 0$ such that

$$\mathbb{E}\,|v^x(t;s,y) - \eta^x(t)|_E^p \leq c_p\, e^{-\delta_p(t-s)} \left(1 + |x|_E^p + |y|_E^p\right).$$

Finally, η^x is a mild solution in $\mathbb{R}$ of the fast equation.

The evolution system of probabilities

For any fixed $x \in E$, we define the transition evolution operator

$$P^x_{s,t}\,\varphi(y) = \mathbb{E}\,\varphi(v^x(t;s,y)), \qquad s < t, \quad y \in E,$$

where $\varphi \in B_b(E)$.

The evolution system of probabilities

For any fixed $x \in E$, we define the transition evolution operator

$$P^x_{s,t}\,\varphi(y) = \mathbb{E}\,\varphi(v^x(t;s,y)), \qquad s < t, \quad y \in E,$$

where $\varphi \in B_b(E)$.

For any $t \in \mathbb{R}$ and $x \in E$, we set

$$\mu^x_t := \mathcal{L}(\eta^x(t)).$$

The evolution system of probabilities

For any fixed $x \in E$, we define the transition evolution operator

$$P^x_{s,t}\,\varphi(y) = \mathbb{E}\,\varphi(v^x(t;s,y)), \qquad s < t, \quad y \in E,$$

where $\varphi \in B_b(E)$.

For any $t \in \mathbb{R}$ and $x \in E$, we set

$$\mu^x_t := \mathcal{L}(\eta^x(t)).$$

We show that the family $\{\mu^x_t\}_{t \in \mathbb{R}}$ defines an evolution system of probability measures on E for the fast equation.

This means that μ^x_t is a probability measure on E, for any $t \in \mathbb{R}$, and for every $\varphi \in C_b(E)$

$$\int_E P^x_{s,t}\varphi(y)\,\mu^x_s(dy) = \int_E \varphi(y)\,\mu^x_t(dy), \qquad s < t.$$

Notice that, due to the previous estimates, for any $p \geq 1$ we have

$$\sup_{t\in\mathbb{R}} \mathbb{E}\,|\eta^x(t)|_E^p \leq c_p\,\left(1+|x|_E^p\right), \qquad x\in\,E,$$

so that

$$\sup_{t\in\mathbb{R}} \int_E |y|_E^p\,\mu_t^x(dy) \leq c_p\,\left(1+|x|_E^p\right).$$

Notice that, due to the previous estimates, for any $p \geq 1$ we have

$$\sup_{t\in\mathbb{R}} \mathbb{E}\,|\eta^x(t)|_E^p \leq c_p\,\left(1+|x|_E^p\right), \qquad x\in\,E,$$

so that

$$\sup_{t\in\mathbb{R}} \int_E |y|_E^p\,\mu_t^x(dy) \leq c_p\,\left(1+|x|_E^p\right).$$

Moreover, we prove that for any $R>0$ there exists $c_R>0$ such that

$$x_1, x_2 \in\,B_E(R) \Longrightarrow \sup_{t\in\mathbb{R}} \mathbb{E}\,|\eta^{x_1}(t)-\eta^{x_2}(t)|_E^2 \leq c_R\,|x_1-x_2|_E^2.$$

The key limiting result

Under the conditions above,

$$\lim_{s\to-\infty} P^x_{s,t}\varphi(y) = \int_E \varphi(y)\,\mu^x_t(dy),$$

for any $\varphi \in C_b(E)$.

The key limiting result

Under the conditions above,

$$\lim_{s\to-\infty} P^x_{s,t}\varphi(y) = \int_E \varphi(y)\,\mu^x_t(dy),$$

for any $\varphi \in C_b(E)$.

Moreover, if $\varphi \in C^1_b(E)$,

$$\left| P^x_{s,t}\varphi(y) - \int_E \varphi(z)\,\mu^x_t(dz) \right| \leq \|\varphi\|_{C^1_b(E)}\, e^{-\delta_1(t-s)}\,(1 + |x|_E + |y|_E)\,.$$

The key limiting result

Under the conditions above,

$$\lim_{s\to-\infty} P^x_{s,t}\varphi(y) = \int_E \varphi(y)\,\mu^x_t(dy),$$

for any $\varphi \in C_b(E)$.

Moreover, if $\varphi \in C^1_b(E)$,

$$\left| P^x_{s,t}\varphi(y) - \int_E \varphi(z)\,\mu^x_t(dz)\right| \leq \|\varphi\|_{C^1_b(E)}\, e^{-\delta_1(t-s)}\,(1+|x|_E+|y|_E)\,.$$

Finally, if $\{\nu^x_t\}_{t\in\mathbb{R}}$ is another evolution family of measures for the same equation, such that

$$\sup_{t\in\mathbb{R}} \int_E |y|_E\, \nu^x_t(dy) < \infty,$$

then

$$\nu^x_t = \mu^x_t, \quad t\in\mathbb{R}, \quad x\in E.$$

Almost periodic functions

Let (X, d_X) and (Y, d_Y) be two complete metric spaces. For any bounded function $f : \mathbb{R} \to Y$ and $\epsilon > 0$, we define

$$T(f, \epsilon) = \{\tau \in \mathbb{R} \, : \, d_Y(f(t+\tau), f(t)) < \epsilon, \text{ for all } t \in \mathbb{R}\} .$$

Almost periodic functions

Let (X, d_X) and (Y, d_Y) be two complete metric spaces. For any bounded function $f : \mathbb{R} \to Y$ and $\epsilon > 0$, we define

$$T(f,\epsilon) = \{\tau \in \mathbb{R} \,:\, d_Y(f(t+\tau), f(t)) < \epsilon, \text{ for all } t \in \mathbb{R}\}\,.$$

- A continuous function $f : \mathbb{R} \to Y$ is said to be almost periodic if, for all $\epsilon > 0$ there exists a number $l_\epsilon > 0$ such that

$$T(f,\epsilon) \cap [a, a + l_\epsilon] \neq \emptyset, \quad a \in \mathbb{R}.$$

Almost periodic functions

Let (X, d_X) and (Y, d_Y) be two complete metric spaces. For any bounded function $f : \mathbb{R} \to Y$ and $\epsilon > 0$, we define

$$T(f, \epsilon) = \{\tau \in \mathbb{R} : d_Y(f(t+\tau), f(t)) < \epsilon, \text{ for all } t \in \mathbb{R}\}.$$

- A continuous function $f : \mathbb{R} \to Y$ is said to be almost periodic if, for all $\epsilon > 0$ there exists a number $l_\epsilon > 0$ such that

$$T(f, \epsilon) \cap [a, a + l_\epsilon] \neq \emptyset, \quad a \in \mathbb{R}.$$

- Let $F \subset X$ and, for any $x \in F$, let $f(\cdot, x) : \mathbb{R} \to Y$ be a continuous function. The family of functions $\{f(\cdot, x)\}_{x \in F}$ is said uniformly almost periodic if for any $\epsilon > 0$ there exists $l_\epsilon > 0$ such that

$$\bigcap_{x \in F} T(f(\cdot, x), \epsilon) \cap [a, a + l_\epsilon] \neq \emptyset, \quad a \in \mathbb{R}.$$

The important consequence of almost periodicity

- There exists the *mean value* in Y of any almost periodic function $f : \mathbb{R} \to Y$, that is

$$\exists \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)\, ds \in Y.$$

Moreover, for every $t \in \mathbb{R}$

$$\lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} f(s)\, ds = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)\, ds,$$

uniformly with respect to $t \in \mathbb{R}$.

- If $\{f(\cdot, x)\}_{x\in F}$ is a uniformly almost periodic family of functions, with $F \subset X$, then

$$\exists \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} f(s,x)\, ds = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s,x)\, ds,$$

uniformly with respect to $t \in \mathbb{R}$ and $x \in F$.

Almost periodicity of the evolution family of measures

We assume the following conditions on $\gamma(t)$ and the coefficients $b_2(t,\xi,\sigma)$ and $g_2(t,\xi,\sigma)$.

- The function $\gamma:\mathbb{R}\to(0,\infty)$ is periodic.
- For every $R>0$, the families of functions

$$\mathcal{B}_R := \left\{ b_2(\cdot,\xi,\sigma)\,:\,\xi\in\bar{D},\ \sigma\in B_{\mathbb{R}^2}(R)\right\},$$

$$\mathcal{G}_R := \left\{ g_2(\cdot,\xi,\sigma)\,:\,\xi\in\bar{D},\ \sigma\in B_{\mathbb{R}}(R)\right\}$$

  are both uniformly almost periodic.

Under these conditions, it is easy to check that for any $R>0$ the family of functions

$$\left\{B_2(\cdot,x,y)\,:\,(x,y)\in B_{E\times E}(R)\right\},\quad \left\{G_2(\cdot,y)\,:\,y\in B_E(R)\right\},$$

are both uniformly almost periodic.

In Da Prato-Tudor (1995), SPDEs with periodic and almost periodic coefficients are studied and it is proven that if

- $\gamma(\cdot)$ is periodic,
- the family of functions

$$\{B_2(\cdot, x, y)\,:\,(x, y) \in\, B_{E\times E}(R)\}\,,\quad \{G_2(\cdot, y)\,:\, y \in\, B_E(R)\}\,,$$

  are both uniformly almost periodic, for any $R > 0$,
- the family of measures $\{\mu_t^x\}_{t\in\mathbb{R}}$ is tight in $\mathcal{P}(E)$,

then the mapping

$$t \in \mathbb{R} \mapsto \mu_t^x \in \mathcal{P}(E) \text{ is almost periodic.}$$

In fact, we prove that if α is sufficiently large and/or L_{g_2} is sufficiently small, there exists $\theta > 0$ such that for any $p \geq 1$ and for any $x \in E$

$$\sup_{t \in \mathbb{R}} \mathbb{E} \, |\eta^x(t)|^p_{C^\theta(\bar{D})} \leq c_p \left(1 + |x|_E^p\right).$$

In particular, the family of measures

$$\Lambda_R := \{\mu_t^x \, ; \, t \in \mathbb{R}, \; x \in B_E(R)\},$$

is tight in $\mathcal{P}(E)$, for any $R > 0$.

In fact, we prove that if α is sufficiently large and/or L_{g_2} is sufficiently small, there exists $\theta > 0$ such that for any $p \geq 1$ and for any $x \in E$

$$\sup_{t \in \mathbb{R}} \mathbb{E} \, |\eta^x(t)|^p_{C^\theta(\bar{D})} \leq c_p \left(1 + |x|_E^p\right).$$

In particular, the family of measures

$$\Lambda_R := \{\mu_t^x \, ; \, t \in \mathbb{R}, \; x \in B_E(R)\},$$

is tight in $\mathcal{P}(E)$, for any $R > 0$.

In view of Da Prato-Tudor result, this implies that the mapping

$$t \in \mathbb{R} \mapsto \mu_t^x \in \mathcal{P}(E),$$

is almost periodic, for any fixed $x \in E$.

The averaged equation

We first show that for every compact set $K \subset E$, the family of functions

$$\left\{ t \in \mathbb{R} \mapsto \int_E B_1(x,z)\, \mu_t^x(dz) \in E \,:\, x \in K \right\}$$

is uniformly almost periodic.

The averaged equation

We first show that for every compact set $K \subset E$, the family of functions

$$\left\{ t \in \mathbb{R} \mapsto \int_E B_1(x,z)\,\mu_t^x(dz) \in E \,:\, x \in K \right\}$$

is uniformly almost periodic.

Due to the almost periodicity of the family above, we can define

$$\bar{B}(x) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_E B_1(x,y)\,\mu_t^x(dy)\,dt, \qquad x \in E.$$

The averaged equation

We first show that for every compact set $K \subset E$, the family of functions

$$\left\{ t \in \mathbb{R} \mapsto \int_E B_1(x,z)\,\mu_t^x(dz) \in E \;:\; x \in K \right\}$$

is uniformly almost periodic.

Due to the almost periodicity of the family above, we can define

$$\bar{B}(x) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_E B_1(x,y)\,\mu_t^x(dy)\,dt, \qquad x \in E.$$

Notice that this, together with the estimates we have for B_1, yields

$$|\bar{B}(x)|_E \leq c\left(1 + |x|_E^{m_1}\right).$$

The fundamental limit

If α is sufficiently large and/or L_{g_2} is sufficiently small, there exist some constants $\kappa_1, \kappa_2 \geq 0$ such that for any $T > 0$, $s \in \mathbb{R}$ and $x, y \in E$

$$\mathbb{E}\left|\frac{1}{T}\int_s^{s+T} B_1(x, v^x(t; s, y))\, dt - \bar{B}(x)\right|_E^2$$

$$\leq \frac{c}{T}\left(1 + |x|_E^{\kappa_1} + |y|_E^{\kappa_2}\right) + \alpha(T, x),$$

for some mapping $\alpha : [0, \infty) \times E \to [0, +\infty)$ such that

$$\sup_{T>0} \alpha(T, x) \leq c\left(1 + |x|_E^{m_1}\right), \qquad x \in E,$$

and, for any compact set $K \subset E$,

$$\lim_{T\to\infty} \sup_{x\in K} \alpha(T, x) = 0.$$

The mapping $\bar{B} : E \to E$ is locally Lipschitz-continuous. Moreover, for any $x, h \in E$ and $\delta \in \mathcal{M}_h$

$$\left\langle \bar{B}(x+h) - \bar{B}(x), \delta \right\rangle_E \leq c\, \left(1 + |h|_E + |x|_E\right).$$

Here $\mathcal{M}_h$ denotes a suitable subset of the subdifferential of the norm of h.

The mapping $\bar{B} : E \to E$ is locally Lipschitz-continuous. Moreover, for any $x, h \in E$ and $\delta \in \mathcal{M}_h$

$$\left\langle \bar{B}(x+h) - \bar{B}(x), \delta \right\rangle_E \leq c\,(1 + |h|_E + |x|_E)\,.$$

Here $\mathcal{M}_h$ denotes a suitable subset of the subdifferential of the norm of h.

Thus, we can introduce the averaged equation

$$du(t) = \left[Au(t) + \bar{B}(u(t))\right]\,dt + G(u(t))\,dw^{Q_1}(t), \qquad u(0) = x \in E.$$

The mapping $\bar{B} : E \to E$ is locally Lipschitz-continuous. Moreover, for any $x, h \in E$ and $\delta \in \mathcal{M}_h$

$$\left\langle \bar{B}(x+h) - \bar{B}(x), \delta \right\rangle_E \leq c \left(1 + |h|_E + |x|_E\right).$$

Here $\mathcal{M}_h$ denotes a suitable subset of the subdifferential of the norm of h.

Thus, we can introduce the averaged equation

$$du(t) = \left[Au(t) + \bar{B}(u(t))\right] dt + G(u(t))\, dw^{Q_1}(t), \qquad u(0) = x \in E.$$

In view of the nice properties of $\bar{B}$,

the equation above admits a unique mild solution
$\bar{u} \in L^p(\Omega; C_b((0, T]; E))$.

The averaging limit

Fix $x \in C^\theta(\bar{D})$, for some $\theta > 0$, and $y \in E$. Then, if α is large enough and/or L_{g_2} is small enough, for any $T > 0$ and $\eta > 0$ we have

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\sup_{t \in [0,T]} |u_\epsilon(t) - \bar{u}(t)|_E > \eta\right) = 0,$$

where $\bar{u}$ is the solution of the averaged equation.

A few comments about the proof

For any $h \in D(A)$ and $\epsilon > 0$, we have

$$\int_D u_\epsilon(t,\xi)h(\xi)\,d\xi = \int_D x(\xi)h(\xi)\,d\xi + \int_0^t \int_D u_\epsilon(s,\xi)Ah(\xi)\,d\xi\,ds$$

$$+\int_0^t \int_D \bar{B}(u_\epsilon(s,\cdot))(\xi)h(\xi)\,d\xi\,ds + \int_0^t \int_D [G_1(u_\epsilon(s)h](\xi)dw^{Q_2}(s,\xi)$$

$$+\int_0^t \int_D \left(B_1(u_\epsilon(s),v_\epsilon(s))(\xi) - \bar{B}(u_\epsilon(s))(\xi)\right) h(\xi)\,d\xi\,ds.$$

Therefore, due to the tightness of the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon\in(0,1]}$ in $\mathcal{P}(C([0,T];E))$, we have to prove

$$\lim_{\epsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}|R_\epsilon(t)|_E = 0,$$

where

$$R_\epsilon(t) := \int_0^t\int_D \left(B_1(u_\epsilon(s),v_\epsilon(s))(\xi) - \bar{B}(u_\epsilon(s))(\xi)\right) h(\xi)\,d\xi\,ds.$$

Clearly, the limit

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} |R_\epsilon(t)|_E = 0, \tag{4}$$

is a consequence of the fundamental result

$$\begin{aligned} &\mathbb{E}\left|\frac{1}{T}\int_s^{s+T} B_1(x, v^x(t; s, y))\, dt - \bar{B}(x)\right|_E^2 \\ &\leq \frac{c}{T}\left(1 + |x|_E^{\kappa_1} + |y|_E^{\kappa_2}\right) + \alpha(T, x) \to 0, \quad \text{as } T \to \infty. \end{aligned} \tag{5}$$

Clearly, the limit

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} |R_\epsilon(t)|_E = 0, \tag{4}$$

is a consequence of the fundamental result

$$\begin{aligned} &\mathbb{E}\left|\frac{1}{T}\int_s^{s+T} B_1(x, v^x(t;s,y))\,dt - \bar{B}(x)\right|_E^2 \\ &\leq \frac{c}{T}\left(1+|x|_E^{\kappa_1}+|y|_E^{\kappa_2}\right)+\alpha(T,x) \to 0, \quad \text{as } T\to\infty. \end{aligned} \tag{5}$$

But going from (5) to (4), is not painless. We use here the Khasminskii method of localization in time, but this requires first a truncation procedure for the coefficients and some uniform estimates.

Thank You