RABI's WORK WITH ED:

A Tribute to Ed Waymire on His Retirement

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Presented By

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1. INTRODUCTION

Ed Waymire has made many significant contributions to Probability theory and its applications, including interacting particles, multiplicative cascades, skew Brownian motion and anomalous diffusions, branching processes, turbulence, nonlinear PDEs including 3D incompressible Navier-Stokes.

- He has also served as the Editor of the Annals of Applied Probability, and of Bernoulli. He is a former President of the Bernoulli Society for Mathematical Statistics and Probability.
- This brief chronological presentation is devoted to some work of Ed that Rabi Bhattacharya has been involved with.

2. HURST EFFECT/EXPONENT

B., Gupta and Waymire (1983). The Hurst effect under trends. *J. Appl. Probab*. 20.

Time series X_n (n=0,1,...). Let $S_n = X_1 + \dots + X_n$, ($n \geqq 1$), $S_0=0$; $\bar{x}_N = S_N/N$; $M_N = \max\{ S_n - n\bar{x}_N : n=0,1,\dots,N\}$, $m_N = \min\{ S_n - n\bar{x}_N : n=0,1,\dots\}$; $R_N = M_N - m_N$; $D_N^2 = \sum_{1\le n\le N} (X_n - \bar{x}_N)^2$.

Hurst Exponent is H (>0) if $W_N \equiv N^{-H} R_N / D_N \Rightarrow$ to a nonzero random variable, as $N \to \infty$.

Named after hydrologist E. Hurst (1951) who estimated H to be 0.73 for annual flows X_n into the Nile River.

The first examples of stationary sequences $\{X_n\}$ for which H>1/2 are due to Mandelbrot and Van Ness (1968). These are increments of so called *fractional Brownian motions* $B_\beta(t)$, $t \geqq 0$, introduced by Kolmogorov (1940), which are Gaussian processes with $B_\beta(0)=0$ and correlation functions

$$\rho_\beta(s,t) \equiv \mathrm{Cov}(B_\beta(t), B_\beta(s)) = \{s^{2\beta} + t^{2\beta} + |t - s|^{2\beta}\}/2$$

$(0 \leq s, t < \infty)$, $(0<\beta<1)$.

$B_{1/2}(t)$ is Brownian motion (Ossiander and Waymire (1989)).The processes $B_\beta(.)$ are *self-similar* ($\forall$ c>0, $\{B_\beta(t): t>0\} =^{L} \{c^{-\beta} B_\beta(ct), t>0\}$). Except for the case β= ½, the increments $\{X_n\}$ of $B_\beta(.)$,over consecutive intervals of equal length, have long range dependence, and are not strong mixing. Originally conceived by Mandelbrot for modeling (a) hydrological time series considered by Hurst and others and (b) financial time series, undoubtedly served as a motivation for his subsequent path breaking theory of the occurrence of *fractals in* nature (Mandelbrot (1967)).

Mandelbrot's theory did not find broad acceptance among hydrologists, who did not accept the property of long range dependence (Klems (1974), (1982), Kottegoda (1970)). A more acceptable view was that Hurst's phenomena were indicative of *pre-asymptotic behavior* exhibiting trends in the range of data observed (Boes and SalasCruz (1978)). In 1980 Vijay Gupta a hydrologist, introduced Rabi to Hurst's work and numerous hydrological phenomena indicative of climatic patterns over long periods of time.

Rabi constructed a sequence: $X_n = Y_n + f(n)$, with Y_n i.i.d., $f(n)$ a slow trend, which has the Hurst effect ($H > 1/2$). *Ed wrote up a complete theory.* Here one assumes that $\{Y_n\}$ is a weakly dependent stationary process (i.e., strongly mixing with finite second moments). Let

$\bar{f}(N) = \sum_{1 \le j \le N} f(j)/N$, $\mu_n(r) = \sum_{1 \le j \le r} f(j) - r\bar{f}(N)$. $\Delta_n = \max_{1 \le r \le N} \mu_n(r) - \min_{1 \le r \le N} \mu_n(r)$.

Theorem (B., Gupta, Waymire (1983)). Let $\{Y_n\}$ be weakly dependent. Hurst effect holds with index $H > 1/2$ *if and only if* Δ_N / N^H converges to a positive constant c’.

EXAMPLE. $f(n) = d + c(m+n)^{\beta}$.

Here $c \neq 0$ and d are constants, and m is the starting point of measurement. Write $W_N = N^{-H} R_N / D_N$,

$R^{\sim} = \max\{B^{\sim}_t : 0 \leq t \leq 1\} - \min\{B^{\sim}_t : 0 \leq t \leq 1\}$, $R^{\#} = \max\{B^{\sim}_t - 2ct(1-t) : 0 \leq t \leq 1\}$ [$B^{\sim}$=Br. Bridge]. Then

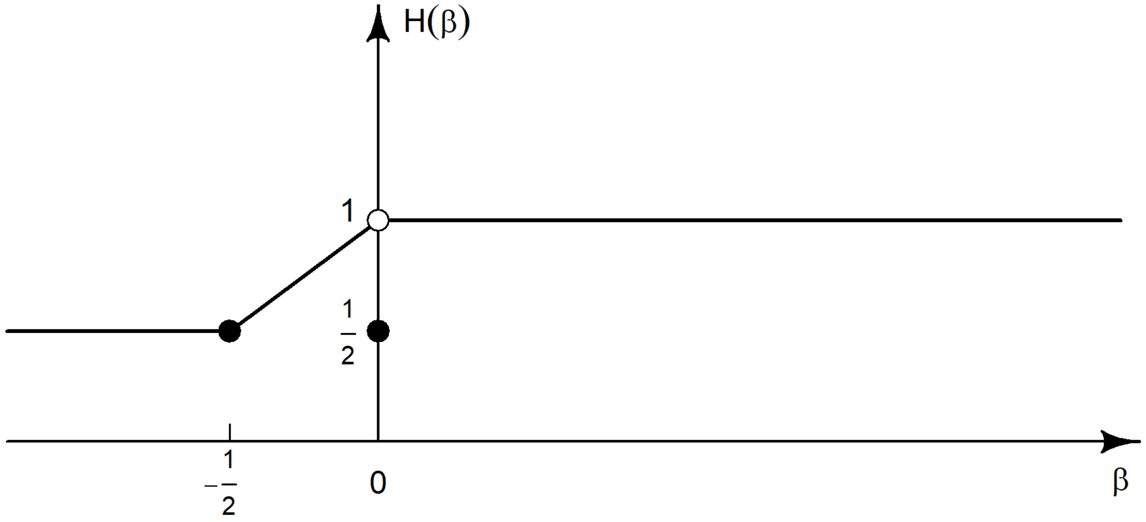
$H =$ ½ $\beta \leq -1/2$, [$W_N \Rightarrow R^{\sim}$ for $\beta < -1/2$,

$W_N \Rightarrow R^{\#}$ for $\beta = -1/2$],

$1+\beta > 1/2$ $-1/2 < \beta < 0$ [$W_N \Rightarrow$ constant $c' > 0$],

½ $\beta = 0$, [$W_N \Rightarrow R^{\sim}$],

1 $\beta > 0$. [$W_N \Rightarrow c'' > 0$]. □


H(β)
1
1/2
−1/2
0
β

REMARK. This is the first demonstration of the appearance of the Hurst effect for weakly dependent sequences with trend and is a highly cited work on the topic. This theorem also highlights the fact that based on usually available data it would be difficult, if not impossible, to distinguish between strictly stationary long range dependence from weakly dependent stationary sequences with an additive slow drift, without making additional (and often ad hoc) assumptions. For detailed history, see Graves, et al. (2017).

3. “Stochastic Processes with Applications” (Wiley (1990), (2009) (B. and Waymire)

This graduate text and reference, reprinted in 2009 in the series SIAM Classics in Applied Mathematics, has received high praise for its comprehensive coverage and rather unique expository style from many distinguished colleagues in the field (including Aldous, Barndorff-Nielsen, Diaconis, Gundy) as well as professionals in diverse areas of science and engineering from the US and abroad

It proceeds in some depth from (1) random walks to Brownian motions as their scaling limits, followed by broad coverages of (2) discrete and (3) continuous parameter Markov chains, (4) martingales, (5) one- and multi-dimensional Brownian motions and diffusions, and their generators, along with numerous applications to elliptic and parabolic PDE’s in applied math. (6) Also presented are stochastic optimization and dynamic programming, including Bellman and Hamilton-Jacobi equations. (7) The final chapter is an introduction to stochastic differential equations. (8) An expository background in measure theoretic probability makes the book self contained.

It seems rather amazing that the authors spent such an enormous amount of time and energy in the creation of this book in the midst of their active research careers!

4. Random Iterations of Quadratic Maps (B.&Waymire (1993))

Bhattacharya, R. and Rao,B.V. (1993) derived a set of sufficient conditions for the stability (i.e. existence and uniqueness of an invariant probability π and convergence to it whatever be the initial point or distribution) of Markov processes obtained by the i.i.d. iterations of two quadratic maps $F_{\theta i}$ (i=1,2), $\theta_1 < \theta_2$, picked with positive probabilities p and q= 1−p. Here

$F_\theta(x) = \theta x(1-x), \; 0<x<1.$ $(\theta \in [1,4])$.

Theorem. Suppose the following *"splitting"* conditions of Dubins and Freedman (1966) hold: (i) the maps $F_{\theta i}$ (i=1,2) are monotone on an invariant interval [a,b] and (ii) there are an integer N and a point x_0 and positive $\delta_i > 0$ (i=1,2) such that after N iterations the random composition $X_N(x)$, starting at x, satisfies

$P(X_N(x) \leq x_0 \ \forall x \in [a,b]) \geqq \delta_1$, and $P(X_N(x) \geqq x_0 \ \forall x \in [a,b]) \geqq \delta_2$. (*)

Then ∃ unique inv. π such that, with $\delta = \min\{\delta_1, \delta_2\}$,

$$\sup_{x,y} |\text{Prob}\,(X_n(x) \leq y) - \pi((0,y])| \leq (1-\delta)^{[n/N]}. \quad (**)$$

Ed introduced the notion of *"strict splitting"* by replacing $\leq$ and $\geqq$ in the events in parentheses in (*) with strict inequalities $<$ and $>$, respectively, and showed that under the strict splitting condition, there is a unique invariant probability π, and (**) holds, for any distribution Q of θ in $[\theta_1, \theta_2]$, instead of just the two-point support $\{\theta_1, \theta_2\}$, provided θ_1, θ_2, belong to the support of Q.

It turns out that the strict splitting condition holds in all the examples in which splitting occurs. This is then a major improvement of the earlier results.

Examples.

(a) $1<\theta_1<\theta_2\leq 2$. **(b)** $2<\theta_1<\theta_2\leq 3$. (c) $2<\theta_1 < 3< \theta_2 \leq 1+\sqrt{5}$, $\theta_1 \in [8/(\theta_2 (4-\theta_2)), \theta_2]$;

for example, $\theta_1= 3.18$, $\theta_2=3.20$.

5. Incompressible Navier-Stokes Equations in 3-D

(Bhattacharya, Chen, Dobson, Guenther, Orum, Ossiander,Thomann,Waymire. (2003), (2005)).

In 2000 Rabi joined Ed and a team at OSU that Ed had organized to work on an NSF FRG project on a class of nonlinear PDEs, especially the 3D incompressible Navier-Stokes (NS) equations.

NS Equations. Consider a homogeneous incompressible fluid with constant density moving in the free space $\mathbb{R}^3$. To determine the velocity field $u(x^1,x^2,x^3, t)$ of the fluid over time $t \geqq 0$, given $u_0 = u(x^1,x^2,x^3, 0)$, along with an external force field g

Think of an “element” of the fluid arriving at $(x^1(t),x^2(t),x^3(t))$ at time t. Its velocity at this time is $u(x^1(t),x^2(t),x^3(t),t)$, and acceleration is given by $d(u(x^1(t),x^2(t),x^3(t)),t)/dt = \partial u(x^1,x^2,x^3,t)/\partial t + \sum_{1\leq i\leq 3} u_i \partial u/\partial x^i = \partial u/\partial t + u.\nabla u$ at $(x^1,x^2,x^{3,}) = (x^1(t),x^2(t),x^3(t))$. The acceleration field on $\mathbb{R}^3$ equals the force (per unit mass) $\nu\Delta u$ + grad p + g, where ν is the coefficient of viscosity of the fluid, p is the pressure (internal force) and g= $g((x^1,x^2,x^{3,},t))$ is the external force, e.g., gravity. One then has the four equations for the four unknown, u_1,u_2,u_3 and p, given by

$\partial u_i/\partial t + u_i. \nabla u = \nu\Delta u_i + \partial p/\partial x^i = g_i((x^1,x^2,x^3,t))$ (i=1,23), and

$\text{Div } u = 0.,$ (5.1)

given $u_0 = u(x^1,x^2,x^3, 0)$, and an external force field $g((x^1,x^2,x^3,t))$.

If $\nu=0$, (5.1) are *Euler's equations*, and for $\nu>0$ they are called the *Navier-Stokes (NS) equations* It is known that in $\mathbb{R}^2$ both Euler's equations and NS equations have unique smooth global (i.e, for all t), for all smooth initial velocities vanishing rapidly at infinity See Ladyzhenskaya (1969), (2003). On $\mathbb{R}^3$ the corresponding problems are open; here unique (smooth) global solutions are known only for u_0

Work of Leray. In a seminal paper Leray (1934) proved the *existence* of weak solutions in $L^2(\mathbb{R}^3)$, but not *uniqueness* for all initial u in $L^2(\mathbb{R}^3)$; and he speculated the existence of initial u with non-unique or *irregular solutions* each of which he proposed to call "*a turbulent solution*".

Fourier Navier-Stokes (FNS) Equations and Majorizing Kernels. Among the massive literature on NS equations, the most relevant work for us is LeJan and Sznitman (1997). For its generalization,

consider the Fourier analytic version of (5.1), with the last equation in the frequency variable ξ becoming

$$\xi . \, u(\xi) = 0. \qquad (5.2)$$

Restrict, or project, u to the space satisfying (5.2). Recallig that (grad p)^(ξ)= $-i p(\xi)\,\xi$, the projection operator cancels p from the first set of equations in (5.1), and u is then governed by the equation in integral form given by

$$u(\xi,t) = \exp\{-\nu|\xi|^2 t\} u_0(\xi) + \int_{[0,t]} \exp\{-\nu|\xi|^2 s\} \big[\{|\xi|(2\pi)^{-3/2} \{\int_{\mathbb{R}3} u(\eta, t-s) \otimes_\xi u(\xi-\eta, t-s) d\eta\}$$

$$+ g(\xi, t-s)\big] ds, \qquad (5.3)$$

where for $w, z \in \mathbb{C}^3$, $w \otimes_\xi z = -i(e_\xi . z)\pi_\xi{}^\perp w$, $e_\xi = \xi / |\xi|$, and $\pi_\xi{}^\perp w$ is the projection of w onto the plane orthogonal to ξ ($\xi \neq 0$).

One assumes that g (ξ) already satisfies the same equation as u(ξ,t), or projects it onto $\xi^{\perp}$. Multiplying u(ξ, t) by the multiplier 1/h(ξ), where h is a *majorizing kernel of exponent* Θ≧0, defined and positive on W= {ξ≠0} and satisfying

$$h*h(\xi) \leq |\xi|^{\Theta} h(\xi), \qquad (5.4)$$

one rewrites (5.3) for χ(ξ,t)= u(ξ,t) /h(ξ) as

$$\chi(\xi,t) = \exp\{-\nu|\xi|^2 t\}\, \chi_0(\xi) + \int_{[0,t]} \nu|\xi|^2 \exp\{-\nu|\xi|^2 s\}$$
$$[\{(1/2)m(\xi) \int_{W\mathrm{x}W} \chi(\eta, t-s) \otimes_{\xi} \chi(\xi-\eta, t-s)d\eta\}$$
$$+(1/2)\ g(\xi, t-s)]ds. \qquad (5.5)$$

Here $m(\xi) = 2h*h(\xi)/[\nu(2\pi)^{3/2}|\xi|h(\xi)]$

Probabilistic View. Think of the time S_ϕ when an exponential clock with parameter $\nu|\xi|^2$ rings. Then, depending on the toss of a fair coin, the tree terminates with probability ½ as a single edge tree of length S_ϕ, and with probability ½ it moves from the root ϕ of a branching binary tree to $\xi_1 = \eta$ and $\xi_2 = \xi - \eta$ with probability determined by the transformed spatial integral kernel on the right side. If the tree does not terminate at time S_ϕ, then the process is repeated at each ξ_i (in place of ξ) as the clock rings at independent exponential times with parameters $\nu|\xi_i|^2$ from the vertices ξ_i ($i=1,2$) and independent coin tosses are made. Continue in this manner until the process terminates at a finite time.

One may think of (5.5) as a random multiplicative functional until termination time, which may be expressed recursively. The expected value of this functional, multiplied by $h(\xi)$, provides the solution to (5.3), provided the expectation is finite. It is an intriguing problem to figure out when the expectation is finite and when it is infinite, which is seemingly related to the fundamental question of global existence.

Alternatively, look at the problem of solution of (5.3) analytically, and use a contraction mapping principle to arrive at the following results (Bhattacharya et al. (2003)) The same results are obtained from the probabilistic description above. To state them define, for $\gamma=0$ or 1, S' space of temp. distributions on $\mathbb{R}^3$, the Banach space $\mathcal{F}_{h,\gamma,T}$ as the completion of

$\{v\in S': v(0,t) = 0, ||v||_{h,\gamma,T}:$

$= \sup_{0\leq t\leq T,\, \xi\neq 0} |v(\xi,t)|\exp\{\gamma|\xi|\sqrt{t}\}/h(\xi) < \infty\}$ (5.6)

Theorem 1. Let h be a majorizing kernel of exponent $\theta=1$. Suppose that $||u_0||_{h,\,0,T} < (2\pi)^{3/2}\nu/2$, and $|(-\Delta)^{-1}g|\mathcal{F}_{h,\,0,T} \leq (2\pi)^{3/2}\nu^2/4$. Then there is a unique solution u to FNS (5.3) lying in the ball B(0, R) in $\mathcal{F}_{h,\,0,T}$ centered at 0 and with radius $R=(2\pi)^{3/2}\nu/2$ $(0<T\leq\infty)$.

The *global existence* corresponds to $T = \infty$. For *local existence*, one has the following result.

Theorem 2. Let h be a majorizing kernel of exponent $\theta < 1$. Assume that $\exp\{\nu t \Delta\}\, u_0 \in \mathcal{F}_{h,\gamma,T}$, and for some β, $1 \le \beta \le 2$, $(-\Delta)^{-\beta/2} g \in \mathcal{F}_{h,\gamma,T}$. Then there is a T^*, $0 < T^* \le T$, for which one has a unique solution $u \in \mathcal{F}_{h,\gamma,T}$. [$\gamma = 0$ or 1]

These are significant extensions of the results of LeJan and Sznitman (1997), who considered cases $h(\xi)=\pi^3/|\xi|$ and $h(\xi) = \alpha\exp\{-\alpha|\xi|\}/2\pi|\xi|$, $\alpha>0$. In these cases one has equality in the definition (5.4) with $\Theta=0$ and $\Theta=1$. These are the boundary points of the class of majorizing kernels

$$h_{\beta,\alpha}(\xi) = c(\beta,\alpha)|\xi|^{\beta-2}\exp\{\alpha|\xi|^{\beta}\} \quad (0\le\beta\le 1, \alpha>0). \qquad (5.7)$$

The idea of mazorizing kernels with inequalities ($\le$) in (5.4) are due to Ed Waymire and colleagues at OSU. For more general results on majorizing kernels see Bhattacharya et al. (2003)

5.Pending Work

Apart from the graduate text A Basic Course in Probability (2007), (2016), Springer, Ed and Rabi have nearly completed a four volume treatise on stochastic processes to be published in the Springer series Universitext. Hopefully they can find interesting new mathematical problems to investigate along the way. It has been great fun for Rabi to work with Ed and have him as a dear friend.

VERY BEST WISHES TO ED ON HIS RETIREMENT