Boundary trace of a class of perturbed Bessel processes

Lidan Wang

University of Washington, Seattle

May 10, 2016

Motivation: Fractional Laplacian

The fractional Laplacian operator $\Delta^\alpha := -(-\Delta)^\alpha$, $0 < \alpha < 1$, is a non-local operator. Analytically, it can be interpreted in two ways, for $f : \mathbb{R}^d \to \mathbb{R}$:

- $(-\Delta)^\alpha f(x) = C_{d,\alpha} \int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+2\alpha}} \mathrm{d}y$, where $C_{d,\alpha}$ is a normalization constant;
- $\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \hat{f}(\xi)$.

Analytic work

In 2007, Caffarelli-Silvestre realized Δ^α as the trace on $\mathbb{R}^d$ of $\nabla \cdot (x_{d+1}^{1-2\alpha} \nabla)$ in $\mathbb{R}^d \times \mathbb{R}_+$, up to a constant, i.e., for a function $f : \mathbb{R}^d \to \mathbb{R}$, consider the extension $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ that satisfies

$$\begin{cases} u(x_1, \cdots, x_d, 0) & = f(x_1, \cdots, x_d) \\ \operatorname{div}(x_{d+1}^{1-2\alpha} \nabla(u)) & = \Delta u + \frac{1-2\alpha}{x_{d+1}} \partial_{d+1} u = 0 \end{cases}$$

It can be shown that up to a constant factor

$$\lim_{x_{d+1} \to 0+} x_{d+1}^{1-2\alpha} \partial_{d+1} u(x_1, \cdots, x_{d+1}) = -(-\Delta)^\alpha f(x_1, \cdots, x_d).$$

This reduction enabled them to use tools from elliptic differential operators to study problems for fractional Laplacians. For example using (local) pde methods, they proved a Harnack inequality and a boundary Harnack inequality for Δ^α.

Probabilistic Approach

In fact, this relation has been known to probabilists for quite a while. Recall that Δ^α is the infinitesimal generator of a 2α-stable process.

Theorem (Molchanov-Ostrovskii, 1969)

Let X_t be a one-dimensional diffusion process on the half-line $[0, \infty)$ (reflected at 0) with generating operator

$$\mathcal{L}_\alpha = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1-2\alpha}{2x}\frac{\mathrm{d}}{\mathrm{d}x}, 0 < \alpha < 1, \tag{1}$$

and let S_t be its inverse local time at zero, then S_t is an α-stable subordinator.
If we let B_t be a d-dimensional Brownian motion independent of X_t, then $Z_t := B_{S_t}$ can be viewed as the trace process on $\mathbb{R}^d$ of the diffusion process (B_t, X_t) in the upper half space $\mathbb{R}^d \times \mathbb{R}_+$, and it is a rotationally symmetric 2α-stable process on $\mathbb{R}^d$.

Interesting Questions

- What kind of non-local operator can be realized as the trace of differential operator in upper half space of one dimension higher?

Interesting Questions

- What kind of non-local operator can be realized as the trace of differential operator in upper half space of one dimension higher?
- Can we represent all the subordinate Brownian motions as traces of processes in the upper half space?

Interesting Questions

- What kind of non-local operator can be realized as the trace of differential operator in upper half space of one dimension higher?
- Can we represent all the subordinate Brownian motions as traces of processes in the upper half space?
- What kinds of subordinators can be realized as inverse local times of **diffusions**?

Interesting Questions

- What kind of non-local operator can be realized as the trace of differential operator in upper half space of one dimension higher?
- Can we represent all the subordinate Brownian motions as traces of processes in the upper half space?
- What kinds of subordinators can be realized as inverse local times of **diffusions**?
- A more modest question: what potential theoretical property can we get for the trace process on $\mathbb{R}^d$, which is obtained from a (reflected) diffusion process in $\mathbb{R}^d \times [0, \infty)$?

Answer to the first and second Questions

Frank Knight proved that any subordinator can be realized as the inverse local times of a generalized diffusion.

Theorem (F. Knight, 1981)

Let X be a generalized diffusion on $\mathbb{R}$ and let S_t be its inverse local time at zero. Then the Lévy measure $\nu(\mathrm{d}x)$ of S_t has a complete monotone density $\nu(x)$ with respect to Lebesgue measure.

Answer to the first and second Questions

Frank Knight proved that any subordinator can be realized as the inverse local times of a generalized diffusion.

Theorem (F. Knight, 1981)

Let X be a generalized diffusion on $\mathbb{R}$ and let S_t be its inverse local time at zero. Then the Lévy measure $\nu(\mathrm{d}x)$ of S_t has a complete monotone density $\nu(x)$ with respect to Lebesgue measure.

Conversely, given any Lévy measure satisfying the above condition, there exists a generalized diffusion such that ν is the Lévy measure of its inverse local time at zero.

(Partial) Answer to the third Question

Only very limited cases have been discussed and explicit diffusions were explored, including Molchanov-Ostrovskii's paper.

- S. A. Molchanov, E. Ostrovskii, Symmetric stable processes as traces of degenerate diffusion processes, 1969;
- C.Donati-Martin, M. Yor, Some explicit Krein representations of certain subordinators, including the Gamma process, 2005;
- C. Donati-Martin, M. Yor, Further examples of explicit Krein representations of certain subordinators, 2006.

Even some simple processes, (for example, mixed BM+stable processes), haven't been realized as the traces of diffusions.

Theorem

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, X_t is a diffusion process, reflecting at 0, determined locally by the generator

$$\mathcal{L} = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}.$$

Let L_t be the local time of X_t at 0. Its inverse local time $S_t = \inf\{s : L_s > t\}$ is a Lévy process (called subordinator). Denote its Laplace exponent by $\phi(\lambda)$. Note that for $t \geq 0$, S_t is a stopping time for $\{\mathcal{F}_t\}$. Now define

$$\frac{d\mathbb{Q}_x}{d\mathbb{P}_x} = \frac{\exp(-mS_t)}{\mathbb{E}_x[\exp(-mS_t)]} \text{ on } \mathcal{F}_{S_t}. \tag{2}$$

Then

(i) (2) defines a new probability measure $\mathbb{Q}_x$ on $\mathcal{F}_\infty$ in a consistent way;

(ii) Under $\mathbb{Q}$, the original diffusion X, write as $X^{(m)}$ for emphasis, is a diffusion determined by the generator

$$\mathcal{L}^{(m)} = \mathcal{L} + 2a(x)\frac{\rho_m'(x)}{\rho_m(x)}\frac{d}{dx}, \quad x > 0.$$

where $\rho_m(x) = \mathbb{E}_x[\exp(-mT_0)]$ and T_0 is the first hitting time at 0 for X_t.

(iii) Denote by $S_t^{(m)}, \phi^{(m)}(\lambda)$ the inverse local time for $X_t^{(m)}$ at 0 and its Laplace exponent, respectively. Then

$$\phi^{(m)}(\lambda) = \phi(\lambda + m) - \phi(m).$$

Remarks

1. If we know a subordinator, S_t, with Lévy measure $\nu(x)$, is the inverse local time at 0 for a diffusion, then we can realize another subordinator, $S_t^{(m)}$, with Lévy measure $\nu^{(m)}(x) = e^{-mx}\nu(x)$, as the inverse local time for some diffusion.
2. $\rho_m(x)$ here can be viewed as the unique solution to

$$\begin{cases} (\mathcal{L} - m)\rho_m = 0; \\ \rho_m(0) = 1, \ \rho_m(\infty) = 0. \end{cases}$$

3. If $X_t^{(\alpha)}$ is a Bessel process on $[0, \infty)$, determined by

$$\mathcal{L}^{(\alpha)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{1-2\alpha}{2x}\frac{d}{dx}, \ 0 < \alpha < 1.$$

   By the above remark, for every $m > 0$, $\rho_m(x) = d_\alpha \widehat{K}_\alpha(\sqrt{2m}x)$, where $\widehat{K}_\alpha = x^\alpha K_\alpha$ and K_α is a modified Bessel function of the second kind. The inverse local time of $X^{(m)}$ is the relativistic α-stable subordinator.

Comparison Theorem

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, X_t and Y_t are diffusions, reflecting at 0, and determined by the local generator

$$\mathcal{L}^X = \frac{1}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx};$$
$$\mathcal{L}^Y = \frac{1}{2}\frac{d^2}{dx^2} + B(x)\frac{d}{dx}.$$

If $X_0 \leq Y_0$ and $f(x) = B(x) - b(x) \geq 0$ and satisfies the condition

$$\sup_{x>0} \mathbb{E}_x \left[\int_0^T |f(X_t)|^2 dt \right] < \infty, \quad \text{for any fixed time } T > 0. \tag{3}$$

Suppose S_t^X, S_t^Y are the corresponding inverse local times at 0, then $S_t^X \leq S_t^Y$, $\mathbb{P}$-a.s.

Sketch of proof.

- Use Girsanov transform to define a new measure, $\mathbb{Q}$. And $(Z^X, \mathbb{Q}) \stackrel{d}{=} (Z^Y, \mathbb{P})$, Z^X, Z^Y are zero sets.
- $M_t := e^{-T_0}\mathbf{1}_{\{T_0 \le t\}} - \int_0^t e^{-s} dL_s^X$ is a $\mathbb{P}$-martingale, and $\mathbb{Q}$-martingale as well.
- $(L_t^X, \mathbb{Q}) \stackrel{d}{=} (L_t^Y, \mathbb{P})$.
- Using Hausdorff measure of zero sets and classic comparison theorem, $L_t^Y \le L_t^X$, $\mathbb{P}$-a.s.

A Counterexample

For Bessel processes with the generator

$$\mathcal{L}^{(\alpha)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{1-2\alpha}{2x}\frac{d}{dx}.$$

With $0 < \beta < \alpha < 1$ and $X_0^{(\alpha)} \leq X_0^{(\beta)}$, we only have $X_t^{(\alpha)} \leq X_t^{(\beta)}$. $S_t^{(\alpha)} \leq S_t^{(\beta)}$ is not true , because their Laplace exponents, $c_\alpha \lambda^\alpha \leq c_\beta \lambda^\beta$ does not hold for all $\lambda > 0$.

Corollary

Suppose X_t, Y_t are as described in Comparison Theorem, ϕ^X and ϕ^Y are Laplace exponents of inverse local times, respectively. Then $\phi^Y - \phi^X$ is completely monotone.

Sketch of proof:

- By regenerative embedding theory, ϕ^Y/ϕ^X is completely monotone.
- By Comparison Theorem, $0 \leq \phi^X \leq \phi^Y$.
- $\left(\frac{\phi^Y}{\phi^X} - 1\right)\phi^X = \phi^Y - \phi^X$ is completely monotone.

Perturbed Bessel processes

Recall that a relativistic α-stable subordinator can be viewed as the inverse local time of the diffusion $X_t^{(\alpha,m)}$, locally determined by

$$\mathcal{L}^{(\alpha,m)} = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{1-2\alpha}{2x} + \frac{\widehat{K}'_\alpha(\sqrt{2m}x)}{\widehat{K}_\alpha(\sqrt{2m}x)}\right)\frac{d}{dx}, \text{ for } x > 0. \tag{4}$$

Lemma

For $m \geq 0$, $0 < \alpha < 1$,

$$\frac{\widehat{K}'_\alpha(\sqrt{2m}x)}{\widehat{K}_\alpha(\sqrt{2m}x)} \sim \begin{cases} -\frac{m^\alpha \Gamma(1-\alpha)}{2^{\alpha-1}\Gamma(\alpha)} x^{2\alpha-1} \text{ as } x \to 0+; \\ -\sqrt{2m} \text{ as } x \to \infty, \end{cases} \tag{5}$$

We would then achieve two main properties for a broad type of perturbed Bessel processes.

Comparison theorem for inverse local times

Theorem

Let Y_t be a diffusion process on $[0, \infty)$ determined by the local generator

$$\mathcal{L} = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{1-2\alpha}{2x} - f(x)\right)\frac{d}{dx},$$

where

$$0 \le f(x) \lesssim \begin{cases} 1 \vee x^{2\alpha-1}, & \text{for } 0 < \alpha < 1/2; \\ 1 \wedge x^{2\alpha-1}, & \text{for } 1/2 \le \alpha < 1. \end{cases}$$

Let S_t be the inverse local time for Y_t at 0. Then there is a constant $m > 0$ so that $S_t^{(\alpha,m)} \le S_t \le S_t^{(\alpha)}$, $\mathbb{P}$-a.s.

Application: Green function estimates

Theorem

Let B_t be a d-dimensional Brownian motion independent of Y_t, and $\mu(x)$ be the density of the Lévy measure of trace process B_{S_t}. Denote the density of the Lévy measure of symmetric 2α-stable process by $\mu^{(\alpha)}(x)$. Then $j(x) = \mu^{(\alpha)}(x) - \mu(x) \geq 0$, and there exists a constant C such that for $|x| \leq 1$,

$$j(x) \leq C|x|^{-d+2-2\alpha}.$$

Let $D \subset \mathbb{R}^d$ be a bounded connected Lipschitz open set. Denote Green functions of D for trace processes by $G_D(x, y)$ and $G_D^{(\alpha)}(x, y)$, respectively. Then there exists a constant $C_1 = C_1(d, \alpha, D, C)$ such that

$$C_1^{-1} G_D^{(\alpha)}(x, y) \leq G_D(x, y) \leq C_1 G_D^{(\alpha)}(x, y),$$

for $x, y \in D$.

Thank you!