The six phases of a two-parameter scaled Brownian penalization

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May 10, 2016

Brownian penalization

Introduced by Roynette-Vallois-Yor in a series of papers beginning around 2005.

Ingredients:

1. canonical Brownian motion: $(C[0,\infty), X_t, \mathcal{F}_t, \mathcal{F}_\infty, \mathbb{W})$
2. adapted weight process $0 \leq \Gamma_t$ with $0 < E[\Gamma_t] =: Z_t^\Gamma < \infty$

If there exists a probability measure Q^Γ on $\mathcal{F}_\infty$ such that for all $s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$

$$\lim_{t \to \infty} \frac{1}{Z_t^\Gamma} E[1_{\Lambda_s} \Gamma_t] = Q^\Gamma(\Lambda_s)$$

then the coordinate process under Q^Γ is called Brownian motion penalized by Γ.

scaled Brownian penalization

By only considering events up to a fixed time in the defining limit, ordinary penalization doesn't "penalize the entire path". A remedy of this situation is the *scaled penalization* which produces a probability measure on $C[0,1]$.

Additional ingredients:

- canonical space of $C[0,1]$ paths: $(C[0,1], Y_s, \mathcal{A})$
- scaling exponent $\alpha \geq 0$

If there exists a non-trivial probability measure $Q^{\alpha,\Gamma}$ on $\mathcal{A}$ such that for all bounded and continuous $F : C[0,1] \to \mathbb{R}$

$$\lim_{t\to\infty} \frac{1}{Z_t^\Gamma} E_0 \left[F\left(\frac{X_{\bullet t}}{t^\alpha} \right) \Gamma_t \right] = Q^{\alpha,\Gamma} \left[F\left(Y_\bullet \right) \right]$$

then the coordinate process under $Q^{\alpha,\Gamma}$ is called Brownian motion penalized by Γ with α scaling.

a two-parameter unscaled Brownian penalization

Introduced in Roynette-Vallois-Yor 2005.

preliminaries:

1. $S_t := \sup_{0 \le s \le t} X_s$ is the running maximum
2. parameters $h, \nu \in \mathbb{R}$, h can be interpreted as drift
3. weight process $\Gamma_t = \exp(hX_t + \nu S_t)$
4. Q^Γ has three different "phases" depending on where (h, ν) lies in the parameter plane

Theorem (Roynette-Vallois-Yor 2005)

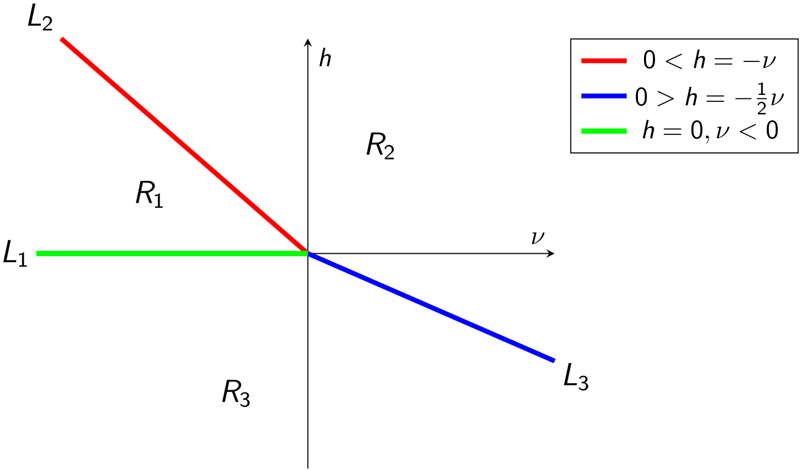
For all $s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$,

$$\lim_{t\to\infty} \frac{1}{Z_t^{h,\nu}} E_0 \left[1_{\Lambda_s} \exp\left(hX_t + \nu S_t\right)\right] = E_0 \left[1_{\Lambda_s} M_s^{h,\nu}\right]$$

where $M_s^{h,\nu}$ is a positive martingale given by

$$M_s^{h,\nu} = \begin{cases} -(h+\nu)e^{(h+\nu)S_s}(S_s - X_s) + e^{(h+\nu)S_s} & : (h,\nu) \in R_1 \\ e^{(h+\nu)X_s - \frac{1}{2}(h+\nu)^2 s} & : (h,\nu) \in R_2 \\ e^{(h+\nu)S_s - \frac{1}{2}h^2 s}\Big(\cosh\left(h(S_s - X_s)\right) - \frac{h+\nu}{h}\sinh\left(h(S_s - X_s)\right)\Big) & : (h,\nu) \in R_3. \end{cases}$$

phase diagram



Remark

In the unscaled penalization, $L_2, L_3 \in R_2$ and $L_1 \in R_1$.

Brownian meander

Path construction:

1. Start with a standard Brownian path of duration 1.
2. Shift the last incomplete excursion so it starts at time 0.
3. Take absolute value.
4. Rescale path to have duration 1.

Theorem (Imhof 1984)

Let m be a Brownian meander and r be a Bessel-3 process of duration 1. Then for measurable $F : C[0,1] \to \mathbb{R}$

$$E[F(m_\bullet)] = \sqrt{\frac{\pi}{2}}\, E_0\left[F(r_\bullet)\frac{1}{r_1}\right]$$

$$E_0[F(r_\bullet)] = \sqrt{\frac{2}{\pi}}\, E\left[F(m_\bullet)m_1\right].$$

Denisov's path decomposition

Theorem (Denisov 1984)

Let W be a standard Brownian path of duration 1 and let θ be the time that its maximum is attained. Then

$$m_s := \frac{1}{\sqrt{\theta}}(W_\theta - W_{\theta(1-s)}),\ 0 \le s \le 1$$

and

$$\tilde{m}_s := \frac{1}{\sqrt{1-\theta}}(W_\theta - W_{\theta+(1-\theta)s}),\ 0 \le s \le 1$$

are independent Brownian meanders.

Critical Line L_1: old result

Theorem (Roynette-Yor 2009)

Let $\nu < 0$ and m be a Brownian meander. Then for bounded and continuous $F : C[0,1] \to \mathbb{R}$

$$\lim_{t\to\infty} \frac{1}{Z_t^\nu} E_0 \left[F\left(\frac{X_{\bullet t}}{\sqrt{t}} \right) \exp\left(\nu S_t\right) \right] = E\left[F(-m_\bullet)\right].$$

Critical Line L_2

Theorem

Let $(h, \nu) \in L_2$ and m be a Brownian meander. For bounded and continuous $F : C[0,1] \to \mathbb{R}$ we have

$$\lim_{t\to\infty} \frac{1}{Z_t^{h,\nu}} E_0 \left[F\left(\frac{X_{\bullet t}}{\sqrt{t}} \right) \exp\left(hX_t + \nu S_t \right) \right] = E\left[F(m_1 - m_{1-\bullet}) \right].$$

Key tool of proof:

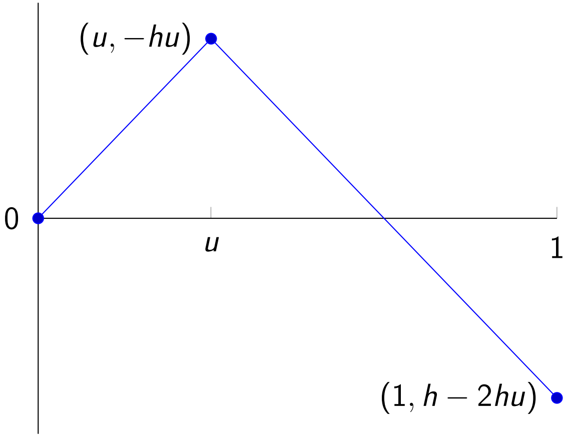
1. Imhof relation

Remark (*Brownian ascent*)

Denisov's path decomposition implies that $m_1 - m_{1-\bullet}$ is a Brownian path conditioned to end at its maximum.

Critical Line L_3: preliminary

Fix $h < 0$. For $0 < u < 1$, consider the path ω_u that linearly interpolates between the points $(0,0)$, $(u, -hu)$ and $(1, h - 2hu)$.



Critical Line L_3: main result

Theorem

Let $(h, \nu) \in L_3$. For bounded and continuous $F : C[0,1] \to \mathbb{R}$ we have

$$\lim_{t\to\infty} \frac{1}{Z_t^{h,\nu}} E_0 \left[F\left(\frac{X_{\bullet t}}{t}\right) \exp\left(hX_t + \nu S_t\right)\right] = \int_0^1 F(\omega_u)\, \mathrm{d}u.$$

Key tools of proof:

1. Denisov's path decomposition
2. multi-variable Laplace method

Region R_1: different scaling for path and endpoint

Theorem

Let $(h,\nu) \in R_1$, $\lambda > 0$ and $\theta \in \mathbb{R}$. Then

$$\lim_{t\to\infty} \frac{1}{Z_t^{h,\nu}} E_0 \left[\exp\left(i\theta X_t - \lambda S_t\right) \exp\left(hX_t + \nu S_t\right)\right] = \frac{h^2(h+\nu)^2(\nu-\lambda)}{\nu(h+i\theta)^2(h+\nu+i\theta-\lambda)^2}.$$

The associated joint density is

$$P(X_\infty \in \mathrm{d}x, S_\infty \in \mathrm{d}y) = \frac{h^2(h+\nu)^2}{-\nu}(2y-x)\exp(hx+\nu y)1_{x<y}1_{y>0}\,\mathrm{d}y\,\mathrm{d}x.$$

Theorem

Let $(h,\nu) \in R_1$ and e be a normalized Brownian excursion. For bounded and continuous $F : C[0,1] \to \mathbb{R}$ we have

$$\lim_{t\to\infty} \frac{1}{Z_t^{h,\nu}} E_0 \left[F\left(\frac{X_{\bullet t}}{\sqrt{t}}\right) \exp\left(hX_t + \nu S_t\right)\right] = E\left[F(-e_\bullet)\right].$$

Regions R_2 and R_3

Theorem

Let $F : C[0,1] \to \mathbb{R}$ be bounded and continuous. Then we have

$$\lim_{t\to\infty} \frac{1}{Z_t^{h,\nu}} E_0\left[F\left(\frac{X_{\bullet t}}{t}\right)\exp\left(hX_t + \nu S_t\right)\right] = \begin{cases} F\big(\bullet\,(h+\nu)\big) & : (h,\nu) \in R_2 \\ F\big(\bullet\, h\big) & : (h,\nu) \in R_3. \end{cases}$$

Remark

When $(h,\nu) \in L_3$, note that $h + \nu = -h$. So the L_3 limit path can be seen to switch from R_2 behavior to R_3 behavior at a random time uniform on $[0,1]$.

FCLT for Regions R_2 and R_3

Theorem

Let W be a Brownian path of duration 1. For bounded and continuous $F : C[0,1] \to \mathbb{R}$, the following limits hold:

$$\left.\begin{array}{l} \text{if } (h,\nu) \in R_2, \text{ then} \\ \displaystyle\lim_{t\to\infty} \frac{1}{Z_t^{h,\nu}} E_0\left[F\left(\frac{X_{\bullet t} - \bullet t(h+\nu)}{\sqrt{t}}\right)\exp\left(hX_t + \nu S_t\right)\right] \\ \\ \text{if } (h,\nu) \in R_3, \text{ then} \\ \displaystyle\lim_{t\to\infty} \frac{1}{Z_t^{h,\nu}} E_0\left[F\left(\frac{X_{\bullet t} - \bullet th}{\sqrt{t}}\right)\exp\left(hX_t + \nu S_t\right)\right] \end{array}\right\} = E_0\left[F(W_\bullet)\right].$$

Thanks for your attention!